# TRIGONOMETRY

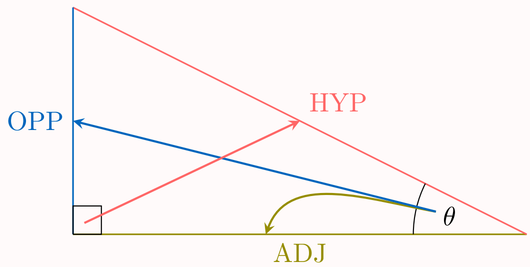
Trigonometry is a branch of mathematics that studies the relationships between the side lengths and angles of triangles, especially right-angled triangles. It is widely used in science, engineering, astronomy, and even video game development. The foundation of trigonometry lies in three main ratios: sine, cosine, and tangent.

## A RIGHT-ANGLED TRIANGLE

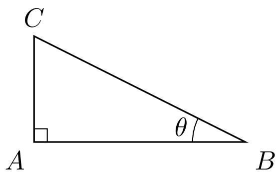
**Definition Right-Angled Triangle**

A **right-angled triangle** is a triangle with one angle equal to 90°. For a given angle $\theta$ (other than the right angle), we define:

- **Hypotenuse (HYP)**: The longest side, opposite the right angle.
- **Adjacent Side (ADJ)**: The side next to the angle $\theta$, which is not the hypotenuse.
- **Opposite Side (OPP)**: The side facing the angle $\theta$.




**Ex:** In the triangle below, identify the hypotenuse, the adjacent side, and the opposite side relative to angle $\theta$.




*Answer:*

- Hypotenuse: $\overline{BC}$
- Adjacent side: $\overline{AC}$
- Opposite side: $\overline{AB}$

## B TRIGONOMETRIC FUNCTIONS

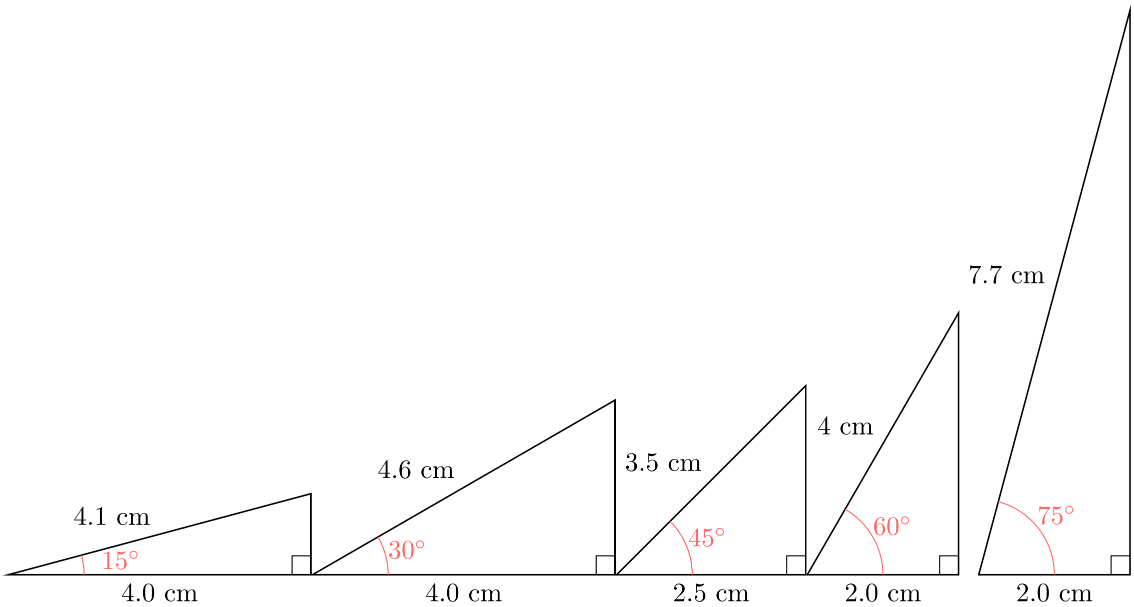
**Discover:**

1. Using a protractor and ruler, draw right-angled triangles with one angle $\theta$ equal to 15°, 30°, 45°, 60°, or 75°. Measure the lengths of the hypotenuse and the adjacent side to the nearest millimeter.
2. Complete the following table with the ratio $\frac{\text{ADJ}}{\text{HYP}}$ for each angle:

| $\theta$ | 15° | 30° | 45° | 60° | 75° |
|---|---|---|---|---|---|
| $\frac{\text{ADJ}}{\text{HYP}}$ | | | | | |

*Answer:*

1. Here are examples of right-angled triangles for the specified angles. In each triangle, the side adjacent to $\theta$ (ADJ) and the hypotenuse (HYP) are labeled so you can measure and verify the ratios.




