# SOLVING QUADRATIC EQUATIONS

Quadratic equations are second-degree equations that model many real situations—from areas and trajectories to profit and optimization. In this chapter you will learn to recognize the standard form $ax^2 + bx + c = 0$, identify the coefficients, and choose an appropriate solving method: factoring (including special products), completing the square, and the quadratic formula. You will also use the discriminant to predict the number of real solutions.

## A QUADRATIC EQUATION

**Definition Quadratic Equation**

A **quadratic equation** is an equation that can be written in the standard form:

$$ax^2 + bx + c = 0$$

where $a, b$, and $c$ are known coefficients, and $a \neq 0$. The condition $a \neq 0$ is what makes the equation quadratic.
A **solution** or **root** of the equation is a value of $x$ that makes the statement true.

**Ex:** Consider the equation $3x^2 + 5x + 4 = 0$. Is it a quadratic equation? If yes, identify the coefficients $a$, $b$, and $c$.

*Answer:* Yes. It is in the form $ax^2 + bx + c = 0$ with $a = 3$, $b = 5$, $c = 4$, and $a \neq 0$, so it is a quadratic equation.

**Ex:** Are 1 and 3 roots of the equation $x^2 - 3x + 2 = 0$?

*Answer:* To check if 1 and 3 are roots, substitute each into the equation:

- For $x = 1$, $1^2 - 3 \cdot 1 + 2 = 1 - 3 + 2 = 0$.
  So 1 is a root.
- For $x = 3$, $3^2 - 3 \cdot 3 + 2 = 9 - 9 + 2 = 2 \neq 0$.
  So 3 is not a root.

A quadratic equation may have no real solution. For example, $x^2 = -1$ has no real solution because the square of a real number cannot be negative.

## B SOLVING BY FACTORIZATION

A primary strategy for solving quadratic equations is to use the **Null Factor Law**. This law allows us to break a single quadratic equation into simpler linear equations. To use it, we must first factorize the quadratic expression.

**Method Solving by Factorization**

The primary strategy for solving quadratic equations is to use the **Null Factor Law**.

1. **Write the equation in standard form**, $ax^2 + bx + c = 0$.
2. **Factorize** the quadratic expression completely.
3. **Apply the Null Factor Law**: set each factor equal to zero.
4. **Solve** each resulting linear equation.

**Proposition Null Factor Law**

If the product of two or more factors is equal to zero, then at least one of the factors must be equal to zero.

$$\text{If } A\,B = 0 \text{ then } A = 0 \textbf{ or } B = 0.$$

Note: one or both of the factors can be zero.

**Ex:** Solve $(x - 1)(x + 2) = 0$.

*Answer:*

$$\begin{aligned}(x-1)(x+2) &= 0 \\ x - 1 = 0 \quad &\text{or} \quad x + 2 = 0 \quad \text{(Null factor law)} \\ x = 1 \quad &\text{or} \quad x = -2 \quad \text{(Solve each equation)}\end{aligned}$$

## C FACTORIZATION TECHNIQUES FOR SPECIAL FORMS OF EQUATIONS

Before learning a general method, we first master solving equations that can be factored using familiar patterns.

**Proposition Common Factor ($c = 0$)**

For equations of the form $ax^2 + bx = 0$, the common factor is $x$:

$$x(ax + b) = 0 \Leftrightarrow x = 0 \text{ or } ax + b = 0$$

**Ex:** Find the roots of $x^2 - 2x = 0$.

*Answer:*

$$\begin{aligned} x^2 - 2x &= 0 \\ x(x-2) &= 0 && \text{(Factor out the common factor } x\text{)} \\ x = 0 \quad \text{or} \quad x - 2 &= 0 && \text{(Null factor law)} \\ x = 0 \quad \text{or} \quad x &= 2 \end{aligned}$$

