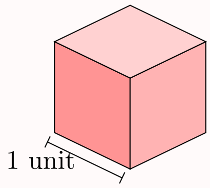
VOLUME

A DEFINITION

Definition Volume

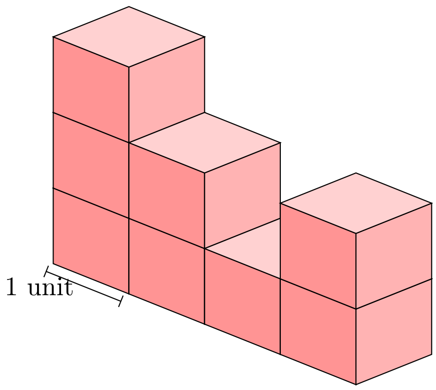
The **volume** of an object is the amount of space it takes up. We measure volume by counting how many **cubic units** can fit inside it. A cubic unit is a cube with sides that are 1 unit long.



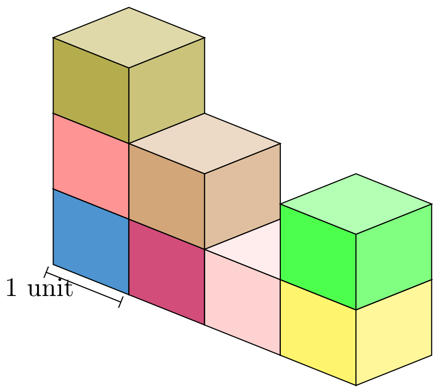
Method Counting Cubes to Find Volume

To find the volume of a shape made of blocks, simply count the total number of blocks (cubic units) it is made from. A good strategy is to count the blocks in each layer.

Ex: Find the volume of the shape below.



Answer: We can find the volume by counting the cubes in the shape. Each small cube has a volume of 1 cubic unit.



There are 8 cubes in total, so:

$$\text{Volume} = 8 \text{ cubic units}$$

B UNITS OF VOLUME

Discover: When we measure volume, it is important to use **standard units** so that everyone gets the same measurement. Non-standard units, like different-sized building blocks, can give different answers.
For volume, we use standard units like the **cubic centimeter**, written cm^3, and the **cubic meter**, written m^3.

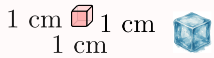
Definition Units of Volume

- **Cubic Millimeter** (mm^3): the volume of a cube with sides 1 m long. This is about the size of a tiny grain of sand.

$$1\ mm^3 = 1\ mm \times 1\ mm \times 1\ mm$$

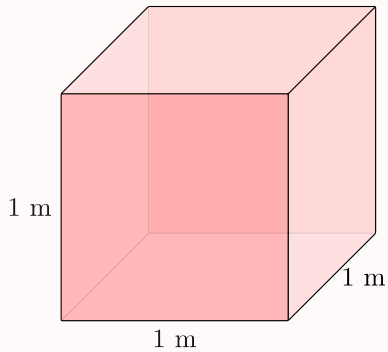
- **Cubic Centimeter** (cm^3): The volume of a cube with sides 1 cm long. This is about the size of an ice cube.

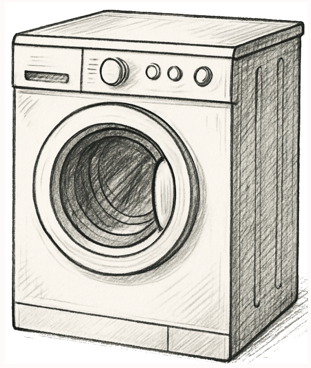
$$1\ cm^3 = 1\ cm \times 1\ cm \times 1\ cm$$



- **Cubic Meter** (m^3): the volume of a cube with sides 1 m long. This is about the volume of a washing machine.

