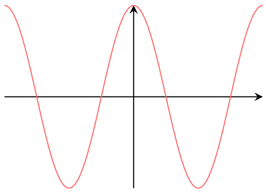
TRIGONOMETRIC FUNCTIONS

A PERIODIC FUNCTION

A.1 IDENTIFYING PERIODIC BEHAVIOUR FROM A GRAPH

MCQ 1: Is the function shown in the graph below periodic?

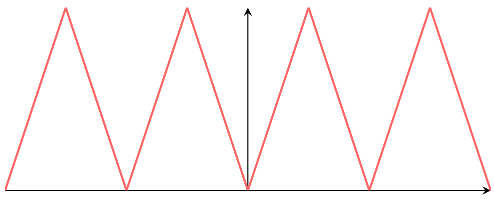


☒ Yes

☐ No

Answer: Yes, this function is periodic. The graph shows a simple wave pattern that repeats. We can see that the shape of the graph from $x = 0$ to $x = \pi$ is identical to the shape from $x = \pi$ to $x = 2\pi$ (and so on). Therefore, the function **is periodic**.

MCQ 2: Is the function shown in the graph below periodic?



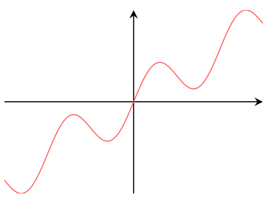
☒ Yes

☐ No

Answer: Yes, this function is periodic. The graph shows a triangular "zig-zag" pattern that repeats itself exactly. One full cycle of the pattern occurs over a horizontal distance of 4 units (for example, from the minimum at $x = 0$ to the next minimum at $x = 4$). Therefore, the function **is periodic**.

MCQ 3: Is the function shown in the graph below periodic?

$$f(x) = \sin(x) + \frac{x}{3}$$



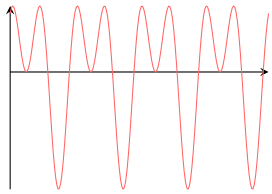
☐ Yes

☒ No

Answer: A function is periodic if its graph shows a pattern that repeats itself exactly. Although this graph has a wave-like shape, it is constantly drifting upwards. The function never returns to its previous y-values, so the pattern does not repeat. Therefore, the function is **not periodic**.

MCQ 4: Is the function shown in the graph below periodic?

$$g(x) = \sin(2x) + \cos(4x)$$



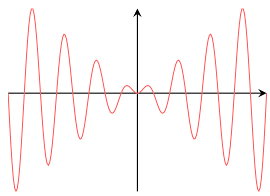
☒ Yes

☐ No

Answer: Yes, this function is periodic. Although the wave shape is complex, we can observe a distinct pattern that repeats itself exactly over a regular horizontal interval. For example, the shape of the graph between $x = 0$ and $x = 2\pi$ is identical to the shape between $x = 2\pi$ and $x = 4\pi$. Therefore, the function **is periodic**.

MCQ 5: Is the function shown in the graph below periodic?

$$h(x) = x\sin(2x)$$



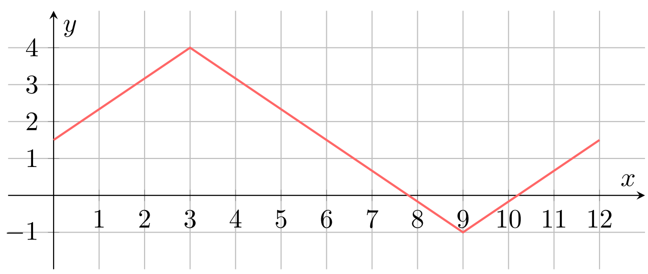
☐ Yes

☒ No

Answer: This function is not periodic. Although it oscillates, the amplitude of the oscillations (the maximum height of the waves) is continuously increasing as $|x|$ increases. For a function to be periodic, its pattern, including its maximum and minimum values, must repeat exactly. Since the amplitude is not constant, the function is **not periodic**.

A.2 IDENTIFYING PROPERTIES OF PERIODIC FUNCTIONS

Ex 6: For the periodic function shown below, find:



1. The period is $\boxed{6}$
2. The equation of the principal axis is $y = \boxed{1.5}$
3. The amplitude is $\boxed{2.5}$

Answer:

1. **Period**: The graph completes one full cycle from the minimum at $x = 9$ to the next point where it would start repeating the downward slope, which appears to be after two peaks. Let's look at the pattern from $x = 0$ to $x = 6$. The pattern from $x = 6$ to $x = 12$ is identical. Therefore, the period is 6.

2. **Principal Axis**: The maximum value is 4 and the minimum value is -1. The principal axis is the line $y = \frac{4+(-1)}{2} = \frac{3}{2} = 1.5$.

3. **Amplitude**: The amplitude is the distance from the principal axis ($y = 1.5$) to a maximum ($y = 4$). The amplitude is $4 - 1.5 = 2.5$. (Alternatively, $\frac{4-(-1)}{2} = \frac{5}{2} = 2.5$).

Ex 7: For the periodic function shown below, find:

1. The period is $\boxed{4}$

2. The equation of the principal axis is $y = \boxed{3}$

3. The amplitude is $\boxed{2}$

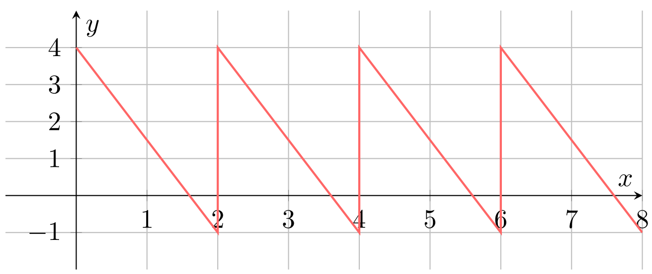
Answer:

1. **Period**: The graph completes one full cycle from the peak at $x = 0$ to the next peak at $x = 4$. The period is $4 - 0 = 4$.

2. **Principal Axis**: The maximum value is 5 and the minimum value is 1. The principal axis is the line $y = \frac{5+1}{2} = 3$.

3. **Amplitude**: The amplitude is the distance from the principal axis ($y = 3$) to a maximum ($y = 5$). The amplitude is $5 - 3 = 2$. (Alternatively, $\frac{5-1}{2} = 2$).

Ex 8: For the periodic function shown below, find:



1. The period is $\boxed{2}$

2. The equation of the principal axis is $y = \boxed{1.5}$

3. The amplitude is $\boxed{2.5}$

Answer:

1. **Period**: The graph shows a repeating "sawtooth" pattern. One full cycle of this pattern occurs from $x = 0$ to $x = 2$, then repeats from $x = 2$ to $x = 4$, and so on. The period is 2.

2. **Principal Axis**: The maximum value is 4 and the minimum value is -1. The principal axis is the line $y = \frac{4+(-1)}{2} = \frac{3}{2} = 1.5$.

3. **Amplitude**: The amplitude is the distance from the principal axis ($y = 1.5$) to a maximum ($y = 4$). The amplitude is $4 - 1.5 = 2.5$. (Alternatively, $\frac{4-(-1)}{2} = 2.5$).

B SINE AND COSINE FUNCTION

B.1 COMPLETING TABLES OF VALUES

Ex 9: For $f(x) = \sin(x)$, complete the table of values for the multiples of $\frac{\pi}{8}$ (rounded to 2 decimal places):

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$\sin(x)$	$\boxed{0}$	$\boxed{0.38}$	$\boxed{0.71}$	$\boxed{0.92}$	$\boxed{1}$

Answer: To calculate these values on your calculator, for each angle:

- If you are in degree mode, first convert the angle to degrees: for example, $\frac{\pi}{4} \times \frac{180°}{\pi} = 45°$, then $\sin(45°) \approx 0.71$.
- If your calculator is set to radians, you can directly compute $\sin\left(\frac{\pi}{4}\right) \approx 0.71$.

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
$\sin(x)$	0	0.38	0.71	0.92	1

Ex 10: Complete the table of values for the multiples of $\frac{\pi}{6}$ (rounded to 2 decimal places):

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
$\cos(x)$	$\boxed{1}$	$\boxed{0.87}$	$\boxed{0.5}$	$\boxed{0}$	$\boxed{-0.5}$	$\boxed{-0.87}$

Answer: To calculate these values on your calculator:

- If you are in degree mode, convert the angle to degrees: e.g., $\frac{\pi}{6} \times \frac{180°}{\pi} = 30°$, then $\cos(30°) \approx 0.87$.
- If your calculator is in radian mode, you can directly compute $\cos\left(\frac{\pi}{6}\right) \approx 0.87$.

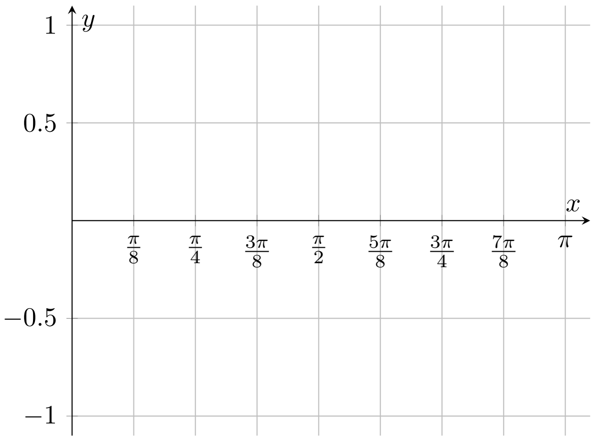
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$
$\cos(x)$	1	0.87	0.5	0	-0.5	-0.87

B.2 PLOTTING GRAPHS

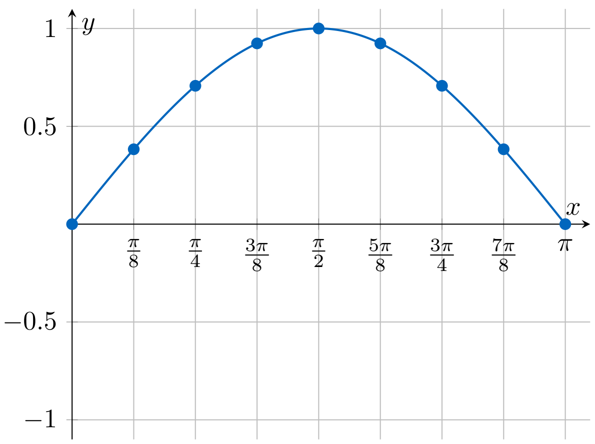
Ex 11: Here is a table of values for the function $f(x) = \sin(x)$:

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
$\sin(x)$	0	0.38	0.71	0.92	1.00	0.92	0.71	0.38	0

Plot the graph of the function.



