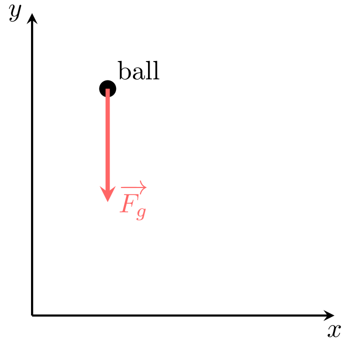
VECTOR APPLICATIONS IN PHYSICS

A NEWTON'S SECOND LAW

A.1 APPLYING NEWTON'S SECOND LAW

Ex 1: A ball of mass $m = 2$ kg is thrown into the air. Near the Earth's surface, the only significant force acting on it is the force of gravity, $\overrightarrow{F_g}$.



Assuming the y-axis points vertically upwards and the acceleration due to gravity is $g = 9.8$ m/s^2, calculate the gravitational force vector $\overrightarrow{F_g}$ and use Newton's Second Law to find the acceleration vector $\overrightarrow{a}$ of the ball.

Answer: First, we calculate the magnitude of the force of gravity (the weight):

$$F_g = mg = 2 \text{ kg} \times 9.8 \text{ m/s}^2 = 19.6 \text{ N}$$

Since this force acts vertically downwards (in the negative y-direction), its vector representation is:

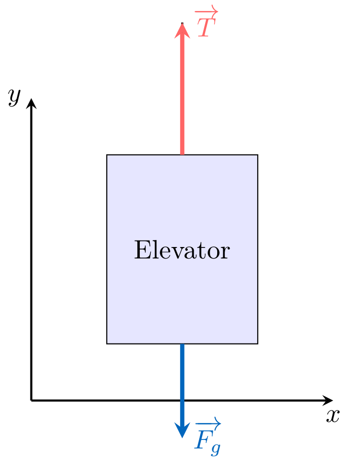
$$\overrightarrow{F_g} = \begin{pmatrix} 0 \\ -19.6 \end{pmatrix} \text{ N}$$

According to Newton's Second Law, $\sum \overrightarrow{F} = m\overrightarrow{a}$. As gravity is the only force, we have:

$$\begin{aligned} \overrightarrow{F_g} &= m\overrightarrow{a} \\ \begin{pmatrix} 0 \\ -19.6 \end{pmatrix} &= 2\overrightarrow{a} \\ \overrightarrow{a} &= \frac{1}{2}\begin{pmatrix} 0 \\ -19.6 \end{pmatrix} \\ \overrightarrow{a} &= \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} \text{ m/s}^2 \end{aligned}$$

The acceleration vector of the ball is $\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$ m/s^2, which is the constant acceleration due to gravity.

Ex 2: An elevator of mass $m = 500$ kg is supported by a cable that exerts an upward tension force $\overrightarrow{T}$ with a magnitude of 5900 N. The force of gravity, $\overrightarrow{F_g}$, acts downwards. The y-axis is oriented vertically upwards.



Using the acceleration due to gravity $g = 9.8$ m/s^2,

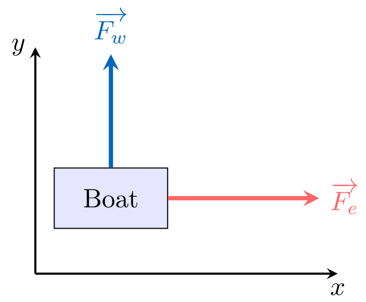
1. Calculate the force of gravity, $F_g = mg$, and write the vectors for the tension force, $\overrightarrow{T}$, and the gravitational force, $\overrightarrow{F_g}$.
2. Calculate the net force $\sum \overrightarrow{F}$ on the elevator.
3. Determine the acceleration vector $\overrightarrow{a}$ of the elevator.

