SOLVING QUADRATIC EQUATIONS

Quadratic equations are second-degree equations that model many real situations—from areas and trajectories to profit and optimization. In this chapter you will learn to recognize the standard form $ax^2 + bx + c = 0$, identify the coefficients, and choose an appropriate solving method: factoring (including special products), completing the square, and the quadratic formula. You will also use the discriminant to predict the number of real solutions.

A QUADRATIC EQUATION

Definition Quadratic Equation

A **quadratic equation** is an equation that can be written in the standard form:

$$ax^2 + bx + c = 0$$

where a, b, and c are known coefficients, and $a \neq 0$. The condition $a \neq 0$ is what makes the equation quadratic. A **solution** or **root** of the equation is a value of x that makes the statement true.

Ex: Consider the equation $3x^2 + 5x + 4 = 0$. Is it a quadratic equation? If yes, identify the coefficients a, b, and c.

Answer: Yes. It is in the form $ax^2 + bx + c = 0$ with $a = 3$, $b = 5$, $c = 4$, and $a \neq 0$, so it is a quadratic equation.

Ex: Are 1 and 3 roots of the equation $x^2 - 3x + 2 = 0$?

Answer: To check if 1 and 3 are roots, substitute each into the equation:

- For $x = 1$, $1^2 - 3 \cdot 1 + 2 = 1 - 3 + 2 = 0$.
  So 1 is a root.
- For $x = 3$, $3^2 - 3 \cdot 3 + 2 = 9 - 9 + 2 = 2 \neq 0$.
  So 3 is not a root.

A quadratic equation may have no real solution. For example, $x^2 = -1$ has no real solution because the square of a real number cannot be negative.

B SOLVING BY FACTORIZATION

A primary strategy for solving quadratic equations is to use the **Null Factor Law**. This law allows us to break a single quadratic equation into simpler linear equations. To use it, we must first factorize the quadratic expression.

Method Solving by Factorization

The primary strategy for solving quadratic equations is to use the **Null Factor Law**.

1. **Write the equation in standard form**, $ax^2 + bx + c = 0$.
2. **Factorize** the quadratic expression completely.
3. **Apply the Null Factor Law**: set each factor equal to zero.
4. **Solve** each resulting linear equation.

Proposition Null Factor Law

If the product of two or more factors is equal to zero, then at least one of the factors must be equal to zero.

$$\text{If } A\,B = 0 \text{ then } A = 0 \textbf{ or } B = 0.$$

Note: one or both of the factors can be zero.

Ex: Solve $(x - 1)(x + 2) = 0$.

Answer:

$$\begin{aligned}
(x-1)(x+2) &= 0 \\
x - 1 = 0 \quad \text{or} \quad x + 2 &= 0 \quad \text{(Null factor law)} \\
x = 1 \quad \text{or} \quad x &= -2 \quad \text{(Solve each equation)}
\end{aligned}$$

C FACTORIZATION TECHNIQUES FOR SPECIAL FORMS OF EQUATIONS

Before learning a general method, we first master solving equations that can be factored using familiar patterns.

Proposition Common Factor ($c = 0$)

For equations of the form $ax^2 + bx = 0$, the common factor is x:

$$x(ax + b) = 0 \Leftrightarrow x = 0 \text{ or } ax + b = 0$$

Ex: Find the roots of $x^2 - 2x = 0$.

Answer:

$$\begin{aligned} x^2 - 2x &= 0 \\ x(x-2) &= 0 && \text{(Factor out the common factor } x\text{)} \\ x = 0 \quad &\text{or} \quad x - 2 = 0 && \text{(Null factor law)} \\ x = 0 \quad &\text{or} \quad x = 2 \end{aligned}$$

Proposition Difference of Squares ($b = 0$)

For equations of the form $x^2 - k = 0$ (where $k > 0$):

$$x^2 - (\sqrt{k})^2 = 0 \Leftrightarrow (x - \sqrt{k})(x + \sqrt{k}) = 0$$

Ex: Solve $x^2 - 9 = 0$.

Answer:

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 - 3^2 &= 0 \\ (x-3)(x+3) &= 0 && \text{(Difference of squares)} \\ x - 3 = 0 \quad &\text{or} \quad x + 3 = 0 && \text{(Null factor law)} \\ x = 3 \quad &\text{or} \quad x = -3 \end{aligned}$$

Proposition Perfect Squares

For equations of the form $x^2 \pm 2ax + a^2 = 0$:

$$x^2 + 2ax + a^2 = (x + a)^2 = 0 \Leftrightarrow x + a = 0,$$
$$x^2 - 2ax + a^2 = (x - a)^2 = 0 \Leftrightarrow x - a = 0.$$

Ex: Solve $x^2 + 2x + 1 = 0$.