2. The ratio $\frac{\text{ADJ}}{\text{HYP}}$ for each angle, using a calculator:

| $\theta$ | 15° | 30° | 45° | 60° | 75° |
|---|---|---|---|---|---|
| $\frac{\text{ADJ}}{\text{HYP}}$ | 0.97 | 0.87 | 0.71 | 0.50 | 0.26 |

**Proposition Trigonometric Ratios**

For any two right-angled triangles with the same angle $\theta$, the ratios $\frac{\text{OPP}}{\text{HYP}}$, $\frac{\text{ADJ}}{\text{HYP}}$, and $\frac{\text{OPP}}{\text{ADJ}}$ are constant.

Proof

Consider two right-angled triangles with the same angle $\theta$. Since their angles are equal, the triangles are similar. Therefore, the ratios are constant for a given angle.

This means that for any right-angled triangle with the same angle $\theta$, the ratio $\frac{\text{ADJ}}{\text{HYP}}$ is always the same. This is why we define the **cosine** function, denoted cos, such that:

$$\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}$$

For example:

| $\theta$ | 15° | 30° | 45° | 60° | 75° |
|---|---|---|---|---|---|
| $\cos(\theta)$ | 0.97 | 0.87 | 0.71 | 0.50 | 0.26 |

For instance, $\cos(45°) \approx 0.71$.

**Definition Trigonometric Functions**

In a right-angled triangle with angle $\theta$:

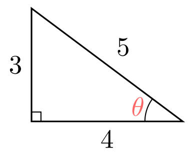
$$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}}, \quad \cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}, \quad \tan(\theta) = \frac{\text{OPP}}{\text{ADJ}}$$

The mnemonic **SOH-CAH-TOA** helps remember the definitions:

- **S**ine = **O**pposite ÷ **H**ypotenuse
- **C**osine = **A**djacent ÷ **H**ypotenuse
- **T**angent = **O**pposite ÷ **A**djacent

To help you memorize, listen to this song: https://www.youtube.com/watch?v=PIWJo5uK3Fo

**Ex:** In the triangle below, find $\cos\theta$, $\sin\theta$, and $\tan\theta$.




*Answer:* Relative to $\theta$:

- Hypotenuse: $BC = 5$
- Adjacent side: $AB = 4$
- Opposite side: $AC = 3$

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{4}{5}$$
$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{3}{5}$$
$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{3}{4}$$

**Proposition Tangent Formula**

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

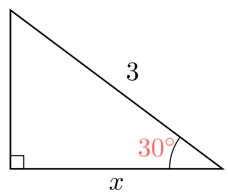
Proof

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{\text{OPP}}{\text{HYP}}}{\frac{\text{ADJ}}{\text{HYP}}} = \frac{\text{OPP}}{\text{HYP}} \times \frac{\text{HYP}}{\text{ADJ}} = \frac{\text{OPP}}{\text{ADJ}} = \tan\theta$$

**Method Using Calculator**

Trigonometric ratios for any angle can be calculated using a calculator in degree mode. Make sure your calculator is set to "degrees" before calculating.

**Ex:** In the triangle below, find $x$.




*Answer:*

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}}$$
$$\cos(30^\circ) = \frac{x}{3}$$
$$x = 3 \times \cos(30^\circ)$$
$$x \approx 3 \times 0.866$$
$$x \approx 2.6\,\text{cm}$$

## C INVERSE TRIGONOMETRIC FUNCTIONS

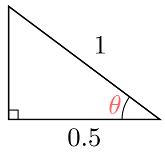
Trigonometric ratios can be used to find unknown angles in right-angled triangles when at least two side lengths are known.

**Definition Inverse Trigonometric Functions**

In a right-angled triangle with an angle $\theta$:

$$\theta = \cos^{-1}\left(\frac{\text{ADJ}}{\text{HYP}}\right), \quad \theta = \sin^{-1}\left(\frac{\text{OPP}}{\text{HYP}}\right), \quad \theta = \tan^{-1}\left(\frac{\text{OPP}}{\text{ADJ}}\right)$$

**Ex:** In the triangle below, find the angle $\theta$.




*Answer:* We know the lengths of the adjacent side ($AB = 0.5$) and the hypotenuse ($BC = 1$) relative to $\theta$. We can use the inverse cosine function:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\text{ADJ}}{\text{HYP}}\right)\\ &= \cos^{-1}\left(\frac{0.5}{1}\right)\\ &= 60^\circ\end{aligned}$$

## D SOLVING REAL-WORLD TRIGONOMETRY PROBLEMS

Trigonometric ratios are powerful tools for solving a wide range of problems involving right-angled triangles, especially in real-world contexts.To solve these problems effectively, follow the structured steps below:

**Method Solving Real-World Trigonometry Problems**

- **Draw a clear diagram** representing the situation described in the problem.
- **Label the unknown** (side or angle) you need to find. Use $x$ for a side and $\theta$ for an angle if possible.
- **Identify a right-angled triangle** within your diagram.
- **Write an equation** relating an angle and two sides of the triangle using the appropriate trigonometric ratio.
- **Solve the equation** to find the unknown value.
- **State your answer clearly**, including appropriate units, in a complete sentence.