Answer:

1. **Force Vectors** First, we calculate the magnitude of the gravitational force:
$$F_g = mg = 500 \text{ kg} \times 9.8 \text{ m/s}^2 = 4900 \text{ N}$$
The tension force acts upwards (positive y-direction) and the gravitational force acts downwards (negative y-direction).
$$\overrightarrow{T} = \begin{pmatrix} 0 \\ 5900 \end{pmatrix} \text{ N} \quad \text{and} \quad \overrightarrow{F_g} = \begin{pmatrix} 0 \\ -4900 \end{pmatrix} \text{ N}$$
2. **Net Force** The net force is the vector sum of all forces.
$$\begin{aligned} \sum \overrightarrow{F} &= \overrightarrow{T} + \overrightarrow{F_g} \\ &= \begin{pmatrix} 0 \\ 5900 \end{pmatrix} + \begin{pmatrix} 0 \\ -4900 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1000 \end{pmatrix} \text{ N} \end{aligned}$$
3. **Acceleration** We use Newton's Second Law, $\sum \overrightarrow{F} = m\overrightarrow{a}$.
$$\begin{aligned} \begin{pmatrix} 0 \\ 1000 \end{pmatrix} &= 500\overrightarrow{a} \\ \overrightarrow{a} &= \frac{1}{500}\begin{pmatrix} 0 \\ 1000 \end{pmatrix} \\ \overrightarrow{a} &= \begin{pmatrix} 0 \\ 2 \end{pmatrix} \text{ m/s}^2 \end{aligned}$$

The acceleration of the elevator is $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ m/s^2, which means it is accelerating upwards at 2 m/s^2.

Ex 3: A boat of mass $m = 200$ kg is moving on the surface of a lake. The boat's engine exerts a force $\overrightarrow{F_e}$ of 600 N towards the East (positive x-direction). The wind exerts a force $\overrightarrow{F_w}$ of 200 N towards the North (positive y-direction). Friction is negligible.



1. Write the column vectors for the engine force, $\overrightarrow{F_e}$, and the wind force, $\overrightarrow{F_w}$.
2. Calculate the net force vector $\sum \overrightarrow{F}$ acting on the boat.
3. Use Newton's Second Law to find the acceleration vector $\overrightarrow{a}$ of the boat.

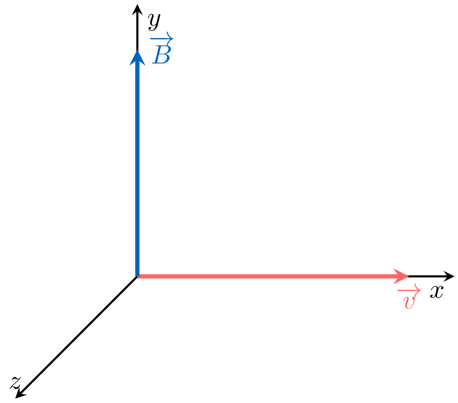
Answer:

1. **Force Vectors** The engine force acts in the positive x-direction, and the wind force acts in the positive y-direction.
$$\overrightarrow{F_e} = \begin{pmatrix} 600 \\ 0 \end{pmatrix} \text{ N} \quad \text{and} \quad \overrightarrow{F_w} = \begin{pmatrix} 0 \\ 200 \end{pmatrix} \text{ N}$$
2. **Net Force** The net force is the vector sum of the individual forces:
$$\begin{aligned} \sum \overrightarrow{F} &= \overrightarrow{F_e} + \overrightarrow{F_w} \\ &= \begin{pmatrix} 600 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 200 \end{pmatrix} \\ &= \begin{pmatrix} 600 \\ 200 \end{pmatrix} \text{ N} \end{aligned}$$
3. **Acceleration** Applying Newton's Second Law, $\sum \overrightarrow{F} = m\overrightarrow{a}$:
$$\begin{aligned} \begin{pmatrix} 600 \\ 200 \end{pmatrix} &= 200\overrightarrow{a} \\ \overrightarrow{a} &= \frac{1}{200}\begin{pmatrix} 600 \\ 200 \end{pmatrix} \\ \overrightarrow{a} &= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \text{ m/s}^2 \end{aligned}$$

The acceleration vector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ m/s².