Answer:

$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x+1)^2 &= 0 && \text{(Perfect square factorization)} \\ x + 1 &= 0 && \text{(Null factor law)} \\ x &= -1 \end{aligned}$$

This is a repeated root (double root).

D THE GENERAL METHOD: COMPLETING THE SQUARE

When a quadratic expression is not one of the special forms, we need a general method. This method is called **completing the square**. The idea is to rewrite the quadratic as a perfect square plus (or minus) a constant. When solving an equation, this eventually leads to a difference of two squares or to taking square roots.

Proposition Completing the Square

Any quadratic expression of the form $x^2 + bx + c$ (leading coefficient 1) can be put into vertex form by completing the square:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c.$$

For a general quadratic $ax^2 + bx + c$ with $a \neq 0$, we first factor out a:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

and then complete the square inside the parentheses.

Ex: Complete the square for $x^2 + 10x + 24$.

Answer: We know that $(x+5)^2 = x^2 + 10x + 25$. So

$$\begin{aligned} x^2 + 10x + 24 &= x^2 + 10x + 25 - 25 + 24 \\ &= (x+5)^2 - 1 && \text{(Complete the square).} \end{aligned}$$

Method General method to solve quadratic equations

Given an equation $ax^2 + bx + c = 0$ with $a \neq 0$:

- Step 1: **Complete the square** to rewrite the left-hand side as a perfect square plus/minus a constant.
- Step 2: **Use the difference of squares** (or take square roots) to isolate x.
- Step 3: **Apply the Null Factor Law** if the expression has been written as a product.
- Step 4: **Solve the linear equations** obtained.

Ex: Solve $x^2 + 10x + 24 = 0$.

Answer: We know that $(x+5)^2 = x^2 + 10x + 25$. So

$$\begin{aligned} x^2 + 10x + 24 &= 0 \\ x^2 + 10x + 25 - 25 + 24 &= 0 \\ (x+5)^2 - 1 &= 0 && \text{(Complete the square)} \\ (x+5)^2 - 1^2 &= 0 && \text{(Difference of squares)} \\ (x+5-1)(x+5+1) &= 0 && \text{(Factorize)} \\ (x+4)(x+6) &= 0 && \text{(Simplify)} \\ x+4 = 0 \text{ or } x+6 &= 0 && \text{(Null factor law)} \\ x = -4 \text{ or } x &= -6 && \text{(Solve).} \end{aligned}$$

E QUADRATIC FORMULA

Applying the method of completing the square to the general equation $ax^2 + bx + c = 0$ gives a formula that solves any quadratic equation. This is the **quadratic formula**.

Proposition Quadratic formula

For any quadratic equation $ax^2 + bx + c = 0$, the discriminant, denoted Δ, is defined as

$$\Delta = b^2 - 4ac.$$

Its sign determines the number of real solutions:

- If $\Delta > 0$, there are two real roots:
$$x = \frac{-b - \sqrt{\Delta}}{2a} \text{ or } x = \frac{-b + \sqrt{\Delta}}{2a}$$
- If $\Delta = 0$, there is one real root (a double root):
$$x = \frac{-b}{2a}.$$
- If $\Delta < 0$, there are no real roots.

Ex: Consider the quadratic equation $x^2 + 2x - 3 = 0$.

1. Find the discriminant.
2. Hence, state the nature of the roots of the equation.
3. Solve the equation.

Answer: $x^2 + 2x - 3 = 0$ has $a = 1$, $b = 2$, $c = -3$.

1. $\Delta = b^2 - 4ac$
   $= (2)^2 - 4(1)(-3)$
   $= 4 + 12$
   $= 16$

2. As $\Delta > 0$, there are two distinct real solutions.

3. $x = \dfrac{-b - \sqrt{\Delta}}{2a}$ or $x = \dfrac{-b + \sqrt{\Delta}}{2a}$
   $x = \dfrac{-2 - \sqrt{16}}{2 \cdot 1}$ or $x = \dfrac{-2 + \sqrt{16}}{2 \cdot 1}$
   $x = \dfrac{-2 - 4}{2}$ or $x = \dfrac{-2 + 4}{2}$
   $x = -3$ or $x = 1$