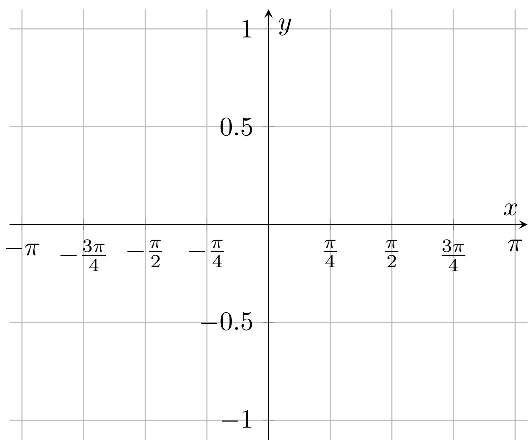
Answer:



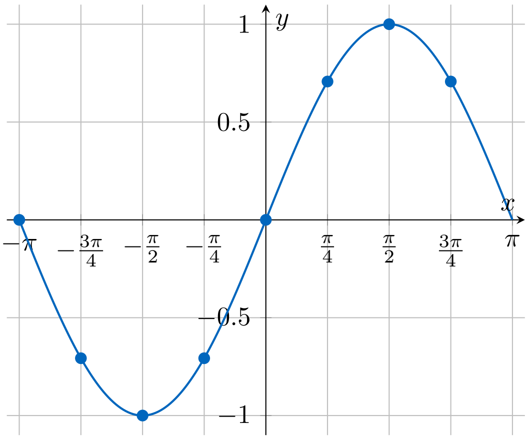
Ex 12: Here is a table of values for the function $f(x) = \sin(x)$:

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$\sin(x)$	0	−0.71	−1.00	−0.71	0	0.71	1	0.71

Plot the graph of the function on the interval $[-\pi\,;\,\pi]$:



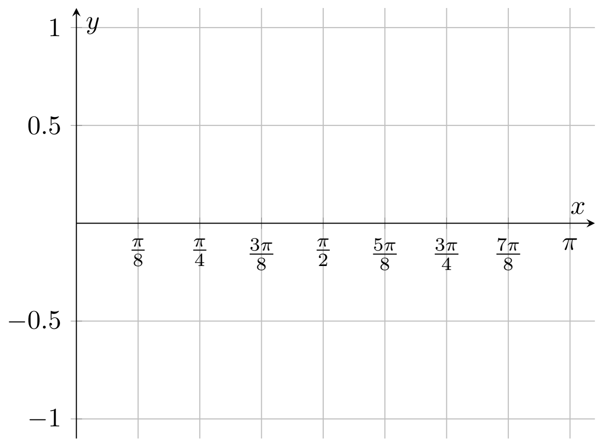
Answer:



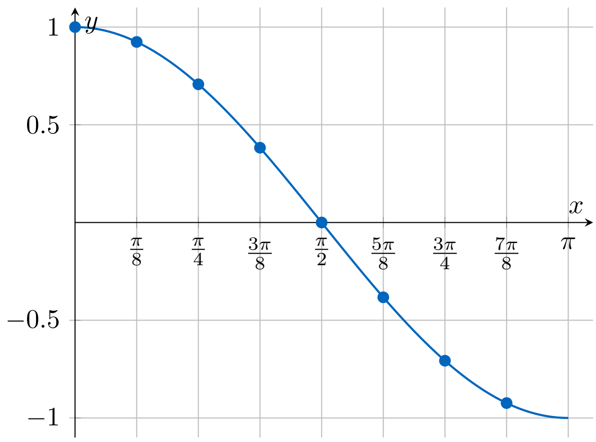
Ex 13: Here is a table of values for the function $f(x) = \cos(x)$:

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$
$\cos(x)$	1	0.92	0.71	0.38	0	−0.38	−0.71	−0.92

Plot the graph of the function.



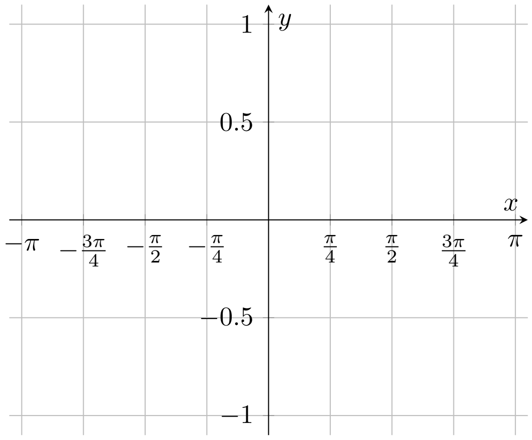
Answer:

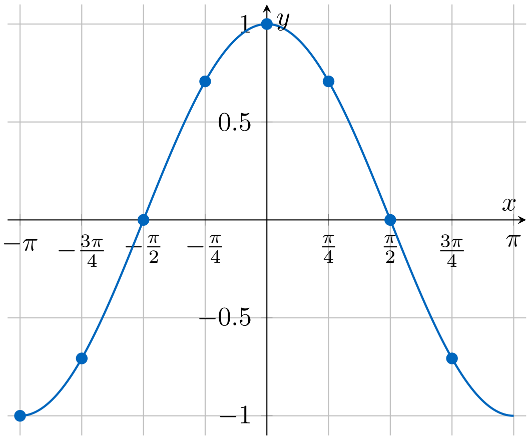


Ex 14: Here is a table of values for the function $f(x) = \cos(x)$:

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
$\cos(x)$	−1	−0.71	0	0.71	1	0.71	0	−0.71

Plot the graph of the function on the interval $[-\pi\,;\,\pi]$:

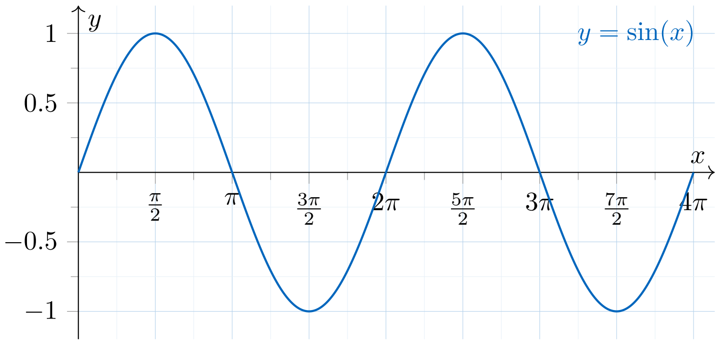




Answer:

B.3 READING GRAPHS

Ex 15: Below is the graph of the function $y = \sin(x)$, for $0 \leq x \leq 4\pi$.



1. Find the y-**intercept** of the graph.

$$(0, \boxed{0})$$

2. Use the graph to determine the values of x in the interval $0 \leq x \leq 4\pi$ such that $\sin(x) = 1$:

$$\boxed{\frac{\pi}{2}}, \boxed{\frac{5\pi}{2}}$$

Answer:

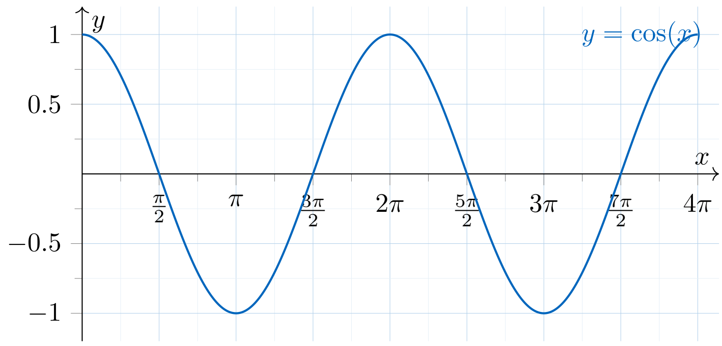
1. The y-intercept of the graph is the point where $x = 0$:

$$\boxed{(0, 0)}$$

2. We are asked to find all values of x in the interval $0 \leq x \leq 4\pi$ for which $\sin(x) = 1$.
From the graph, $\sin(x) = 1$ when $x = \frac{\pi}{2}$ and again one full period later, at $x = \frac{5\pi}{2}$. These are the only two values within the interval $[0, 4\pi]$.

$$\boxed{x = \frac{\pi}{2} \quad \text{and} \quad x = \frac{5\pi}{2}}$$

Ex 16: Below is the graph of the function $y = \cos(x)$, for $0 \leq x \leq 4\pi$.



1. Find the y-**intercept** of the graph.

$$(0, \boxed{1})$$

2. Use the graph to determine the values of x in the interval $0 \leq x \leq 4\pi$ such that $\cos(x) = 0$:

$$\boxed{\frac{\pi}{2}}, \boxed{\frac{3\pi}{2}}, \boxed{\frac{5\pi}{2}}, \boxed{\frac{7\pi}{2}}$$

Answer:

1. The y-intercept of the graph is the point where $x = 0$:

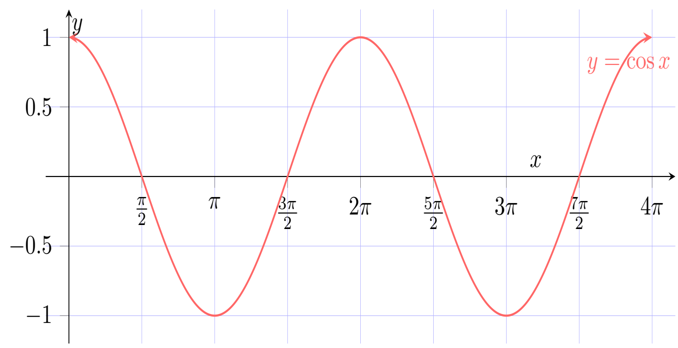
$$\boxed{(0, 1)}$$

2. We are asked to find all values of $x \in [0, 4\pi]$ such that $\cos(x) = 0$. From the graph, $\cos(x) = 0$ at:

$$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}}$$

B.4 READING KEY FEATURES FROM A GRAPH

Ex 17: Below is an accurate graph of the function $y = \cos(x)$, for $0 \leq x \leq 4\pi$.