A.2 APPLYING THE VECTOR PRODUCT IN PHYSICS

Ex 4: The magnetic force $\overrightarrow{F}$ exerted on a particle with electric charge q moving with velocity $\overrightarrow{v}$ through a magnetic field $\overrightarrow{B}$ is given by the Lorentz force formula:
$$\overrightarrow{F} = q(\overrightarrow{v} \times \overrightarrow{B})$$
A particle with charge $q = 2$ C enters a uniform magnetic field $\overrightarrow{B} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}$ T with a velocity of $\overrightarrow{v} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ m/s, as shown below.

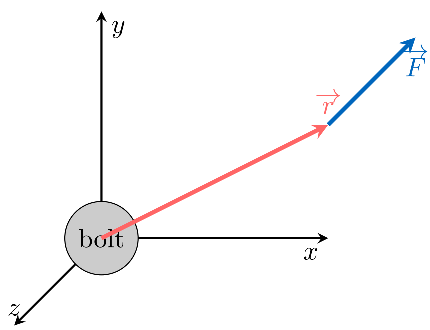


Calculate the magnetic force $\overrightarrow{F}$ acting on the particle. The resulting force will be in Newtons (N).

Answer:
$$\begin{aligned} \overrightarrow{F} &= q(\overrightarrow{v} \times \overrightarrow{B}) \\ &= 2\left(\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix}\right) \\ &= 2\begin{pmatrix} (0)(0)-(0)(5) \\ (0)(0)-(3)(0) \\ (3)(5)-(0)(0) \end{pmatrix} \\ &= 2\begin{pmatrix} 0 \\ 0 \\ 15 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix} \end{aligned}$$

The magnetic force acting on the particle is $\overrightarrow{F} = \begin{pmatrix} 0 \\ 0 \\ 30 \end{pmatrix}$ N. The force has a magnitude of 30 N and acts in the positive z-direction, perpendicular to both velocity and magnetic field.

Ex 5: Torque, or the turning moment of a force, is a measure of how much a force acting on an object causes that object to rotate. The torque vector $\overrightarrow{\tau}$ is calculated using the vector product of the position vector $\overrightarrow{r}$ (from the axis of rotation to the point where the force is applied) and the force vector $\overrightarrow{F}$.
$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F}$$
A force of $\overrightarrow{F} = \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix}$ N is applied to a wrench at a position $\overrightarrow{r} = \begin{pmatrix} 0.2 \\ 0.1 \\ 0 \end{pmatrix}$ m relative to the center of a bolt, as shown.



Calculate the torque vector $\overrightarrow{\tau}$ on the bolt. The resulting torque will be in Newton-meters (Nm).

Answer:

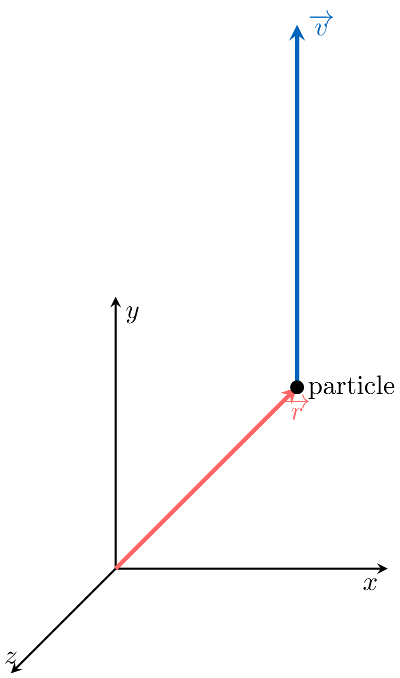
$$\begin{aligned}\vec{\tau} &= \vec{r} \times \vec{F} \\ &= \begin{pmatrix}0.2\\0.1\\0\end{pmatrix} \times \begin{pmatrix}0\\0\\-10\end{pmatrix} \\ &= \begin{pmatrix}(0.1)(-10)-(0)(0)\\(0)(0)-(0.2)(-10)\\(0.2)(0)-(0.1)(0)\end{pmatrix} \\ &= \begin{pmatrix}-1\\2\\0\end{pmatrix}\end{aligned}$$

The torque on the bolt is $\vec{\tau} = \begin{pmatrix}-1\\2\\0\end{pmatrix}$ Nm. This vector indicates the axis and direction of the resulting rotation.