**Proposition Difference of Squares ($b = 0$)**

For equations of the form $x^2 - k = 0$ (where $k > 0$):

$$x^2 - (\sqrt{k})^2 = 0 \Leftrightarrow (x - \sqrt{k})(x + \sqrt{k}) = 0$$

**Ex:** Solve $x^2 - 9 = 0$.

*Answer:*

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 - 3^2 &= 0 \\ (x-3)(x+3) &= 0 && \text{(Difference of squares)} \\ x - 3 = 0 \quad \text{or} \quad x + 3 &= 0 && \text{(Null factor law)} \\ x = 3 \quad \text{or} \quad x &= -3 \end{aligned}$$

**Proposition Perfect Squares**

For equations of the form $x^2 \pm 2ax + a^2 = 0$:

$$x^2 + 2ax + a^2 = (x + a)^2 = 0 \Leftrightarrow x + a = 0,$$
$$x^2 - 2ax + a^2 = (x - a)^2 = 0 \Leftrightarrow x - a = 0.$$

**Ex:** Solve $x^2 + 2x + 1 = 0$.

*Answer:*

$$\begin{aligned} x^2 + 2x + 1 &= 0 \\ (x+1)^2 &= 0 && \text{(Perfect square factorization)} \\ x + 1 &= 0 && \text{(Null factor law)} \\ x &= -1 \end{aligned}$$

This is a repeated root (double root).

## D THE GENERAL METHOD: COMPLETING THE SQUARE

When a quadratic expression is not one of the special forms, we need a general method. This method is called **completing the square**. The idea is to rewrite the quadratic as a perfect square plus (or minus) a constant. When solving an equation, this eventually leads to a difference of two squares or to taking square roots.

**Proposition Completing the Square**

Any quadratic expression of the form $x^2 + bx + c$ (leading coefficient 1) can be put into vertex form by completing the square:

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c.$$

For a general quadratic $ax^2 + bx + c$ with $a \neq 0$, we first factor out $a$:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

and then complete the square inside the parentheses.

Proof

We start with the expression $x^2+bx+c$. Our goal is to manipulate the first two terms to create a perfect square trinomial of the form $(x+k)^2 = x^2+2kx+k^2$.

$$x^2+bx+c = (x^2+bx)+c \quad \text{(Group the } x\text{-terms)}$$

Comparing $x^2+bx$ to $x^2+2kx$, we see that we need $b=2k$, so $k=\frac{b}{2}$. The term needed to complete the square is $k^2 = \left(\frac{b}{2}\right)^2$. We add and subtract this term inside the parentheses.

$$\begin{aligned}
x^2+bx+c &= \left(x^2+bx+\left(\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2\right)+c && \text{(Add and subtract the "magic" term)}\\
&= \left(x^2+bx+\left(\frac{b}{2}\right)^2\right)-\left(\frac{b}{2}\right)^2+c && \text{(Group the perfect square trinomial)}\\
&= \left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2+c && \text{(Factor the trinomial)}
\end{aligned}$$

This completes the proof.

**Ex:** Complete the square for $x^2+10x+24$.

*Answer:* We know that $(x+5)^2 = x^2+10x+25$. So

$$\begin{aligned}
x^2+10x+24 &= x^2+10x+25-25+24\\
&= (x+5)^2-1 && \text{(Complete the square).}
\end{aligned}$$

Method **General method to solve quadratic equations**

Given an equation $ax^2+bx+c=0$ with $a\neq 0$:

- Step 1: **Complete the square** to rewrite the left-hand side as a perfect square plus/minus a constant.
- Step 2: **Use the difference of squares** (or take square roots) to isolate $x$.
- Step 3: **Apply the Null Factor Law** if the expression has been written as a product.
- Step 4: **Solve the linear equations** obtained.

**Ex:** Solve $x^2+10x+24=0$.

*Answer:* We know that $(x+5)^2 = x^2+10x+25$. So

$$\begin{aligned}
x^2+10x+24 &= 0\\
x^2+10x+25-25+24 &= 0\\
(x+5)^2-1 &= 0 && \text{(Complete the square)}\\
(x+5)^2-1^2 &= 0 && \text{(Difference of squares)}\\
(x+5-1)(x+5+1) &= 0 && \text{(Factorize)}\\
(x+4)(x+6) &= 0 && \text{(Simplify)}\\
x+4=0 \text{ or } x+6 &= 0 && \text{(Null factor law)}\\
x=-4 \text{ or } x &= -6 && \text{(Solve).}
\end{aligned}$$

## E QUADRATIC FORMULA

Applying the method of completing the square to the general equation $ax^2+bx+c=0$ gives a formula that solves any quadratic equation. This is the **quadratic formula**.