1. Find the y-intercept of the graph.
2. Find the values of x on $0 \leq x \leq 4\pi$ for which:

(a) $\cos x = 1$
(b) $\cos x = 0$

3. Find the intervals on $0 \leq x \leq 4\pi$ where $\cos x$ is:

(a) non negative.
(b) non positive

4. Find the range of the function.

Answer:

1. The y-intercept is the value of y when $x = 0$. From the graph, this is 1.

2. By reading the x-coordinates from the graph at the required heights:

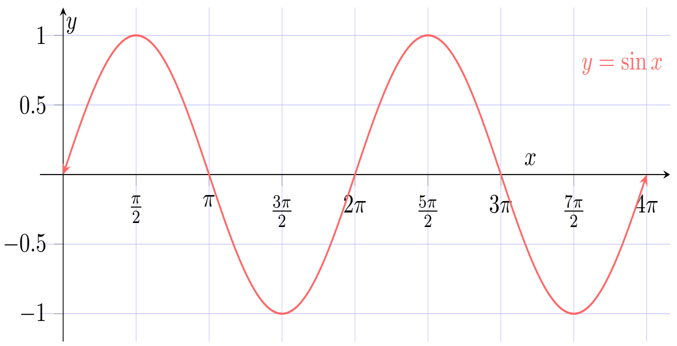
(a) $\cos x = 1$ when $x = 0, 2\pi, 4\pi$.
(b) $\cos x = 0$ when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

3. By observing where the graph is above or below the x-axis:

(a) Non negative for $x \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, \frac{5\pi}{2}] \cup [\frac{7\pi}{2}, 4\pi]$.
(b) Non positive for $x \in [\frac{\pi}{2}, \frac{3\pi}{2}] \cup [\frac{5\pi}{2}, \frac{7\pi}{2}]$.

4. The minimum value of the function is -1 and the maximum is 1. The range is $[-1, 1]$.

Ex 18: Below is an accurate graph of the function $y = \sin(x)$, for $0 \leq x \leq 4\pi$.



1. Find the y-intercept of the graph.

2. Find the values of x on $0 \leq x \leq 4\pi$ for which:

(a) $\sin x = 1$
(b) $\sin x = 0$

3. Find the intervals on $0 \leq x \leq 4\pi$ where $\sin x$ is:

(a) non-negative
(b) non-positive.

4. Find the range of the function.

Answer:

1. The y-intercept is the value of y when $x = 0$. From the graph, this is 0.

2. By reading the x-coordinates from the graph at the required heights:

(a) $\sin x = 1$ when $x = \frac{\pi}{2}, \frac{5\pi}{2}$.
(b) $\sin x = 0$ when $x = 0, \pi, 2\pi, 3\pi, 4\pi$.

3. By observing where the graph is on or above/below the x-axis:

(a) Non-negative for $x \in [0, \pi] \cup [2\pi, 3\pi]$.
(b) Non-positive for $x \in [\pi, 2\pi] \cup [3\pi, 4\pi]$.

4. The minimum value of the function is -1 and the maximum is 1. The range is $[-1, 1]$.

C GENERAL SINE AND COSINE FUNCTIONS

C.1 IDENTIFYING PROPERTIES FROM AN EQUATION

Ex 19: For the function $y = 4\cos(x) - 2$, state:

1. The amplitude. $\boxed{4}$

2. The period. $\boxed{2\pi}$

3. The phase shift. $\boxed{0}$

4. The principal axis. $y = \boxed{-2}$

Answer: The function is in the form $y = a\cos(b(x - c)) + d$.For $y = 4\cos(x) - 2$, we can identify the parameters as $a = 4, b = 1, c = 0, d = -2$.

1. **Amplitude**: $|a| = |4| = 4$.

2. **Period**: $\frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$.

3. **Phase Shift**: $c = 0$. There is no horizontal shift.

4. **Principal Axis**: $d = -2$. The principal axis is the line $y = -2$.

Ex 20: For the function $y = 2\cos(3x) + 1$, state:

1. The amplitude. $\boxed{2}$

2. The period. $\boxed{2\pi/3}$

3. The phase shift. $\boxed{0}$

4. The principal axis. $y = \boxed{1}$

Answer: The function is in the form $y = a\cos(b(x - c)) + d$.For $y = 2\cos(3x) + 1$, we can identify the parameters as $a = 2, b = 3, c = 0, d = 1$.

1. **Amplitude**: $|a| = |2| = 2$.

2. **Period**: $\frac{2\pi}{|b|} = \frac{2\pi}{3}$.

3. **Phase Shift**: $c = 0$. There is no horizontal shift.

4. **Principal Axis**: $d = 1$. The principal axis is the line $y = 1$.

Ex 21: For the function $y = 3\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$, state:

1. The amplitude. $\boxed{3}$

2. The period. $\boxed{\pi}$

3. The phase shift. $\boxed{\frac{\pi}{4}}$

4. The principal axis. $y = \boxed{1}$

Answer: The function is in the form $y = a\sin(b(x-c)) + d$.

1. **Amplitude**: $a = 3$. The amplitude is $|a| = 3$.
2. **Period**: $b = 2$. The period is $\frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi$.
3. **Phase Shift**: $c = \frac{\pi}{4}$. The shift is $\frac{\pi}{4}$ to the right.
4. **Principal Axis**: $d = 1$. The principal axis is the line $y = 1$.

Ex 22: For the function $y = -5\sin(3x + \pi) + 7$, state:

1. The amplitude. $\boxed{5}$
2. The period. $\boxed{2\pi/3}$
3. The phase shift. $\boxed{-\pi/3}$
4. The principal axis. $y = \boxed{7}$

Answer: First, we must write the function in the standard form $y = a\sin(b(x-c)) + d$ by factoring out the coefficient of x inside the sine function.

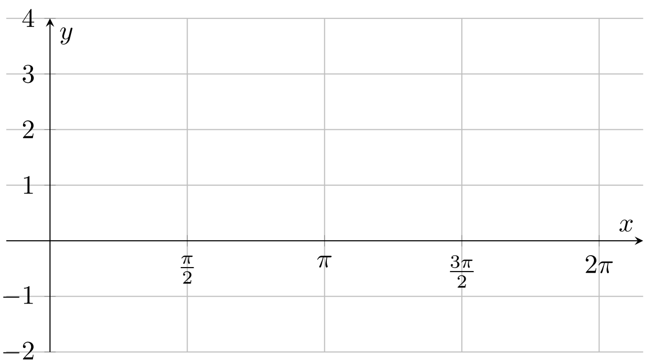
$$y = -5\sin\left(3\left(x + \frac{\pi}{3}\right)\right) + 7$$

From this form, we identify the parameters: $a = -5, b = 3, c = -\frac{\pi}{3}, d = 7$.

1. **Amplitude**: $|a| = |-5| = 5$.
2. **Period**: $\frac{2\pi}{|b|} = \frac{2\pi}{3}$.
3. **Phase Shift**: $c = -\frac{\pi}{3}$. The shift is $\frac{\pi}{3}$ to the left.
4. **Principal Axis**: $d = 7$. The principal axis is the line $y = 7$.

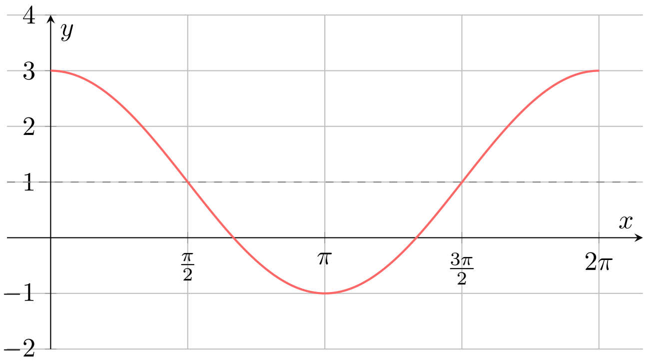
C.2 SKETCHING TRANSFORMED FUNCTIONS

Ex 23: Sketch the graph of $y = 2\cos(x) + 1$ for $0 \leq x \leq 2\pi$.

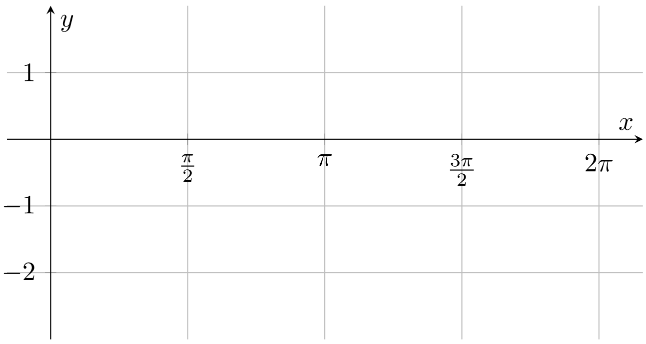


Answer: This is a transformation of $y = \cos(x)$ with a vertical stretch of factor 2 ($a = 2$) and a vertical shift of 1 unit up ($d = 1$).

- **Principal Axis**: $y = 1$.
- **Amplitude**: 2.
- **Range**: The graph will oscillate between $1 - 2 = -1$ and $1 + 2 = 3$.
- **Period**: 2π.