Ex 6: The angular momentum $\vec{L}$ of a particle relative to a point of origin is a measure of its rotational motion. It is defined by the vector product of the particle's position vector $\vec{r}$ and its linear momentum vector $\vec{p}$, where $\vec{p} = m\vec{v}$.

$$\vec{L} = \vec{r} \times \vec{p}$$

A particle of mass $m = 3$ kg is at position $\vec{r} = \begin{pmatrix}2\\2\\0\end{pmatrix}$ m and is moving with a velocity $\vec{v} = \begin{pmatrix}0\\4\\0\end{pmatrix}$ m/s.



Calculate the angular momentum vector $\vec{L}$ of the particle. The units will be kg·m²/s.

Answer: First, we find the linear momentum $\vec{p}$:

$$\vec{p} = m\vec{v} = 3\begin{pmatrix}0\\4\\0\end{pmatrix} = \begin{pmatrix}0\\12\\0\end{pmatrix}$$

Now, we calculate the angular momentum $\vec{L}$ using the cross product:

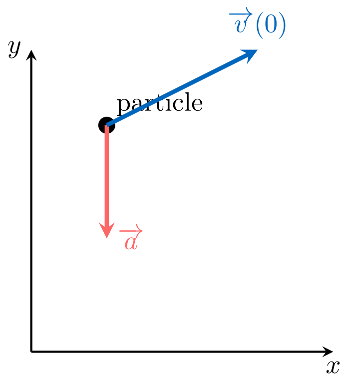
$$\begin{aligned}\vec{L} &= \vec{r} \times \vec{p} \\ &= \begin{pmatrix}2\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\12\\0\end{pmatrix} \\ &= \begin{pmatrix}(2)(0)-(0)(12)\\(0)(0)-(2)(0)\\(2)(12)-(2)(0)\end{pmatrix} \\ &= \begin{pmatrix}0\\0\\24\end{pmatrix}\end{aligned}$$

The angular momentum of the particle is $\vec{L} = \begin{pmatrix}0\\0\\24\end{pmatrix}$ kg·m²/s. This indicates a rotation around the z-axis.

B VELOCITY AND ACCELERATION WITH CALCULUS

B.1 APPLYING CALCULUS TO VECTOR KINEMATICS

Ex 7: A particle of mass m is launched with an initial velocity of $\vec{v}(0) = \begin{pmatrix}20\\30\end{pmatrix}$ m/s. The only force acting on it is gravity, which results in a constant acceleration of $\vec{a} = \begin{pmatrix}0\\-9.8\end{pmatrix}$ m/s².



Find the velocity vector $\vec{v}(t)$ of the particle at time t.

Answer: To find the velocity vector $\vec{v}(t)$, we integrate the constant acceleration vector $\vec{a} = \begin{pmatrix}0\\-9.8\end{pmatrix}$ with respect to time:

$$\begin{aligned}\vec{v}(t) &= \int \vec{a}\,dt \\ &= \int \begin{pmatrix}0\\-9.8\end{pmatrix} dt \\ &= \begin{pmatrix}C_1\\-9.8t + C_2\end{pmatrix}\end{aligned}$$

where $\begin{pmatrix}C_1\\C_2\end{pmatrix}$ is the constant vector of integration. We find its value using the initial condition, $\vec{v}(0) = \begin{pmatrix}20\\30\end{pmatrix}$.

$$\begin{aligned}\vec{v}(0) &= \begin{pmatrix}C_1\\-9.8(0) + C_2\end{pmatrix} \\ \begin{pmatrix}20\\30\end{pmatrix} &= \begin{pmatrix}C_1\\C_2\end{pmatrix}\end{aligned}$$

Substituting the constant back into the velocity equation gives:

$$\vec{v}(t) = \begin{pmatrix} 20 \\ 30 - 9.8t \end{pmatrix} \text{ m/s}$$

Ex 8: The position vector of a particle at time t seconds is given by:

$$\vec{r}(t) = \begin{pmatrix} 3t^2 - 4t \\ 8t + 1 \end{pmatrix} \text{ m}$$

1. Find the velocity vector, $\vec{v}(t)$.
2. Calculate the speed of the particle at time $t = 2$ seconds.

Answer:

1. **Velocity Vector** The velocity vector is the derivative of the position vector with respect to time.

