SOLVING EQUATIONS

A WHAT IS AN EQUATION?

Definition Equation and Solution

An **equation** is a mathematical statement that says two expressions are equal. It often contains a **variable** (or **unknown**), which is a symbol (like x or $\triangle$) representing a number we do not yet know.
Solving an equation means finding all value(s) of the variable that make the equation a true statement. Each of these values is called a **solution** of the equation.

Ex: Show that $x = 2$ is a solution to the equation $3 + x = 5$.

Answer: We substitute $x = 2$ into the equation and check if the left-hand side equals the right-hand side:

$$\begin{aligned} 3 + (2) &= 5 \\ 5 &= 5 \qquad \text{(This is a true statement.)} \end{aligned}$$

Since the statement is true, $x = 2$ is a solution.

Ex: Show that $x = 1$ is **not** a solution to $3 + x = 5$.

Answer: We substitute $x = 1$ into the equation:

$$\begin{aligned} 3 + (1) &= 5 \\ 4 &= 5 \quad \text{(This is a false statement.)} \end{aligned}$$

Since the statement is false, $x = 1$ is not a solution.

B SOLVING BY INSPECTION AND TRIAL-AND-ERROR

Method Trial and Error

Trial and error is a basic problem-solving method where we test different values for the variable until we find one that makes the equation true. For each value, we substitute it into the equation and check whether the left-hand side equals the right-hand side.
For simple equations, we can sometimes see the solution just by looking at the equation. This is called **solving by inspection**.

Ex: Consider the equation $2x + 3 = 11$.
Use the trial and error method to find the solution.

Answer: We test different integer values for x to see which one makes the equation true.

- Try $x = 2$:

$$\begin{aligned} 2 \times (2) + 3 &= 11 \qquad \text{(Substitute)} \\ 4 + 3 &= 11 \\ 7 &= 11 \qquad \text{(False)} \end{aligned}$$

- Try $x = 3$:

$$\begin{aligned} 2 \times (3) + 3 &= 11 \qquad \text{(Substitute)} \\ 6 + 3 &= 11 \\ 9 &= 11 \qquad \text{(False)} \end{aligned}$$

- Try $x = 4$:

$$\begin{aligned} 2 \times (4) + 3 &= 11 \qquad \text{(Substitute)} \\ 8 + 3 &= 11 \\ 11 &= 11 \qquad \text{(True)} \end{aligned}$$

Therefore, a solution to the equation $2x + 3 = 11$ is $x = 4$.

C THE PRINCIPLE OF BALANCE

Definition Equivalent Equations

Two equations are **equivalent** if they have exactly the same solution(s). To solve an equation, we transform it into a series of simpler, equivalent equations until the solution is evident.

Proposition Golden Rules of Solving Equations

To create an equivalent equation, what you do to one side of the equation, you must do to the other side.

- **Addition/Subtraction Property**: Adding or subtracting the same number from both sides produces an equivalent equation.
$$A = B \iff A + c = B + c$$
(and similarly $A - c = B - c$)
- **Multiplication/Division Property**: Multiplying or dividing both sides by the same non-zero number produces an equivalent equation.
$$A = B \iff cA = cB \quad (c \neq 0)$$
$$A = B \iff \frac{A}{c} = \frac{B}{c} \quad (c \neq 0)$$

D SOLVING BY REVERSING OPERATIONS

An algebraic expression like $2x + 1$ can be seen as a sequence of operations applied to a variable x. To solve an equation involving this expression, our goal is to *reverse* this sequence to get back to x.

Definition Inverse Operations

An **inverse operation** is an operation that "undoes" another operation.

- Addition and subtraction are inverse operations. $\boxed{x} \xrightarrow{+a} \boxed{x+a} \xrightarrow{-a} \boxed{x}$
- Multiplication and division are inverse operations. $\boxed{x} \xrightarrow{\times a} \boxed{ax} \xrightarrow{\div a} \boxed{x}$ with $a \neq 0$

Method Solving by Reversing

In many linear equations, we can isolate the variable by applying **inverse operations in the reverse order**.

1. Identify the sequence of operations used to build the expression containing the variable.
2. Reverse this sequence using the corresponding inverse operations, applying each step to both sides of the equation.

Ex: Solve for x:
$$2x + 1 = 7$$

Answer:

- **Sequence of operations on** x: First, x is multiplied by 2, then 1 is added.
$$\boxed{x} \xrightarrow{\times 2} \boxed{2x} \xrightarrow{+1} \boxed{2x+1}$$
- **Reverse sequence with inverse operations**: To isolate x, we must first subtract 1, then divide by 2.
$$\boxed{2x+1} \xrightarrow{-1} \boxed{2x} \xrightarrow{\div 2} \boxed{x}$$
- **Applying the steps**:
$$\begin{aligned} 2x + 1 &= 7 \\ 2x + 1 - 1 &= 7 - 1 && \text{(Subtract 1 from both sides)} \\ 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} && \text{(Divide both sides by 2)} \\ x &= 3 \end{aligned}$$

  The solution is $x = 3$.

E SOLVING PRODUCT OF LINEAR FACTORS

In the previous chapter, we learned how to factorize expressions. We will now see how this skill is essential for solving non-linear equations, especially when an equation can be written as a product of linear factors equal to zero. The power of factorization comes from a simple but fundamental property of the number zero.

Proposition Null Factor Law

If the product of two or more factors is equal to zero, then at least one of the factors must be equal to zero.

$$\text{If } A\,B = 0 \text{ then } A = 0 \textbf{ or } B = 0.$$

Note: one or both of the factors can be zero.

Method Solving by Factorization

The Null Factor Law gives us a powerful strategy for solving equations of the form "product of factors $=0$":

1. Rearrange the equation so that all the terms are on one side and zero is on the other side.
2. Factorize completely the side that contains the variable(s).
3. Set each factor equal to zero and solve the resulting linear equations.

Ex: Solve for x: $x(x+1)=0$

Answer: The expression is already factored and equal to zero, so we can apply the Null Factor Law directly by setting each factor equal to zero:

$$\begin{aligned} x\,(x+1) &= 0 \\ x = 0 \text{ or } (x+1) &= 0 \quad \text{(Null Factor Law)} \\ x = 0 \text{ or } x &= -1\,. \end{aligned}$$

The equation has two solutions: $x=0$ and $x=-1$.

F SOLVING BASIC QUADRATIC EQUATIONS

Proposition Solutions of $x^2 = k$

Consider the equation $x^2 = k$, where k is a real number. Looking for *real* solutions, we obtain:

$$\begin{cases} x = \pm\sqrt{k} & \text{if } k > 0 \quad \text{(two real solutions)} \\ x = 0 & \text{if } k = 0 \quad \text{(one real solution)} \\ \text{No real solutions} & \text{if } k < 0 \end{cases}$$

In other words, solving $x^2 = k$ means taking the square root of both sides and remembering that there are two opposite roots when $k>0$.

Ex: Solve for x: $x^2 = 3$

Answer: Here $k = 3 > 0$, so there are two real solutions:

$$x = \sqrt{3} \quad \text{or} \quad x = -\sqrt{3}.$$

This can be written more concisely as $x = \pm\sqrt{3}$.