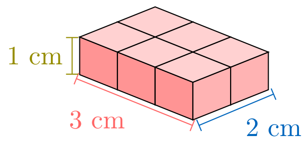
$$1\ m^3 = 1\ m \times 1\ m \times 1\ m$$



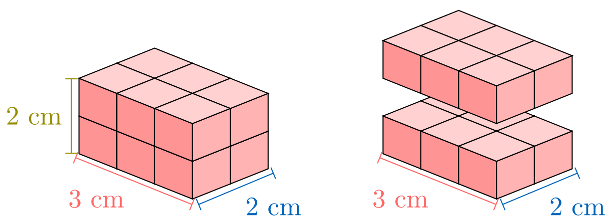


C VOLUME OF A RECTANGULAR CUBOID

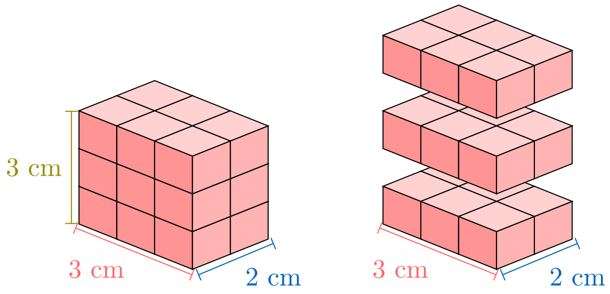
Discover: Counting every little cube inside a rectangular box (rectangular cuboid) gives its volume, but that is slow. Instead, imagine making the box taller *one layer at a time* and watching how the volume grows.
Each new layer adds the same number of cubes. By counting layer by layer, we spot a pattern and get a quick rule for volume: we can multiply the *length*, the *width*, and the *height*.

- 

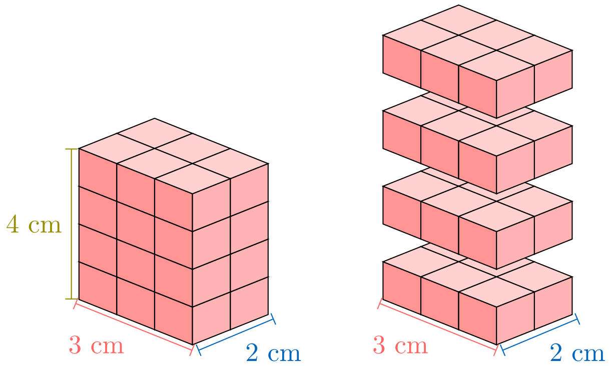
$$\begin{aligned} \text{Volume} &= 3 \times 2 \times 1 \\ &= 6\,cm^3 \end{aligned}$$

- 

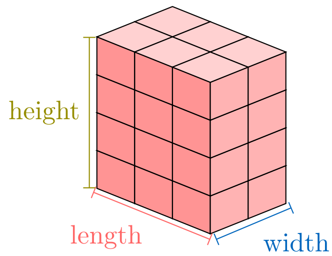
$$\begin{aligned} \text{Volume} &= (3 \times 2) + (3 \times 2) \\ &= (3 \times 2) \times 2 \\ &= 12\,cm^3 \end{aligned}$$

- 

$$\begin{aligned}\text{Volume} &= (3 \times 2) + (3 \times 2) + (3 \times 2)\\ &= (3 \times 2) \times 3\\ &= 18\,\text{cm}^3\end{aligned}$$

- 

$$\begin{aligned}\text{Volume} &= (3 \times 2) + (3 \times 2) + (3 \times 2) + (3 \times 2)\\ &= (3 \times 2) \times 4\\ &= 24\,\text{cm}^3\end{aligned}$$

- 

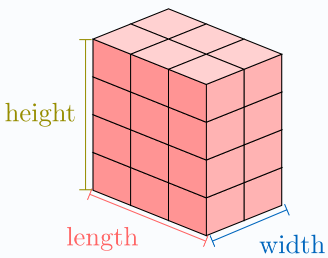
$$\text{Volume} = \text{length} \times \text{width} \times \text{height}$$

Proposition Volume of a Rectangular Cuboid

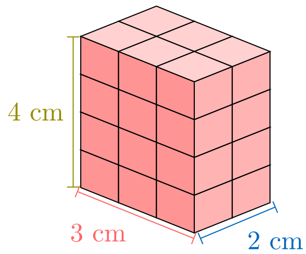
The volume of a rectangular cuboid (also called a rectangular prism) is found by multiplying its length, width, and height:

$$\text{Volume} = \text{length} \times \text{width} \times \text{height}$$

$$V = l \times w \times h$$



Ex: Find the volume of this rectangular cuboid.