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\begin{pmatrix} 3t^2 - 4t \\ 8t + 1 \end{pmatrix} \\ &= \begin{pmatrix} 6t - 4 \\ 8 \end{pmatrix} \text{ m/s} \end{aligned}$$

2. **Speed at t=2 s** First, we find the velocity vector at $t = 2$:

$$\vec{v}(2) = \begin{pmatrix} 6(2) - 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 12 - 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \end{pmatrix} \text{ m/s}$$

The speed is the magnitude of this velocity vector.

$$\begin{aligned} \text{Speed} = |\vec{v}(2)| &= \left\| \begin{pmatrix} 8 \\ 8 \end{pmatrix} \right\| \\ &= \sqrt{8^2 + 8^2} \\ &= \sqrt{64 + 64} \\ &= \sqrt{128} = \sqrt{64 \times 2} \\ &= 8\sqrt{2} \text{ m/s} \end{aligned}$$

The speed at $t = 2$ seconds is $8\sqrt{2}$ m/s (approx. 11.3 m/s).

Ex 9: A particle starts from rest at the point $(5, 1)$. Its acceleration vector at time t is given by $\vec{a}(t) = \begin{pmatrix} 3t \\ -2 \end{pmatrix}$ m/s².

1. Find the velocity vector, $\vec{v}(t)$.
2. Find the position vector, $\vec{r}(t)$.

Answer:

1. **Velocity Vector** We find the velocity by integrating the acceleration vector.

$$\vec{v}(t) = \int \begin{pmatrix} 3t \\ -2 \end{pmatrix} dt = \begin{pmatrix} \frac{3}{2}t^2 \\ -2t \end{pmatrix} + \vec{C}$$

The particle "starts from rest," which means its initial velocity is zero: $\vec{v}(0) = \vec{0}$. We use this to find the constant vector $\vec{C}$.

$$\vec{v}(0) = \begin{pmatrix} \frac{3}{2}(0)^2 \\ -2(0) \end{pmatrix} + \vec{C} \implies \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \vec{C} \implies \vec{C} = \vec{0}$$

So, the velocity vector is $\vec{v}(t) = \begin{pmatrix} \frac{3}{2}t^2 \\ -2t \end{pmatrix}$ m/s.

2. **Position Vector** We find the position by integrating the velocity vector.

$$\vec{r}(t) = \int \begin{pmatrix} \frac{3}{2}t^2 \\ -2t \end{pmatrix} dt = \begin{pmatrix} \frac{1}{2}t^3 \\ -t^2 \end{pmatrix} + \vec{D}$$

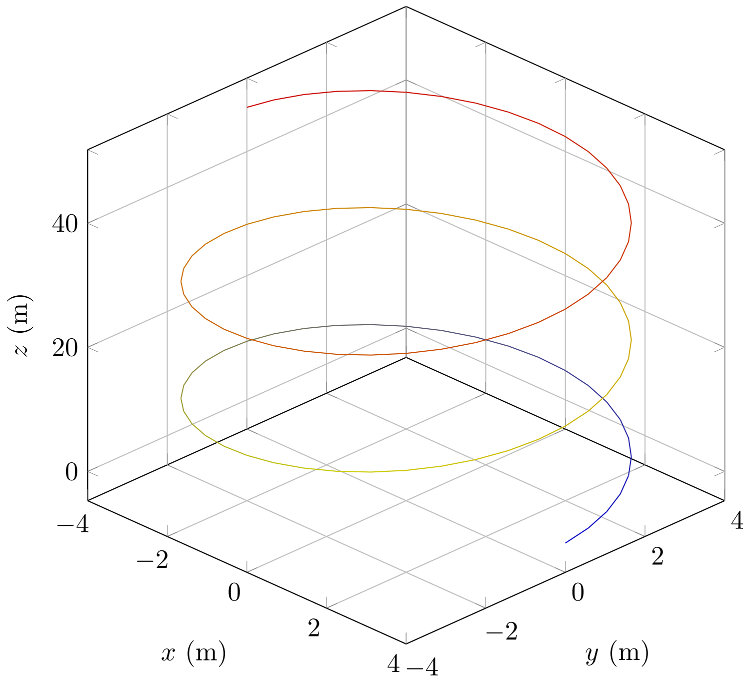
The particle starts at the point $(5, 1)$, so its initial position is $\vec{r}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. We use this to find the constant vector $\vec{D}$.