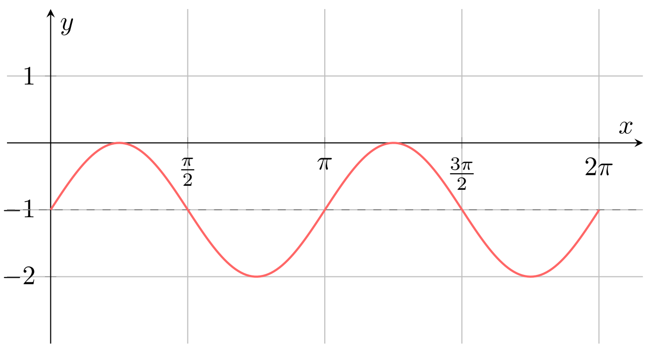


Ex 24: Sketch the graph of $y = \sin(2x) - 1$ for $0 \leq x \leq 2\pi$.

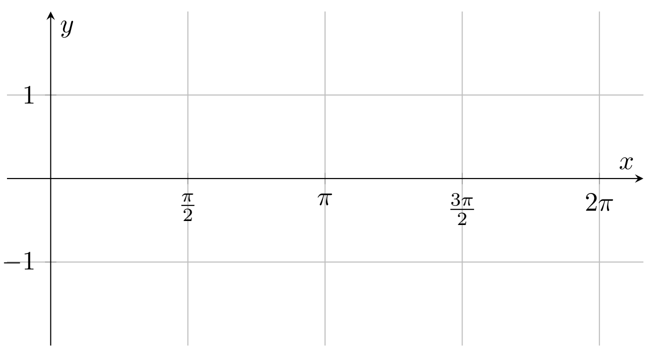


Answer: This is a transformation of $y = \sin(x)$ with a horizontal stretch of factor $\frac{1}{2}$ ($b = 2$) and a vertical shift of 1 unit down ($d = -1$).

- **Principal Axis**: $y = -1$.
- **Amplitude**: $a = 1$.
- **Range**: The graph will oscillate between $-1 - 1 = -2$ and $-1 + 1 = 0$.
- **Period**: $\frac{2\pi}{2} = \pi$. The function will complete two cycles in the domain $[0, 2\pi]$.

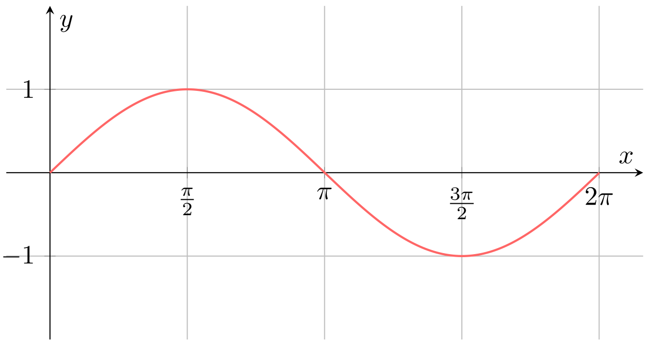


Ex 25: Sketch the graph of $y = \cos\left(x - \frac{\pi}{2}\right)$ for $0 \leq x \leq 2\pi$.



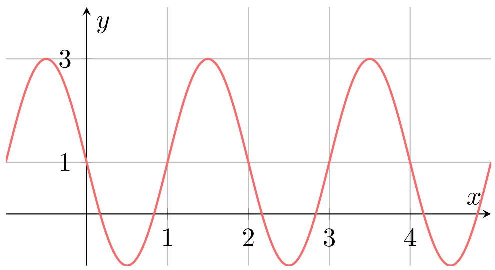
Answer: This is a transformation of $y = \cos(x)$ with a horizontal shift of $\frac{\pi}{2}$ units to the right ($c = \frac{\pi}{2}$).

- **Principal Axis**: $y = 0$.
- **Amplitude**: 1.
- **Range**: $[-1, 1]$.
- **Period**: 2π.
- **Note**: The graph of $y = \cos(x - \frac{\pi}{2})$ is identical to the graph of $y = \sin(x)$.



C.3 FINDING THE EQUATION FROM A GRAPH

MCQ 26: Which of the following equations best describes the graph shown below?



☒ $y = 2\sin(\pi(x-1)) + 1$

☐ $y = 2\sin(2\pi(x-1)) + 1$

☐ $y = 3\sin(\pi(x-1)) - 1$

☐ $y = \sin(\pi(x+1)) + 2$

Answer: We identify the key parameters of the sine function from the graph.

- **Principal Axis** (d): The maximum value is 3 and the minimum value is -1. The principal axis is the line $y = \frac{3+(-1)}{2} = 1$. So, $d = 1$.
- **Amplitude** (a): The amplitude is the distance from the principal axis to a maximum: $a = 3 - 1 = 2$.
- **Period** (b): The graph completes one full cycle from $x = 1$ to $x = 3$. The period is $P = 3 - 1 = 2$. We use the formula $P = \frac{2\pi}{b}$ to find b:

$$2 = \frac{2\pi}{b} \implies b = \pi$$

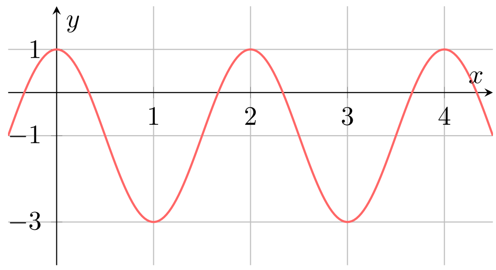
- **Phase Shift** (c): A standard sine wave starts at its principal axis ($y = 1$) and goes up. On the graph, this starting point of a cycle is at $x = 1$. So, we can choose a phase shift of $c = 1$.

Assembling these parameters into the form $y = a\sin(b(x-c)) + d$ gives:

$$y = 2\sin(\pi(x-1)) + 1$$

This matches the first option.

MCQ 27: Which of the following equations best describes the graph shown below?



☐ $y = 2\cos(\pi x) + 1$

☐ $y = \cos(2\pi x) - 1$

☒ $y = 2\cos(\pi x) - 1$

☐ $y = 2\cos(x) - 1$

Answer: We identify the key parameters from the graph, assuming a cosine function of the form $y = a\cos(b(x-c)) + d$.

- **Principal Axis** (d): Max value is 1, min value is -3. The principal axis is $y = \frac{1+(-3)}{2} = -1$. So, $d = -1$.
- **Amplitude** (a): $a = \frac{1-(-3)}{2} = 2$.
- **Period** (b): The graph completes one full cycle from the peak at $x = 0$ to the next peak at $x = 2$. The period is $P = 2$. So, $b = \frac{2\pi}{P} = \frac{2\pi}{2} = \pi$.
- **Phase Shift** (c): A standard cosine wave starts at a maximum. This graph has a maximum at $x = 0$, so there is no phase shift. We can choose $c = 0$.

Assembling these parameters gives:

$$y = 2\cos(\pi(x-0)) - 1 = 2\cos(\pi x) - 1$$

This matches the third option.

MCQ 28: Which of the following equations best describes the graph shown below?

☐ $y = 2\sin(\frac{\pi}{2}x) + 2$

☐ $y = -2\sin(\pi x) + 2$

☐ $y = -4\sin(\frac{\pi}{2}x) + 2$

☒ $y = -2\sin(\frac{\pi}{2}x) + 2$

Answer: We identify the key parameters from the graph, assuming a sine function of the form $y = a\sin(b(x-c)) + d$.

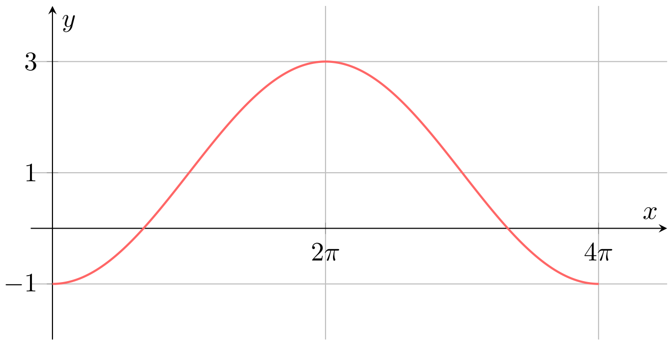
- **Principal Axis** (d): Max value is 4, min value is 0. The principal axis is $y = \frac{4+0}{2} = 2$. So, $d = 2$.
- **Amplitude** (a): $a = \frac{4-0}{2} = 2$.
- **Period** (b): The graph completes one full cycle from $x = 0$ to $x = 4$. The period is $P = 4$. So, $b = \frac{2\pi}{P} = \frac{2\pi}{4} = \frac{\pi}{2}$.
- **Reflection and Phase Shift** (a, c): At $x = 0$, the graph is on its principal axis ($y = 2$) and is decreasing. A standard sine wave starts at the principal axis and increases. This indicates a reflection in the x-axis, so the value of a must be negative. Thus, $a = -2$. Since the cycle starts at $x = 0$, there is no phase shift ($c = 0$).

Assembling these parameters gives:

$$y = -2\sin(\frac{\pi}{2}(x-0)) + 2 = -2\sin(\frac{\pi}{2}x) + 2$$

This matches the fourth option.

MCQ 29: Which of the following equations best describes the graph shown below?



- ☐ $y = 2\cos(2x) + 1$
- ☐ $y = -2\cos(x) + 1$
- ☐ $y = 2\cos(0.5x) - 1$
- ☒ $y = -2\cos(0.5x) + 1$

Answer: We identify the key parameters from the graph, assuming a cosine function of the form $y = a\cos(b(x-c)) + d$.

- **Principal Axis** (d): The maximum value is 3 and the minimum value is -1. The principal axis is the line $y = \frac{3+(-1)}{2} = 1$. So, $d = 1$.
- **Amplitude** ($|a|$): The amplitude is the distance from the principal axis to a maximum: $|a| = 3 - 1 = 2$.
- **Period** (b): The graph completes one full cycle from the minimum at $x = 0$ to the next minimum at $x = 4\pi$. The period is $P = 4\pi$. So, $b = \frac{2\pi}{P} = \frac{2\pi}{4\pi} = \frac{1}{2} = 0.5$.
- **Reflection and Phase Shift** (a, c): A standard cosine function ($a > 0$) starts at a maximum. This graph starts at a minimum (at $x = 0$), which indicates a reflection in the principal axis. Therefore, the value of a must be negative, so $a = -2$. Since the cycle starts at an extremum on the y-axis, there is no phase shift ($c = 0$).