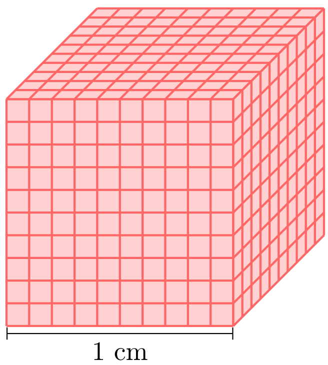


Answer: Using the formula for the volume of a rectangular cuboid:

$$\begin{aligned}\text{Volume} &= \text{length} \times \text{width} \times \text{height}\\ &= 3 \times 2 \times 4\\ &= 24\,\text{cm}^3\end{aligned}$$

D CONVERSION OF VOLUME UNITS

Discover: Let's explore how volume units are related. Consider a cube with a volume of **1** cm^3. Since $1\,\text{cm} = 10\,\text{mm}$, each side of this cube is 10 mm long.



The volume of this cube is $10\,\text{mm} \times 10\,\text{mm} \times 10\,\text{mm}$.
The bottom layer has $10 \times 10 = 100$ small cubes. Since the height is 10 mm, there are 10 layers.
Therefore, the total number of $1\,\text{mm}^3$ cubes is $100 \times 10 = 1\,000$.

$$\begin{aligned}1\,\text{cm}^3 &= 1\,\text{cm} \times 1\,\text{cm} \times 1\,\text{cm}\\ &= 10\,\text{mm} \times 10\,\text{mm} \times 10\,\text{mm} \quad (1\,\text{cm} = 10\,\text{mm})\\ &= 1\,000\,\text{mm}^3\end{aligned}$$

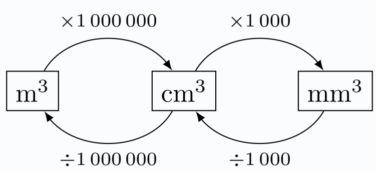
So $1\,\text{cm}^3$ is the same as $1\,000\,\text{mm}^3$. This shows that when converting volume units, the conversion factor is **cubed** (for example, 10 becomes $10^3 = 1\,000$).

Proposition Conversion of Volume Units

- $1\,\text{cm}^3 = (10 \times 10 \times 10)\,\text{mm}^3 = \mathbf{1\,000}\,\text{mm}^3$
- $1\,\text{m}^3 = (100 \times 100 \times 100)\,\text{cm}^3 = \mathbf{1\,000\,000}\,\text{cm}^3$

Method Converting Using Multiplication or Division

- Use **multiplication** to go from a larger unit to a smaller one (like cubic meters to cubic centimeters).
- Use **division** to go from a smaller unit to a larger one (like cubic centimeters to cubic meters).



Method Converting Using a Table

For volume, each unit in the place value table is split into **three columns**. Let's convert **10.5** m^3 to cm^3.

1. **Draw the volume conversion table.** Each unit has three columns.

m^3						cm^3			mm^3		

2. **Place the number in the table.** The rule is: the digit in the **ones place** goes into the **right-hand column** of the starting unit. For $10.5\,\mathrm{m}^3$, the ones digit is **0**, so it goes in the right-hand column of m^3.

m^3						cm^3			mm^3		
	1	**0**	**5**								

3. **Move the decimal point** to the right side of your target unit's columns. Our target is cm^3. Fill any empty columns with zeros.

m^3						cm^3			mm^3		
	1	0	5	**0**	**0**	**0**	**0**	**0.**			

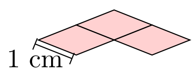
4. **Read the final number.**

So, $10.5\,\mathrm{m}^3 = 10\,500\,000\,\mathrm{cm}^3$.