$$\vec{r}(0) = \begin{pmatrix} \frac{1}{2}(0)^3 \\ -(0)^2 \end{pmatrix} + \vec{D} \implies \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \vec{D} \implies \vec{D} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Therefore, the position vector is $\vec{r}(t) = \begin{pmatrix} \frac{1}{2}t^3 + 5 \\ -t^2 + 1 \end{pmatrix}$ m.

Ex 10: The position vector of a particle at time t seconds is given by:

$$\vec{r}(t) = \begin{pmatrix} 4\cos(t) \\ 4\sin(t) \\ 3t \end{pmatrix} \text{ m}$$



1. Find the velocity vector, $\vec{v}(t)$.
2. Calculate the speed of the particle at any time t.

Answer:

1. **Velocity Vector** The velocity vector is the derivative of the position vector with respect to time.

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}\begin{pmatrix} 4\cos(t) \\ 4\sin(t) \\ 3t \end{pmatrix} \\ &= \begin{pmatrix} -4\sin(t) \\ 4\cos(t) \\ 3 \end{pmatrix} \text{ m/s} \end{aligned}$$

2. **Speed at time t** The speed is the magnitude of the velocity vector.

$$\begin{aligned}\text{Speed} = |\overrightarrow{v}(t)| &= \left\| \begin{pmatrix} -4\sin(t) \\ 4\cos(t) \\ 3 \end{pmatrix} \right\| \\ &= \sqrt{(-4\sin(t))^2 + (4\cos(t))^2 + 3^2} \\ &= \sqrt{16\sin^2(t) + 16\cos^2(t) + 9} \\ &= \sqrt{16(\sin^2(t) + \cos^2(t)) + 9} \\ &= \sqrt{16(1) + 9} \quad (\text{since } \sin^2(t) + \cos^2(t) = 1) \\ &= \sqrt{25} \\ &= 5 \text{ m/s}\end{aligned}$$

The speed of the particle is constant at 5 m/s. (Note: This describes a particle moving in a helical path).

C MOTION WITH CONSTANT VELOCITY

C.1 CALCULATING VELOCITY AND SPEED FROM DISPLACEMENT

Ex 11: A helicopter travels at a constant velocity. It is initially at position $A(6, 9, 3)$, and 10 minutes later it is at position $B(3, 10, 2.5)$. Distances are measured in kilometres.

1. Find the velocity vector of the helicopter in km/h.
2. Find the speed of the helicopter in km/h, correct to one decimal place.

Answer:

1. **Velocity Vector** We start with the equation of motion for constant velocity, where $\overrightarrow{r_0}$ is the initial position and $\overrightarrow{r}(t)$ is the final position after time t. The time taken is 10 minutes, which is $t = \frac{10}{60} = \frac{1}{6}$ hours.

$$\begin{aligned}\overrightarrow{r}(t) &= \overrightarrow{r_0} + t\overrightarrow{v} \\ t\overrightarrow{v} &= \overrightarrow{r}(t) - \overrightarrow{r_0} \\ \overrightarrow{v} &= \frac{\overrightarrow{r}(t) - \overrightarrow{r_0}}{t} \\ &= \frac{1}{1/6}\left(\begin{pmatrix} 3 \\ 10 \\ 2.5 \end{pmatrix} - \begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}\right) \\ &= 6\begin{pmatrix} -3 \\ 1 \\ -0.5 \end{pmatrix} \\ &= \begin{pmatrix} -18 \\ 6 \\ -3 \end{pmatrix} \text{ km/h}\end{aligned}$$