Assembling these parameters gives:

$$y = -2\cos(0.5x) + 1$$

This matches the fourth option.

D TANGENT FUNCTION

D.1 GRAPHING THE TANGENT FUNCTION FROM VALUES

Ex 30: For $f(x) = \tan(x)$, complete the table of values (rounded to 2 decimal places).

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\tan(x)$	0	0.58	1	1.73

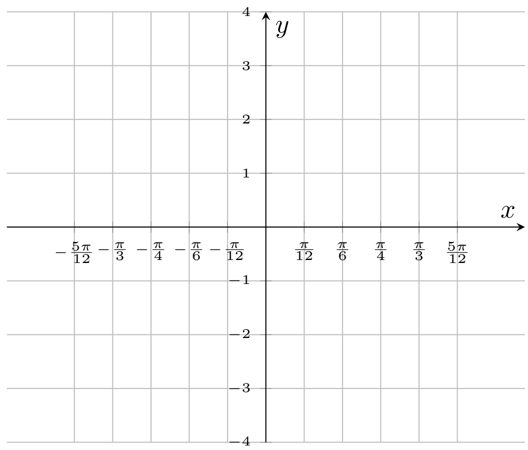
Answer: Ensure your calculator is in radian mode.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\tan(x)$	0

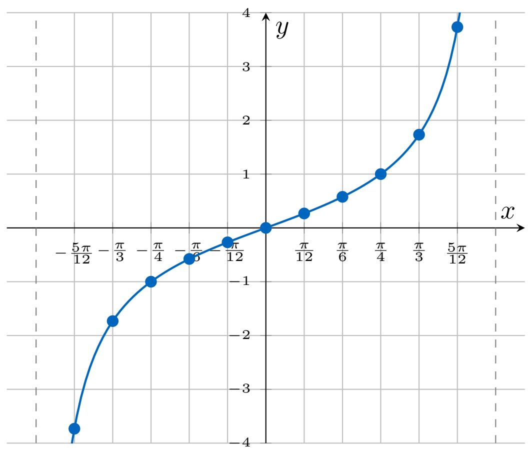
Ex 31: Here is a table of values for the function $f(x) = \tan(x)$ (rounded to 2 decimal places):

x	$-\frac{5\pi}{12}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	$-\frac{\pi}{12}$	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$
$\tan(x)$	-3.73	-1.73	-1.00	-0.58	-0.27	0	0.27	0.58	1.00	1.73	3.73

Plot the graph of the function on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.



Answer:



E RECIPROCAL TRIGONOMETRIC FUNCTIONS

E.1 FINDING DOMAINS AND ASYMPTOTES

MCQ 32: The function $y = \sec(x)$ is undefined for which of the following values?

- ☐ $x = 0$
- ☐ $x = \pi$
- ☒ $x = \frac{\pi}{2}$
- ☐ $x = \frac{\pi}{4}$

Answer: The secant function is defined as $\sec(x) = \frac{1}{\cos(x)}$. It is undefined whenever its denominator is zero, i.e., when $\cos(x) = 0$.The values of $\cos(x)$ at the given points are:

- $\cos(0) = 1$
- $\cos(\pi) = -1$
- $\cos(\frac{\pi}{2}) = 0$
- $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

The function is undefined at $x = \frac{\pi}{2}$.

MCQ 33: The function $y = \csc(x)$ is undefined for which of the following values?

- ☐ $x = \frac{\pi}{2}$
- ☐ $x = \frac{3\pi}{2}$
- ☐ $x = \frac{\pi}{4}$
- ☒ $x = \pi$

Answer: The cosecant function is defined as $\csc(x) = \frac{1}{\sin(x)}$. It is undefined whenever its denominator is zero, i.e., when $\sin(x) = 0$. The values of $\sin(x)$ at the given points are:

- $\sin(\frac{\pi}{2}) = 1$
- $\sin(\frac{3\pi}{2}) = -1$
- $\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
- $\sin(\pi) = 0$

The function is undefined at $x = \pi$.

MCQ 34: The function $y = \cot(x)$ has the same vertical asymptotes as which other function?

- ☐ $y = \sin(x)$
- ☒ $y = \csc(x)$
- ☐ $y = \cos(x)$
- ☐ $y = \sec(x)$

Answer: The cotangent function is defined as $\cot(x) = \frac{\cos(x)}{\sin(x)}$. It has vertical asymptotes whenever its denominator, $\sin(x)$, is equal to zero.
The cosecant function is defined as $\csc(x) = \frac{1}{\sin(x)}$. It also has vertical asymptotes whenever its denominator, $\sin(x)$, is equal to zero.
Therefore, $\cot(x)$ and $\csc(x)$ have the same vertical asymptotes.

E.2 SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

Ex 35: Express the function $f(x) = \frac{1}{\csc(x)}$ in terms of a primary trigonometric function.

$$f(x) = \boxed{\sin(x)}$$

Answer: By definition, the cosecant function is the reciprocal of the sine function: $\csc(x) = \frac{1}{\sin(x)}$.
Therefore, the reciprocal of the cosecant function is:

$$f(x) = \frac{1}{\csc(x)} = \frac{1}{1/\sin(x)} = \sin(x)$$

Ex 36: Express the function $f(x) = \tan(x) \cdot \sec(x)$ in terms of sine and cosine.

$$f(x) = \boxed{\sin(x)/\cos^2(x)}$$

Answer: We use the definitions $\tan(x) = \frac{\sin x}{\cos x}$ and $\sec(x) = \frac{1}{\cos x}$.

$$\begin{aligned} f(x) &= \tan(x) \cdot \sec(x) \\ &= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\ &= \frac{\sin x}{\cos^2 x} \end{aligned}$$

Ex 37: Express $\sec^2(x)$ in terms of $\tan^2(x)$.

$$\sec^2(x) = \boxed{1 + \tan^2(x)}$$

Answer: This relationship comes directly from one of the Pythagorean identities $\sin^2(x) + \cos^2(x) = 1$.

$$\begin{aligned} \sec^2(x) &= \frac{1}{\cos^2(x)} \\ \sec^2(x) &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ \sec^2(x) &= 1 + \frac{\sin^2(x)}{\cos^2(x)} \\ \sec^2(x) &= 1 + \tan^2(x) \end{aligned}$$

MCQ 38: The expression $\sin(x) \cdot \cot(x)$ simplifies to:

- ☐ $\sin^2(x)$
- ☒ $\cos^2(x)$
- ☐ $\cos(x)$
- ☐ 1

Answer: We use the definition of the cotangent function, $\cot(x) = \frac{\cos x}{\sin x}$.

$$\begin{aligned} \sin(x) \cdot \cot(x) &= \sin(x) \cdot \frac{\cos x}{\sin x} \\ &= \cos x \end{aligned}$$

(assuming $\sin x \neq 0$).

E.3 EVALUATING RECIPROCAL FUNCTIONS

Ex 39: Find the exact value of $\cot(\frac{\pi}{6})$.

$$\cot(\tfrac{\pi}{6}) = \boxed{\sqrt{3}}$$

Answer: The cotangent function is defined as $\cot(x) = \frac{\cos(x)}{\sin(x)}$. We use the known values for the angle $\frac{\pi}{6}$:

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \quad \text{and} \quad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Therefore:

$$\cot\left(\frac{\pi}{6}\right) = \frac{\cos(\pi/6)}{\sin(\pi/6)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

Ex 40: Find the exact value of $\sec(\pi)$.

$$\sec(\pi) = \boxed{-1}$$

Answer: The secant function is defined as $\sec(x) = \frac{1}{\cos(x)}$. We use the known value for the angle π:

$$\cos(\pi) = -1$$

Therefore:

$$\sec(\pi) = \frac{1}{\cos(\pi)} = \frac{1}{-1} = -1$$

Ex 41: Find the exact value of $\csc(\frac{3\pi}{2})$.

$$\csc(\tfrac{3\pi}{2}) = \boxed{-1}$$

Answer: The cosecant function is defined as $\csc(x) = \frac{1}{\sin(x)}$. We use the known value for the angle $\frac{3\pi}{2}$:

$$\sin\left(\frac{3\pi}{2}\right) = -1$$

Therefore:

$$\csc\left(\frac{3\pi}{2}\right) = \frac{1}{\sin(3\pi/2)} = \frac{1}{-1} = -1$$

Ex 42: Find the exact value of $\sec\left(\frac{5\pi}{4}\right)$.

$$\sec\left(\tfrac{5\pi}{4}\right) = \boxed{-\sqrt{2}}$$

Answer: The secant function is defined as $\sec(x) = \frac{1}{\cos(x)}$. We first find the value of $\cos\left(\frac{5\pi}{4}\right)$.
The angle $\frac{5\pi}{4}$ is in the third quadrant, where cosine is negative. The reference angle is $\frac{\pi}{4}$.

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

Therefore:

$$\sec\left(\frac{5\pi}{4}\right) = \frac{1}{-\sqrt{2}/2} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

F INVERSE TRIGONOMETRIC FUNCTIONS

F.1 EVALUATING INVERSE TRIGONOMETRIC FUNCTIONS AT SPECIAL ANGLES

Ex 43: Find the angle in radians:

$$\cos^{-1}(1) = \boxed{0}$$

Answer: As $\cos 0 = 1$, $\cos^{-1}(1) = 0$.

Ex 44: Find the angle in radians:

$$\sin^{-1}(1) = \boxed{\frac{\pi}{2}}$$

Answer: As $\sin\frac{\pi}{2} = 1$, $\sin^{-1}(1) = \frac{\pi}{2}$.