E VOLUMES OF SOLIDS WITH UNIFORM CROSS-SECTION

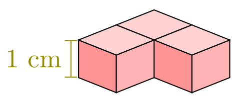
Discover: Counting the cubes inside a solid to find its volume can be time-consuming, especially for solids with a uniform cross-section. Instead, we can explore a faster method by examining the solid layer by layer to identify a pattern for calculating the volume.

- **Area of the Base:**



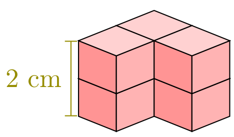
$$\text{Area} = 3\,\mathrm{cm}^2$$

- **Volume with Height of 1 cm:**



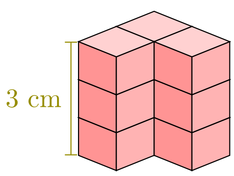
$$\begin{aligned}\text{Volume} &= \text{Area} \times 1\\ &= 3\,\mathrm{cm}^3\end{aligned}$$

- **Volume with Height of 2 cm:**



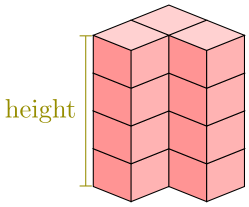
$$\begin{aligned}\text{Volume} &= \text{Area} \times 2\\ &= 6\,\mathrm{cm}^3\end{aligned}$$

- **Volume with Height of 3 cm:**



$$\begin{aligned}\text{Volume} &= \text{Area} \times 3 \\ &= 9\,\text{cm}^3\end{aligned}$$

- **General Formula:**

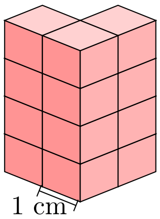


$$\text{Volume} = \text{Area} \times \text{height}$$

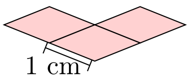
Proposition Volume of Solid of Uniform Cross-Section

$$\text{Volume} = \text{area of base (cross-section)} \times \text{height}$$

Ex: Find the volume of the figure.

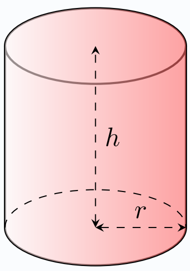


Answer:

- Area of  $= 3\,\text{cm}^2$
- height $= 4\,\text{cm}$
- $$\begin{aligned}V &= \text{area of base} \times \text{height} \\ &= 3\,\text{cm}^2 \times 4\,\text{cm} \\ &= 12\,\text{cm}^3\end{aligned}$$

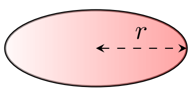
Proposition Volume of a Cylinder

$$V = \pi r^2 h$$



Proof

A cylinder has a uniform cross-section.

- Area of base = Area of  = Area of a circle $= \pi r^2$
- height $= h$

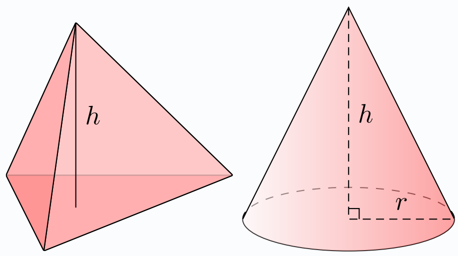
- $$V = \text{area of base} \times \text{height}$$
  $$= \pi r^2 \times h$$

F VOLUMES OF TAPERED SOLIDS AND SPHERES

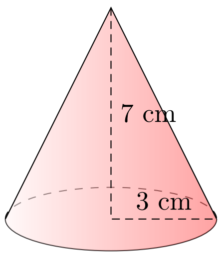
Proposition Volume of a Pyramid or Cone

The volume of any tapered solid (a pyramid or a cone) that comes to a point (apex) is one-third of the area of its base multiplied by its perpendicular height.

$$V = \frac{1}{3} \times A_{\text{base}} \times h$$



Ex: Find the volume of the cone.