Proposition **Quadratic formula**

For any quadratic equation $ax^2+bx+c=0$, the discriminant, denoted $\Delta$, is defined as

$$\Delta = b^2-4ac.$$

Its sign determines the number of real solutions:

- If $\Delta > 0$, there are two real roots:
$$x = \frac{-b-\sqrt{\Delta}}{2a} \text{ or } x = \frac{-b+\sqrt{\Delta}}{2a}$$
- If $\Delta = 0$, there is one real root (a double root):
$$x = \frac{-b}{2a}.$$
- If $\Delta < 0$, there are no real roots.

Proof

Suppose $ax^2+bx+c=0$, where $a \neq 0$.

$$\begin{aligned}
ax^2+bx+c &= 0 \\
x^2+\frac{b}{a}x+\frac{c}{a} &= 0 && \text{(Divide each term by } a\text{, since } a \neq 0) \\
x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a} &= 0 && \text{(Complete the square)} \\
\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2} &= 0 && \text{(Simplify)} \\
\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2} &= 0 && \text{(Where } \Delta = b^2-4ac\text{).}
\end{aligned}$$

Now, consider the cases based on the discriminant $\Delta$:

- **Case** $\Delta \geq 0$: Since $\dfrac{\Delta}{4a^2} \geq 0$, a real square root exists.
$$\begin{aligned}
\left(x+\frac{b}{2a}\right)^2 &= \frac{\Delta}{4a^2} \\
\left(x+\frac{b}{2a}\right)^2-\left(\sqrt{\frac{\Delta}{4a^2}}\right)^2 &= 0 \\
\left(x+\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}}\right)\left(x+\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}}\right) &= 0 && \text{(Difference of squares).}
\end{aligned}$$
Applying the Null Factor Law:
$$x+\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}} = 0 \quad \text{or} \quad x+\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}} = 0.$$
Solving these linear equations:
$$x = -\frac{b}{2a}+\sqrt{\frac{\Delta}{4a^2}} \quad \text{or} \quad x = -\frac{b}{2a}-\sqrt{\frac{\Delta}{4a^2}}.$$
Simplifying:
$$x = \frac{-b \pm \sqrt{\Delta}}{2a}.$$
  - If $\Delta > 0$, there are two distinct real roots.
  - If $\Delta = 0$, there is one real root (double root): $x = -\dfrac{b}{2a}$.
- **Case** $\Delta < 0$: Then $\dfrac{\Delta}{4a^2} < 0$, so
$$\left(x+\frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} < 0.$$
Since the square of a real number is non-negative, there are no real solutions.

**Ex:** Consider the quadratic equation $x^2+2x-3=0$.

1. Find the discriminant.

2. Hence, state the nature of the roots of the equation.
3. Solve the equation.

*Answer:* $x^2 + 2x - 3 = 0$ has $a = 1$, $b = 2$, $c = -3$.

1. $\Delta = b^2 - 4ac$
   $= (2)^2 - 4(1)(-3)$
   $= 4 + 12$
   $= 16$
2. As $\Delta > 0$, there are two distinct real solutions.
3. $$x = \frac{-b - \sqrt{\Delta}}{2a} \quad \text{or} \quad x = \frac{-b + \sqrt{\Delta}}{2a}$$
   $$x = \frac{-2 - \sqrt{16}}{2 \cdot 1} \quad \text{or} \quad x = \frac{-2 + \sqrt{16}}{2 \cdot 1}$$
   $$x = \frac{-2 - 4}{2} \quad \text{or} \quad x = \frac{-2 + 4}{2}$$
   $$x = -3 \quad \text{or} \quad x = 1$$