Ex 45: Find the angle in radians:

$$\sin^{-1}\left(\tfrac{1}{2}\right) = \boxed{\frac{\pi}{6}}$$

Answer: As $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Ex 46: Find the angle in radians:

$$\cos^{-1}\left(\tfrac{1}{2}\right) = \boxed{\frac{\pi}{3}}$$

Answer: As $\cos\frac{\pi}{3} = \frac{1}{2}$, $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

Ex 47: Find the angle in radians:

$$\sin^{-1}\left(\tfrac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$$

Answer: As $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

Ex 48: Find the angle in radians:

$$\cos^{-1}\left(\tfrac{\sqrt{2}}{2}\right) = \boxed{\frac{\pi}{4}}$$

Answer: As $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, $\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

Ex 49: Find the angle in radians:

$$\cos^{-1}\left(-\tfrac{\sqrt{3}}{2}\right) = \boxed{\frac{5\pi}{6}}$$

Answer: As $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\frac{5\pi}{6} \in [0,\pi]$, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

Ex 50: Find the angle in radians:

$$\tan^{-1}(1) = \boxed{\frac{\pi}{4}}$$

Answer: As $\tan\frac{\pi}{4} = \frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{4}} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$, $\tan^{-1}(1) = \frac{\pi}{4}$.

Ex 51: Find the angle in radians:

$$\tan^{-1}\left(\sqrt{3}\right) = \boxed{\frac{\pi}{3}}$$

Answer: As $\tan\frac{\pi}{3} = \frac{\sin\frac{\pi}{3}}{\sin\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$, $\tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$.

Ex 52: Find the angle in radians:

$$\tan^{-1}\left(-\tfrac{1}{\sqrt{3}}\right) = \boxed{-\frac{\pi}{6}}$$

Answer: As $\tan\left(-\frac{\pi}{6}\right) = \frac{\sin\left(-\frac{\pi}{6}\right)}{\cos\left(-\frac{\pi}{6}\right)} = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}$, $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$.

F.2 SIMPLIFYING EXPRESSIONS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Ex 53: Simplify:

$$\arccos\left(\cos\left(-\tfrac{\pi}{4}\right)\right) = \boxed{\frac{\pi}{4}}$$

Answer:

$$\begin{aligned}\arccos\left(\cos\left(-\frac{\pi}{4}\right)\right) &= \arccos\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\pi}{4}\end{aligned}$$

Ex 54: Simplify:

$$\arccos\left(\sin\left(\tfrac{2\pi}{3}\right)\right) = \boxed{\frac{\pi}{6}}$$

Answer:

$$\begin{aligned}\arccos\left(\sin\left(\frac{2\pi}{3}\right)\right) &= \arccos\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\pi}{6}\end{aligned}$$

Ex 55: Simplify:

$$\arctan\left(\cos\left(4\pi\right)\right) = \boxed{\frac{\pi}{4}}$$

Answer:

$$\begin{aligned}\arctan\left(\cos\left(4\pi\right)\right) &= \arctan\left(1\right)\\ &= \frac{\pi}{4}\end{aligned}$$

Ex 56: Simplify:

$$\arccos\left(\sin\left(\tfrac{\pi}{3}\right)\right) = \boxed{\frac{\pi}{6}}$$

Answer:

$$\begin{aligned}\arccos\left(\sin\left(\frac{\pi}{3}\right)\right) &= \arccos\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\pi}{6}\end{aligned}$$

Ex 57: Simplify:

$$\arcsin\left(\cos\left(\tfrac{\pi}{6}\right)\right) = \boxed{\frac{\pi}{3}}$$

Answer:

$$\begin{aligned}\arcsin\left(\cos\left(\frac{\pi}{6}\right)\right) &= \arcsin\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\pi}{3}\end{aligned}$$

Ex 58: Simplify:

$$\arctan\left(-\tan\left(\tfrac{\pi}{6}\right)\right) = \boxed{-\frac{\pi}{6}}$$

Answer:

$$\begin{aligned}\arctan\left(-\tan\left(\frac{\pi}{6}\right)\right) &= \arctan\left(-\frac{1}{\sqrt{3}}\right)\\ &= -\frac{\pi}{6}\end{aligned}$$

G SOLVING TRIGONOMETRIC EQUATIONS

G.1 SOLVING BASIC TRIGONOMETRIC EQUATIONS

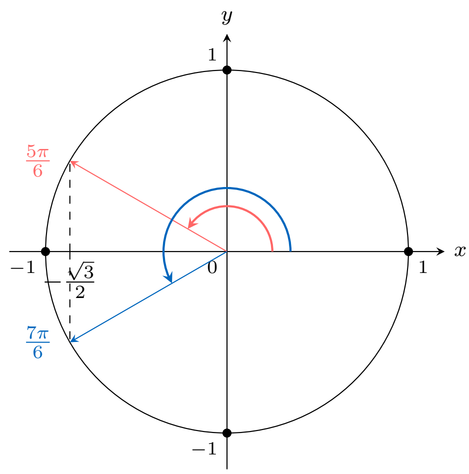
Ex 59: Solve for x on the domain $0 \leq x \leq 2\pi$:

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$x = \boxed{5\pi/6} < x = \boxed{7\pi/6}$$

Answer: We are looking for angles on the unit circle where the x-coordinate is $-\frac{\sqrt{3}}{2}$.
The reference angle for which $\cos(x) = \frac{\sqrt{3}}{2}$ is $\theta = \frac{\pi}{6}$.Cosine is negative in the second and third quadrants.

- Second quadrant solution: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.
- Third quadrant solution: $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$.



The solutions on the domain $0 \leq x \leq 2\pi$ are $x = \frac{5\pi}{6}$ or $x = \frac{7\pi}{6}$.

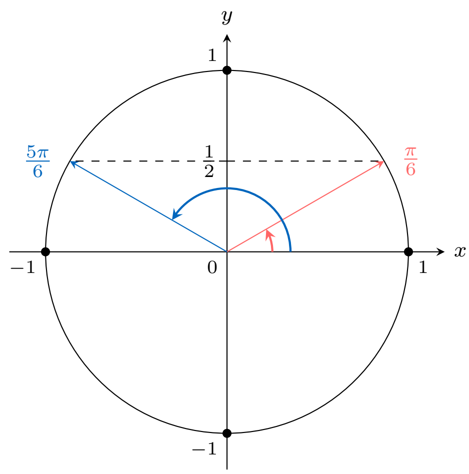
Ex 60: Solve for x on the domain $0 \leq x \leq 2\pi$:

$$\sin x = \frac{1}{2}$$

$$x = \boxed{\pi/6} < x = \boxed{5\pi/6}$$

Answer: We are looking for angles on the unit circle where the y-coordinate is $\frac{1}{2}$.
The reference angle for which $\sin(\theta) = \frac{1}{2}$ is $\theta = \frac{\pi}{6}$.Sine is positive in the first and second quadrants.

- First quadrant solution: $x = \frac{\pi}{6}$.
- Second quadrant solution: $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.



The solutions on the domain $0 \leq x \leq 2\pi$ are $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.

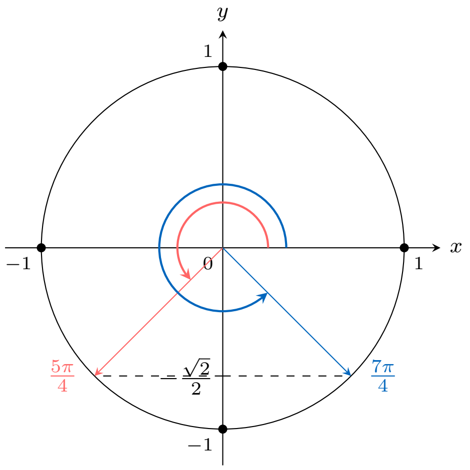
Ex 61: Solve for x on the domain $0 \leq x \leq 2\pi$:

$$\sin x = -\frac{\sqrt{2}}{2}$$

$$x = \boxed{5\pi/4} < x = \boxed{7\pi/4}$$

Answer: We are looking for angles on the unit circle where the y-coordinate is $-\frac{\sqrt{2}}{2}$.
The reference angle for which $\sin(\theta) = \frac{\sqrt{2}}{2}$ is $\theta = \frac{\pi}{4}$.Sine is negative in the third and fourth quadrants.

- Third quadrant solution: $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$.
- Fourth quadrant solution: $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.



The solutions on the domain $0 \leq x \leq 2\pi$ are $x = \frac{5\pi}{4}$ or $x = \frac{7\pi}{4}$.

Ex 62: Solve for x on the domain $0 \leq x \leq 2\pi$:

$$2\cos x = 1$$

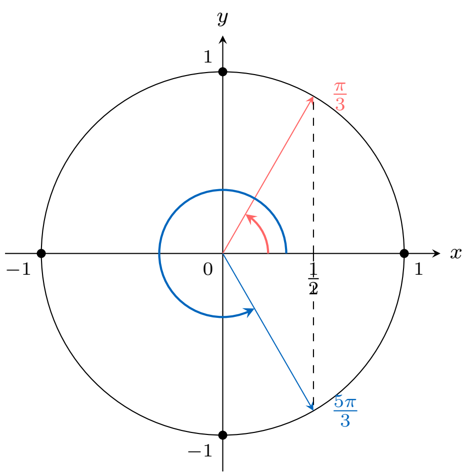
$$x = \boxed{\pi/3} < x = \boxed{5\pi/3}$$

Answer: First, we rearrange the equation to isolate $\cos x$:

$$\cos x = \frac{1}{2}$$

We are looking for angles on the unit circle where the x-coordinate is $\frac{1}{2}$.
The reference angle for which $\cos(\theta) = \frac{1}{2}$ is $\theta = \frac{\pi}{3}$.Cosine is positive in the first and fourth quadrants.

- First quadrant solution: $x = \frac{\pi}{3}$.
- Fourth quadrant solution: $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.



The solutions on the domain $0 \leq x \leq 2\pi$ are $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$.

G.2 SOLVING EQUATIONS OF QUADRATIC FORM

Ex 63: Solve for x on the domain $0 \leq x \leq 2\pi$:

$$\sin^2 x = \frac{1}{2}$$

Answer: First, we take the square root of both sides to isolate $\sin x$.