Answer:

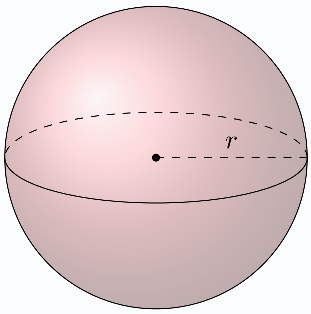
1. **Find the area of the circular base:**
$$\begin{aligned} A_{\text{base}} &= \pi r^2 \\ &= \pi(3)^2 \\ &= 9\pi \,\text{cm}^2 \end{aligned}$$

2. **Calculate the volume of the cone:**
$$\begin{aligned} V &= \frac{1}{3} \times A_{\text{base}} \times h \\ &= \frac{1}{3} \times 9\pi \times 7 \\ &= 21\pi \,\text{cm}^3 \\ &\approx 65.97 \,\text{cm}^3 \end{aligned}$$

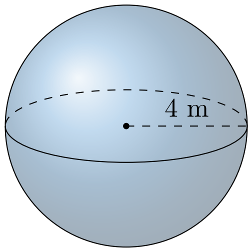
Proposition Volume of a Sphere

The volume of a sphere with radius r is given by the formula:

$$V = \frac{4}{3}\pi r^3$$



Ex: Find the volume of the sphere.



Answer:

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(4)^3 \\ &= \frac{4}{3}\pi(64) \\ &= \frac{256}{3}\pi\,\mathrm{m}^3 \\ &\approx 268.08\,\mathrm{m}^3 \end{aligned}$$

G CAPACITY

Discover: We often need to measure liquids like water, milk, or juice. Instead of using cubic centimeters, there's an **easier way** to talk about these amounts: we use the **liter (L)**.
One liter is the same as **1 000 cubic centimeters** ($1\,000\,\mathrm{cm}^3$). So **1 milliliter** ($1\,\mathrm{mL}$) is the same as **1 cubic centimeter** ($1\,\mathrm{cm}^3$).
Using liters and milliliters makes liquid amounts easier to compare and understand.

Definition Liter

A **liter** is a unit we use to measure the volume (capacity) of liquids.

- 1 **liter** is the volume of a cube that measures 10 cm on each side.

$$1\,\mathrm{L} = 1\,000\,\mathrm{cm}^3 \quad \text{and} \quad 1\,000\,\mathrm{L} = 1\,\mathrm{m}^3$$

- We write it with the symbol **L** (a capital "L").
- A smaller unit, the **centiliter** (cL), is often used for smaller volumes: $1\,\mathrm{L} = 100\,\mathrm{cL}$.
- An even smaller unit, the **milliliter** (mL), is used for very small volumes:

$$1\,\mathrm{L} = 1\,000\,\mathrm{mL} \quad \text{and} \quad 1\,\mathrm{cL} = 10\,\mathrm{mL} \quad \text{and} \quad 1\,\mathrm{mL} = 1\,\mathrm{cm}^3$$

Ex:

- A big water bottle holds about 1 L of water, which is 100 cL or 1 000 mL: 

- A small soda can holds about 0.33 L, which is 33 cL (about 330 mL): 

H DENSITY

Discover: Two blocks are the same size, but one is wood and the other is metal. Why is one much heavier? The material itself matters—this is captured by *density*.

Definition Density

The **density** ρ of a substance is its mass per unit volume:

$$\rho = \frac{m}{V}.$$

Common units: kg/m^3 or g/cm^3.

Proposition Rearranging the Density Formula

$$m = \rho V \qquad \text{and} \qquad V = \frac{m}{\rho}.$$

Method Unit Conversions

$$1\ \text{g/cm}^3 = 1000\ \text{kg/m}^3 \qquad \text{and} \qquad 1\ \text{kg/m}^3 = 0.001\ \text{g/cm}^3.$$

Ex: An aluminium block measures 3 cm $\times$ 2 cm $\times$ 4 cm. Aluminium has density $\rho = 2.7\ \text{g/cm}^3$. Find its mass.