2. **Speed** The speed is the magnitude of the velocity vector.

$$\begin{aligned}\text{Speed} &= |\overrightarrow{v}| \\ &= \sqrt{(-18)^2 + 6^2 + (-3)^2} \\ &= \sqrt{324 + 36 + 9} \\ &= \sqrt{369} \\ &\approx 19.2 \text{ km/h}\end{aligned}$$

Ex 12: A car travels at a constant velocity. At 1:00 PM, its position is $A(15, 20)$. At 1:30 PM, its position is $B(55, -10)$. Distances are measured in kilometres.

1. Find the velocity vector of the car in km/h.
2. Find the speed of the car in km/h.

Answer:

1. **Velocity Vector** The initial position is $\overrightarrow{r_0} = \begin{pmatrix} 15 \\ 20 \end{pmatrix}$ and the final position is $\overrightarrow{r}(t) = \begin{pmatrix} 55 \\ -10 \end{pmatrix}$. The time taken is 30 minutes, which is $t = 0.5$ hours.

$$\begin{aligned}\overrightarrow{r}(t) &= \overrightarrow{r_0} + t\overrightarrow{v} \\ \overrightarrow{v} &= \frac{\overrightarrow{r}(t) - \overrightarrow{r_0}}{t} \\ &= \frac{1}{0.5}\left(\begin{pmatrix} 55 \\ -10 \end{pmatrix} - \begin{pmatrix} 15 \\ 20 \end{pmatrix}\right) \\ &= 2\begin{pmatrix} 40 \\ -30 \end{pmatrix} \\ &= \begin{pmatrix} 80 \\ -60 \end{pmatrix} \text{ km/h}\end{aligned}$$

2. **Speed** The speed is the magnitude of the velocity vector.

$$\begin{aligned}\text{Speed} &= |\overrightarrow{v}| \\ &= \sqrt{80^2 + (-60)^2} \\ &= \sqrt{6400 + 3600} \\ &= \sqrt{10000} \\ &= 100 \text{ km/h}\end{aligned}$$

Ex 13: An airplane travels at a constant velocity. At 8:00 AM, it is at position $A(100, 200, 10)$. At 8:15 AM, it is at position $B(160, 170, 9.5)$. Distances are measured in kilometres.

1. Find the velocity vector of the airplane in km/h.
2. Find the speed of the airplane in km/h, correct to one decimal place.

Answer:

1. **Velocity Vector** The initial position is $\overrightarrow{r_0} = \begin{pmatrix} 100 \\ 200 \\ 10 \end{pmatrix}$ and the final position is $\overrightarrow{r}(t) = \begin{pmatrix} 160 \\ 170 \\ 9.5 \end{pmatrix}$. The time taken is 15 minutes, which is $t = \frac{15}{60} = 0.25$ hours.

$$\begin{aligned}\overrightarrow{v} &= \frac{\overrightarrow{r}(t) - \overrightarrow{r_0}}{t} \\ &= \frac{1}{0.25}\left(\begin{pmatrix} 160 \\ 170 \\ 9.5 \end{pmatrix} - \begin{pmatrix} 100 \\ 200 \\ 10 \end{pmatrix}\right) \\ &= 4\begin{pmatrix} 60 \\ -30 \\ -0.5 \end{pmatrix} \\ &= \begin{pmatrix} 240 \\ -120 \\ -2 \end{pmatrix} \text{ km/h}\end{aligned}$$

2. **Speed** The speed is the magnitude of the velocity vector.

$$\begin{aligned}
\text{Speed} &= |\overrightarrow{v}| \\
&= \sqrt{240^2 + (-120)^2 + (-2)^2} \\
&= \sqrt{57600 + 14400 + 4} \\
&= \sqrt{72004} \\
&\approx 268.3 \text{ km/h}
\end{aligned}$$