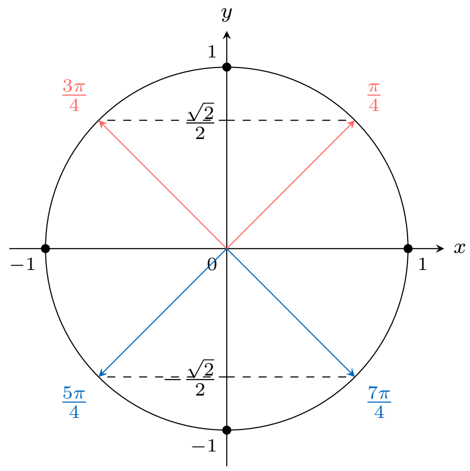
$$\sin x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$$

This gives us two separate equations to solve:

1. $\sin x = \frac{\sqrt{2}}{2}$
2. $\sin x = -\frac{\sqrt{2}}{2}$

The reference angle for which $\sin(\theta) = \frac{\sqrt{2}}{2}$ is $\theta = \frac{\pi}{4}$.

- For $\sin x = \frac{\sqrt{2}}{2}$ (positive y-coordinate), solutions are in the first and second quadrants:
  - $x = \frac{\pi}{4}$
  - $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$
- For $\sin x = -\frac{\sqrt{2}}{2}$ (negative y-coordinate), solutions are in the third and fourth quadrants:
  - $x = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$
  - $x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$



The four solutions on the domain $0 \leq x \leq 2\pi$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Ex 64: Solve for x on the domain $0 \leq x \leq 2\pi$:

$$\sin^2 x = \frac{3}{4}$$

Answer: First, we take the square root of both sides to isolate $\sin x$.

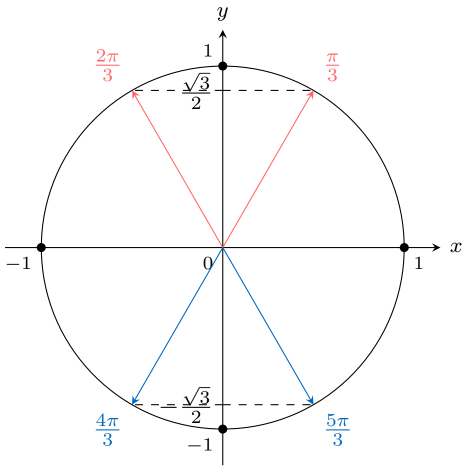
$$\sin x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

This gives us two separate equations to solve:

1. $\sin x = \frac{\sqrt{3}}{2}$
2. $\sin x = -\frac{\sqrt{3}}{2}$

The reference angle for which $\sin(\theta) = \frac{\sqrt{3}}{2}$ is $\theta = \frac{\pi}{3}$.

- For $\sin x = \frac{\sqrt{3}}{2}$ (positive y-coordinate), solutions are in the first and second quadrants:
  - $x = \frac{\pi}{3}$
  - $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$
- For $\sin x = -\frac{\sqrt{3}}{2}$ (negative y-coordinate), solutions are in the third and fourth quadrants:
  - $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
  - $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$



The four solutions on the domain $0 \leq x \leq 2\pi$ are $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

G.3 SOLVING EQUATIONS WITH TRANSFORMED ARGUMENTS

Ex 65: Consider the solution of trigonometric equations.

1. Find all solutions to the equation $\sin(x) = \frac{\sqrt{2}}{2}$ on the domain $0 \leq x \leq 2\pi$.
2. Hence, find all solutions to the equation $\sin(2x) = \frac{\sqrt{2}}{2}$ on the domain $0 \leq x \leq \pi$.

Answer:

1. We are looking for angles on the unit circle in the interval $[0, 2\pi]$ where the y-coordinate is $\frac{\sqrt{2}}{2}$.
   The reference angle is $\arcsin(\frac{\sqrt{2}}{2}) = \frac{\pi}{4}$.
   Sine is positive in the first and second quadrants.
   - First quadrant solution: $x = \frac{\pi}{4}$.
   - Second quadrant solution: $x = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

   The solutions are $\boldsymbol{x = \frac{\pi}{4}}$ and $\boldsymbol{x = \frac{3\pi}{4}}$.
2. To solve $\sin(2x) = \frac{\sqrt{2}}{2}$, we let $u = 2x$.
   First, we must adjust the domain for the new variable u.
   If $0 \leq x \leq \pi$, then $2 \times 0 \leq 2x \leq 2 \times \pi$, so the new domain is $\boldsymbol{0 \leq u \leq 2\pi}$.
   We need to find all values of u in this new domain for which $\sin(u) = \frac{\sqrt{2}}{2}$.
   This is precisely the problem we solved in part (1). The solutions for u in the interval $[0, 2\pi]$ are:
   $$u = \frac{\pi}{4} \quad \text{and} \quad u = \frac{3\pi}{4}$$
   Finally, we substitute back $u = 2x$ and solve for x:
   - $2x = \frac{\pi}{4} \implies x = \frac{\pi}{8}$
   - $2x = \frac{3\pi}{4} \implies x = \frac{3\pi}{8}$

   Both of these values lie within the required domain $0 \leq x \leq \pi$.
   The solutions are $\boldsymbol{x = \frac{\pi}{8}}$ and $\boldsymbol{x = \frac{3\pi}{8}}$.

Ex 66: Consider the solution of trigonometric equations.

1. Find all solutions to the equation $\cos(x) = -\frac{1}{2}$ on the domain $0 \leq x \leq 2\pi$.
2. Hence, find all solutions to the equation $\cos(x - \frac{\pi}{3}) = -\frac{1}{2}$ on the domain $0 \leq x \leq 2\pi$.

Answer:

1. We are looking for angles on the unit circle in the interval $[0, 2\pi]$ where the x-coordinate is $-\frac{1}{2}$.
   The reference angle is $\arccos(\frac{1}{2}) = \frac{\pi}{3}$.
   Cosine is negative in the second and third quadrants.
   - Second quadrant solution: $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
   - Third quadrant solution: $x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

   The solutions are $\boldsymbol{x = \frac{2\pi}{3}}$ and $\boldsymbol{x = \frac{4\pi}{3}}$.
2. To solve $\cos(x - \frac{\pi}{3}) = -\frac{1}{2}$, we let $u = x - \frac{\pi}{3}$.
   First, we must adjust the domain for the new variable u.
   If $0 \leq x \leq 2\pi$, then $0 - \frac{\pi}{3} \leq x - \frac{\pi}{3} \leq 2\pi - \frac{\pi}{3}$, so the new domain is $\boldsymbol{-\frac{\pi}{3} \leq u \leq \frac{5\pi}{3}}$.
   We need to find all values of u in this new domain for which $\cos(u) = -\frac{1}{2}$.
   From part (1), the solutions in the interval $[0, 2\pi]$ are $u = \frac{2\pi}{3}$ and $u = \frac{4\pi}{3}$. Both of these lie within our domain for u.
   We must also check for negative solutions. The solution equivalent to $\frac{4\pi}{3}$ in the negative direction is $\frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$. This is outside our domain.
   The solutions for u in the interval $[-\frac{\pi}{3}, \frac{5\pi}{3}]$ are therefore $u = \frac{2\pi}{3}$ and $u = \frac{4\pi}{3}$.
   Finally, we substitute back $u = x - \frac{\pi}{3}$ and solve for x:
   - $x - \frac{\pi}{3} = \frac{2\pi}{3} \implies x = \frac{3\pi}{3} = \pi$
   - $x - \frac{\pi}{3} = \frac{4\pi}{3} \implies x = \frac{5\pi}{3}$

   Both values lie within the required domain $0 \leq x \leq 2\pi$.
   The solutions are $\boldsymbol{x = \pi}$ and $\boldsymbol{x = \frac{5\pi}{3}}$.

Ex 67: Solve for x on the domain $0 \leq x < 2\pi$:
$$\cos\left(x - \frac{\pi}{5}\right) = 0$$

Answer:

1. **Substitution**: Let $u = x - \frac{\pi}{5}$. The equation becomes $\cos(u) = 0$.
2. **Adjust the domain**: We transform the domain for x into a new domain for u. If $0 \leq x < 2\pi$, then:
   $$0 - \frac{\pi}{5} \leq x - \frac{\pi}{5} < 2\pi - \frac{\pi}{5}$$
   $$-\frac{\pi}{5} \leq u < \frac{9\pi}{5}$$
   The new domain for u is $[-\frac{\pi}{5}, \frac{9\pi}{5})$.

3. **Solve for** u: We need to find all values of u in this new domain for which $\cos(u) = 0$.
The general solution for $\cos(u) = 0$ is $u = \frac{\pi}{2} + k\pi$ for any integer k. We find the solutions within our domain for u:
   - $k = 0 : u = \frac{\pi}{2}$. (This is in the interval $[-\frac{\pi}{5}, \frac{9\pi}{5})$)
   - $k = 1 : u = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$. (This is also in the interval)
   - $k = 2 : u = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$. (This is outside the interval as $\frac{5\pi}{2} = 2.5\pi$ and $\frac{9\pi}{5} = 1.8\pi$)
   - $k = -1 : u = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$. (This is outside the interval as $-\frac{\pi}{2} < -\frac{\pi}{5}$)

   The solutions for u are $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

4. **Solve for** x: Now we substitute back $u = x - \frac{\pi}{5}$ and solve for x.
   - $x - \frac{\pi}{5} = \frac{\pi}{2} \implies x = \frac{\pi}{2} + \frac{\pi}{5} = \frac{5\pi+2\pi}{10} = \frac{7\pi}{10}$
   - $x - \frac{\pi}{5} = \frac{3\pi}{2} \implies x = \frac{3\pi}{2} + \frac{\pi}{5} = \frac{15\pi+2\pi}{10} = \frac{17\pi}{10}$

   Both values are in the original domain $[0, 2\pi)$.
The solutions are $\boldsymbol{x = \frac{7\pi}{10}}$ and $\boldsymbol{x = \frac{17\pi}{10}}$.

Ex 68: Consider the solution of trigonometric equations.

1. Find all solutions to the equation $\tan(x) = 1$ on the domain $0 \leq x \leq \pi$.
2. Hence, find all solutions to the equation $\tan(2x) = 1$ on the domain $0 \leq x \leq 2\pi$.