Answer: Volume $V = 3 \cdot 2 \cdot 4 = 24\ \text{cm}^3$. Then

$$m = \rho V = 2.7 \times 24 = 64.8\ \text{g}.$$

I SURFACE AREA

Definition Surface Area

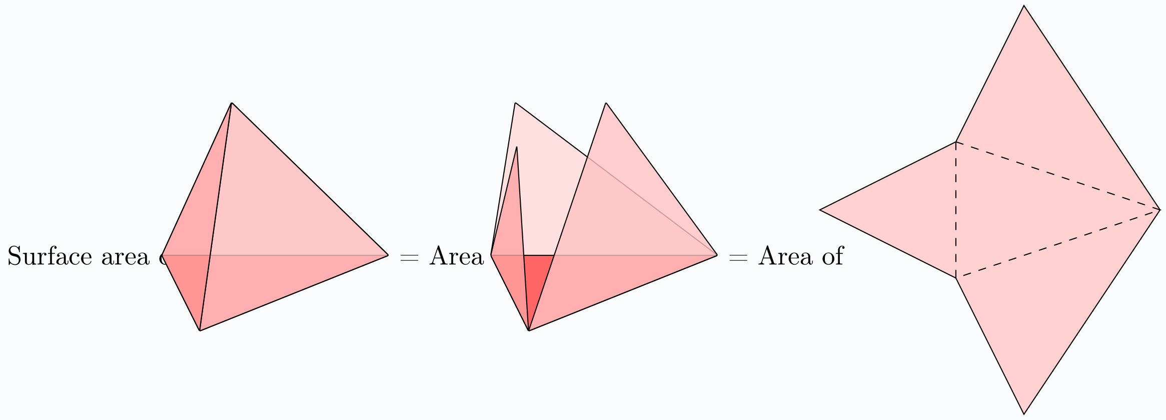
The **surface area** of a 3D shape is the total area of all its faces. It is measured in square units (cm^2, m^2, ...).

Proposition Surface area of Polyhedra

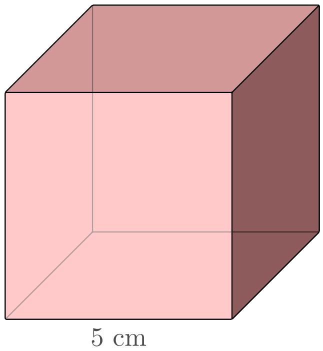
The surface area of a three-dimensional figure with plane faces is the sum of the areas of the faces.

Method Collapse the 3D figure into its net

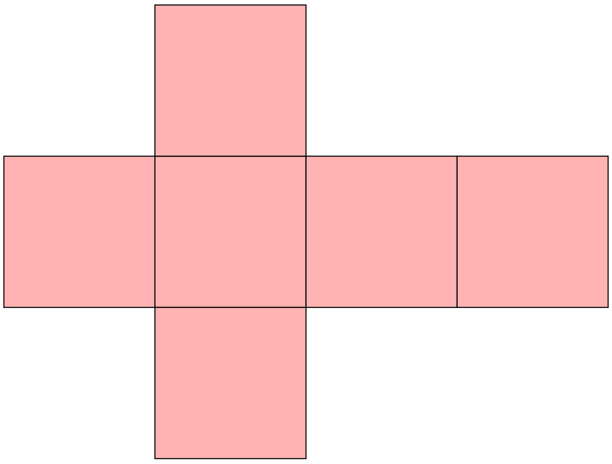
If we collapse the figure into its net, the surface area is the area of the net.



Ex: Find the surface area of the cube:

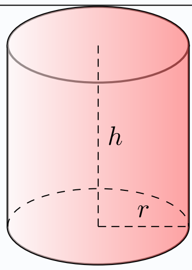
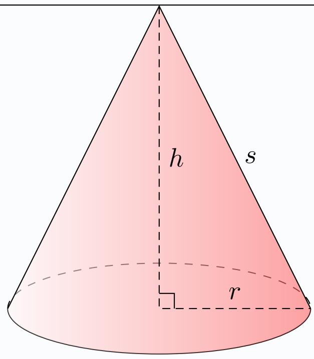
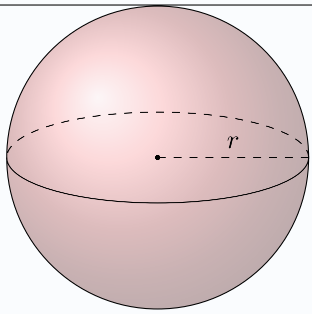


Answer: A cube has 6 identical square faces. The area of one face is the side length squared (s^2).



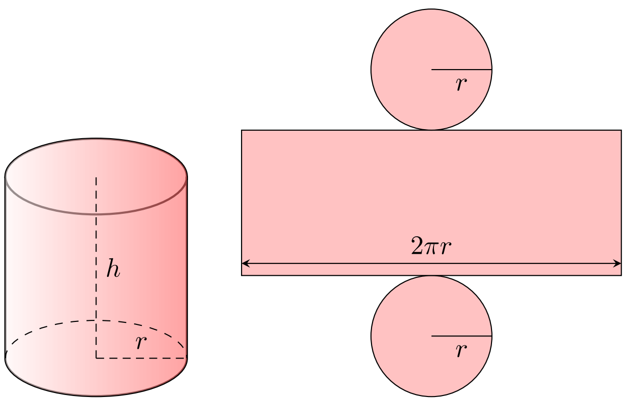
$$\begin{aligned}
\text{Surface Area} &= 6 \times (\text{Area of one square}) \\
&= 6 \times s^2 \\
&= 6 \times 5^2 \\
&= 6 \times 25 \\
&= 150\,\text{cm}^2
\end{aligned}$$

Proposition Surface area of solids with curved surfaces

Name	Solid	Surface area
Cylinder		$A =$ curved surface $+ 2$ circular ends $= 2\pi rh + 2\pi r^2$
Cone		$A =$ curved surface $+$ circular base $= \pi rs + \pi r^2$
Sphere		$A = 4\pi r^2$

Proof

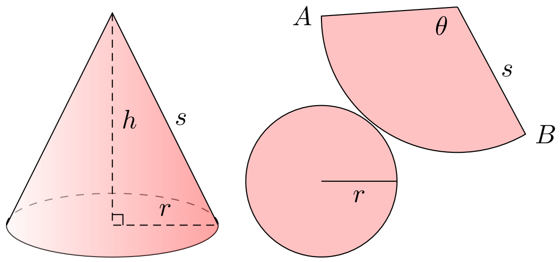
1. **Curved surface area of the cylinder.**
   If we *unroll* the cylinder, its lateral surface becomes a rectangle of height h and length equal to the circumference of the base, $2\pi r$:



   Therefore,

$$\text{area of the curved surface} = \text{area of the rectangle} = 2\pi rh.$$

2. **Curved surface area of the cone.**
   If we *unroll* the cone's lateral surface, we obtain a circular sector of radius s (the slant height) and central angle θ:



The arc length of the sector equals the circumference of the base:

$$\text{arc } AB = 2\pi r.$$

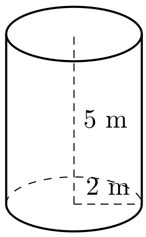
Since a sector of radius s and angle θ has arc length $\left(\frac{\theta}{360}\right)2\pi s$, we get

$$\text{arc } AB = \left(\frac{\theta}{360}\right)2\pi s \quad \Rightarrow \quad \frac{\theta}{360} = \frac{r}{s}.$$

Hence the sector area (the cone's curved surface area) is

$$\text{area of the curved surface} = \left(\frac{\theta}{360}\right)\pi s^2 = \frac{r}{s}\pi s^2 = \pi r s.$$

Ex: Find the surface area of the cylinder:



Answer:

$$\begin{aligned} A &= 2\pi r h + 2\pi r^2 \\ &= 2\pi(2)(5) + 2\pi(2)^2 \\ &= 20\pi + 8\pi \\ &= 28\pi \,\text{m}^2 \\ &\approx 87.96\,\text{m}^2 \end{aligned}$$