Answer:

1. The reference angle for which $\tan(x) = 1$ is $x = \frac{\pi}{4}$.
Since the period of the tangent function is π, this is the only solution in the interval $[0, \pi]$.
The solution is $\boldsymbol{x = \frac{\pi}{4}}$.
2. To solve $\tan(2x) = 1$, we let $u = 2x$.
First, we adjust the domain for the new variable u.
If $0 \leq x \leq 2\pi$, then $0 \leq 2x \leq 4\pi$, so the new domain is $\boldsymbol{0 \leq u \leq 4\pi}$.
We need to find all values of u in this new domain for which $\tan(u) = 1$. From part (1), the base solution is $u = \frac{\pi}{4}$.
Since the period of tangent is π, the general solution for u is $u = \frac{\pi}{4} + k\pi$, where k is an integer.
We find the solutions for u that are in the interval $[0, 4\pi]$:
   - $k = 0 : u = \frac{\pi}{4}$
   - $k = 1 : u = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$
   - $k = 2 : u = \frac{\pi}{4} + 2\pi = \frac{9\pi}{4}$
   - $k = 3 : u = \frac{\pi}{4} + 3\pi = \frac{13\pi}{4}$

   (For $k = 4$, $u = \frac{17\pi}{4}$ which is greater than 4π).
Finally, we substitute back $u = 2x$ and solve for x by dividing each solution by 2:
$$x = \frac{\pi}{8}, \quad x = \frac{5\pi}{8}, \quad x = \frac{9\pi}{8}, \quad x = \frac{13\pi}{8}$$
These are the four solutions in the domain $0 \leq x \leq 2\pi$.

Ex 69: Consider the solution of trigonometric equations.

1. Find all solutions to the equation $\cos(x) = \frac{1}{2}$ on the domain $0 \leq x \leq 2\pi$.
2. Hence, find all solutions to the equation $\cos(\frac{x}{2}) = \frac{1}{2}$ on the domain $0 \leq x \leq 4\pi$.

Answer:

1. We are looking for angles on the unit circle in the interval $[0, 2\pi]$ where the x-coordinate is $\frac{1}{2}$. The reference angle is $\arccos(\frac{1}{2}) = \frac{\pi}{3}$. Cosine is positive in the first and fourth quadrants.
   - First quadrant solution: $x = \frac{\pi}{3}$.
   - Fourth quadrant solution: $x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

   The solutions are $\boldsymbol{x = \frac{\pi}{3}}$ and $\boldsymbol{x = \frac{5\pi}{3}}$.
2. To solve $\cos(\frac{x}{2}) = \frac{1}{2}$, we let $u = \frac{x}{2}$. First, we must adjust the domain for the new variable u. If $0 \leq x \leq 4\pi$, then $\frac{0}{2} \leq \frac{x}{2} \leq \frac{4\pi}{2}$, so the new domain is $\boldsymbol{0 \leq u \leq 2\pi}$. We need to find all values of u in this new domain for which $\cos(u) = \frac{1}{2}$. This is precisely the problem we solved in part (1). The solutions for u in the interval $[0, 2\pi]$ are:
$$u = \frac{\pi}{3} \quad \text{and} \quad u = \frac{5\pi}{3}$$
Finally, we substitute back $u = \frac{x}{2}$ and solve for x by multiplying by 2:
   - $\frac{x}{2} = \frac{\pi}{3} \implies x = \frac{2\pi}{3}$
   - $\frac{x}{2} = \frac{5\pi}{3} \implies x = \frac{10\pi}{3}$

   Both of these values lie within the required domain $0 \leq x \leq 4\pi$. The solutions are $\boldsymbol{x = \frac{2\pi}{3}}$ and $\boldsymbol{x = \frac{10\pi}{3}}$.

H MODELING PERIODIC DATA WITH A SINE FUNCTION

H.1 MODELING REAL-WORLD PHENOMENA

Ex 70: The horizontal displacement, D cm, of the bob of a pendulum from its central position is modelled by a sine function of time, t seconds. The bob is released from its maximum displacement of 10 cm at $t = 0.25$ seconds. It swings to a minimum displacement of -10 cm and first returns to its maximum displacement at $t = 1.25$ seconds.
Find a sine function of the form $D(t) = a\sin(b(t - c)) + d$ to model this motion.

Answer:

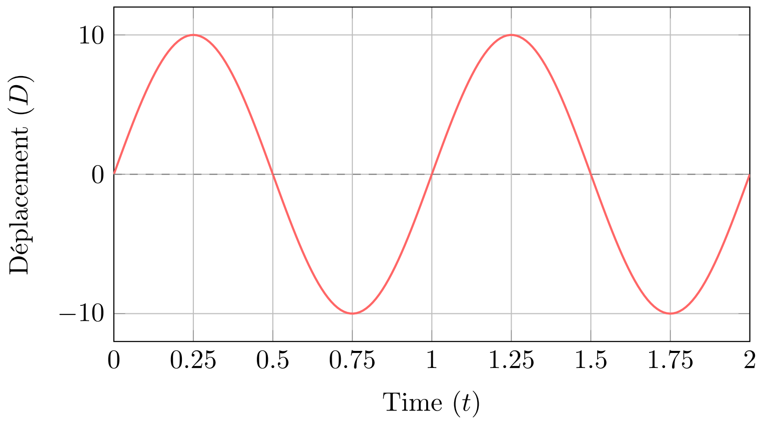
1. **Find Vertical Shift** (d): The max displacement is 10 and the min is -10.
$$d = \frac{10 + (-10)}{2} = 0$$
2. **Find Amplitude** (a):
$$a = \frac{10 - (-10)}{2} = 10$$
3. **Find Period** (b): The time from a maximum ($t = 0.25$) to the next maximum ($t = 1.25$) is one full period. The period is $P = 1.25 - 0.25 = 1$ second.
$$b = \frac{2\pi}{P} = \frac{2\pi}{1} = 2\pi$$

4. **Find Phase Shift** (c): Our model is $D(t) = 10\sin(2\pi(t - c))$. A standard sine function starts at its principal axis ($y = 0$) and is increasing. The time from a starting point to a maximum is one quarter of a period. Since the maximum is at $t = 0.25$ and the period is 1, the starting point of the sine cycle must be a quarter of a period earlier.

$$c = 0.25 - \frac{1}{4}P = 0.25 - \frac{1}{4}(1) = 0$$

5. **Final Model**:

$$D(t) = 10\sin(2\pi t)$$



Ex 71: The height H (in metres) of a rider on a Ferris wheel after t seconds is recorded. The wheel rotates at a constant speed. The maximum height is 25 metres and the minimum height is 1 metre. The wheel completes one full revolution every 20 seconds. At $t = 0$, the rider is at the bottom of the wheel.
Find a cosine function of the form $H(t) = a\cos(b(t - c)) + d$ to model the rider's height.

Answer:

1. **Find Vertical Shift** (d): The max value is 25 and the min value is 1.

$$d = \frac{25 + 1}{2} = 13$$

2. **Find Amplitude** (a):

$$|a| = \frac{25 - 1}{2} = 12$$

Since the rider starts at the bottom of the wheel (a minimum), we should use a reflected cosine function, so we choose $a = -12$.

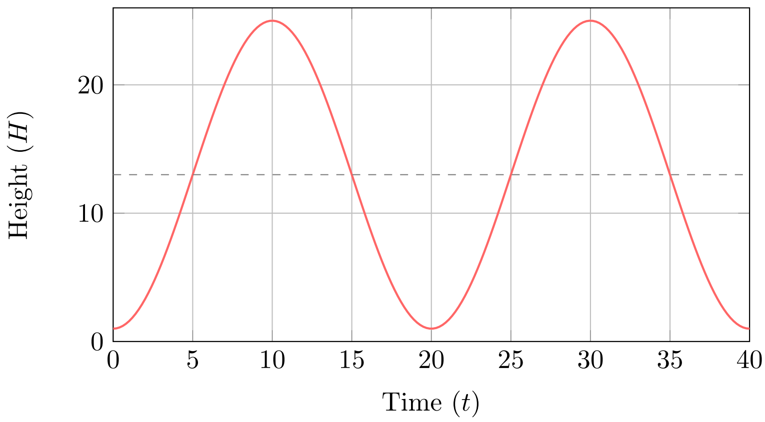
3. **Find Period** (b): The period is given as $P = 20$ seconds.

$$b = \frac{2\pi}{P} = \frac{2\pi}{20} = \frac{\pi}{10}$$

4. **Find Phase Shift** (c): A standard reflected cosine function ($a < 0$) starts at a minimum on the y-axis. Since the rider is at the minimum at $t = 0$, there is no phase shift. We can set $c = 0$.

5. **Final Model**:

$$H(t) = -12\cos\left(\frac{\pi}{10}t\right) + 13$$



Ex 72: The depth of water, D metres, in a harbour can be modelled by a sinusoidal function of time, t hours after midnight. The depth has a maximum of 14m at 3:00 am and a minimum of 2m at 9:00 am.
Find a cosine function of the form $D(t) = a\cos(b(t - c)) + d$ to model the water depth.

Answer:

1. **Find Vertical Shift** (d): Max value = 14, min value = 2.

$$d = \frac{14 + 2}{2} = 8$$

2. **Find Amplitude** (a):

$$a = \frac{14 - 2}{2} = 6$$

Since we are modeling with a standard (non-reflected) cosine, we use $a = 6$.

3. **Find Period** (b): The time from a maximum (3:00) to the next minimum (9:00) is $9 - 3 = 6$ hours. This is half a period.
The full period is $P = 2 \times 6 = 12$ hours.

$$b = \frac{2\pi}{P} = \frac{2\pi}{12} = \frac{\pi}{6}$$

4. **Find Phase Shift** (c): A standard cosine function starts at a maximum. The first maximum occurs at $t = 3$ (3:00 am). Therefore, the graph is shifted 3 units to the right.
We set the phase shift $c = 3$.

5. **Final Model**:

$$D(t) = 6\cos\left(\frac{\pi}{6}(t - 3)\right) + 8$$