SEQUENCES

A NUMERICAL SEQUENCE

In mathematics, a sequence is more than just a pattern; it's an ordered list of numbers where each number has a specific position (its place in the list). To work with sequences effectively, we use a special notation to distinguish a term's position from its value.

Definition Numerical Sequence

A **numerical sequence** is an ordered list of numbers, called **terms**. We use the notation u_n to describe a term in the sequence.

- The subscript n is the **index**, which tells us the **position** of the term (often starting from $n = 0$). The index n usually takes integer values: $0, 1, 2, \ldots$
- u_n represents the **value** of the term at that specific position.

So, u_0 is the value of the term at position 0, u_1 is the value at position 1, and so on.

Index (n)	0	1	2	...
Term (u_n)	u_0	u_1	u_2	...

Ex: Given the sequence defined by the table below, find the value of the term u_4.

n	0	1	2	3	4	5	...
u_n	3	5	7	9	11	13	...

Answer: To find u_4, we look in the table for the column where the index is $n = 4$. The value in the row below it is 11. Therefore, $u_4 = 11$.

B RECURSIVE DEFINITION

Definition Recursive Definition

A **recursive definition** of a sequence includes two parts:

- The **first term** (or initial term), denoted for example by u_0 or u_1. This is the starting point.
- The **recursive rule** (or recurrence relation), which is a formula that connects the next term, u_{n+1}, to the current term, u_n, for all appropriate integers n (such as $n \geq 0$ or $n \geq 1$).

Once these two parts are known, every term in the sequence can be calculated step by step.

Ex: A sequence is defined recursively by:

- $u_1 = 5$ (The first term is 5).
- $u_{n+1} = u_n + 3$ (The rule is to add 3 to the previous term, for all integers $n \geq 1$).

Find the first five terms of this sequence.

Answer: Let's build the sequence step-by-step using the recursive definition.

- **1st term**: The starting term is given: $\boldsymbol{u_1 = 5}$.
- **2nd term**: Use the rule with $n = 1$. $\boldsymbol{u_2} = u_1 + 3 = 5 + 3 = \mathbf{8}$.
- **3rd term**: Use the rule with $n = 2$. $\boldsymbol{u_3} = u_2 + 3 = 8 + 3 = \mathbf{11}$.
- **4th term**: Use the rule with $n = 3$. $\boldsymbol{u_4} = u_3 + 3 = 11 + 3 = \mathbf{14}$.
- **5th term**: Use the rule with $n = 4$. $\boldsymbol{u_5} = u_4 + 3 = 14 + 3 = \mathbf{17}$.

$$5 \xrightarrow{+3} 8 \xrightarrow{+3} 11 \xrightarrow{+3} 14 \xrightarrow{+3} 17$$

The first five terms are: $5, 8, 11, 14, 17$.

C EXPLICIT DEFINITION

While a recursive rule tells you how to get from one term to the next, it is not very efficient if you want to find a term far into the sequence (like the 100^{th} term), because you would have to calculate all the terms before it.
An alternative and often more powerful way to define a sequence is with an **explicit rule**. An explicit rule is a formula that gives the value of u_n directly when you know the position n in the sequence. You choose n, and the rule gives you u_n immediately.

Definition Explicit Rule

An **explicit rule** (or **explicit formula**) defines the nth term of a sequence, u_n, directly as a function of its position n.

$$u_n = f(n)$$

Note The key advantage is that it allows us to calculate any term in the sequence directly, without first having to find all the previous terms one by one.

Ex: Consider the sequence defined by the explicit formula:

$$u_n = 3n + 2$$

Calculate u_{100}.

Answer: To find the value of u_{100}, we substitute $n = 100$ into the explicit formula:

$$\begin{aligned} u_{100} &= 3(100) + 2 \\ &= 300 + 2 \\ &= \mathbf{302} \end{aligned}$$

We did not need to know any of the previous terms in the sequence.

D ARITHMETIC SEQUENCE

An arithmetic sequence is the most common type of sequence that follows a recursive rule. It is defined by a starting term and a constant change between consecutive terms.

Definition Arithmetic Sequence

An **arithmetic sequence** is a sequence where each term after the first is found by adding a constant value to the previous term. This constant is called the **common difference**, denoted by d.

- **Recursive Definition:** The rule for finding the next term from the previous one is:

$$u_{n+1} = u_n + d$$

- **Explicit Formula:** We can also find any term directly using its position, n.
  - If the sequence starts with u_1:

$$u_n = u_1 + (n-1)d$$

  - If the sequence starts with u_0:

$$u_n = u_0 + nd$$

Ex: An arithmetic sequence has a first term $u_1 = 5$ and a common difference $d = 3$.
Find the first five terms of this sequence.

Answer: The recursive rule is $u_{n+1} = u_n + 3$. We start with $u_1 = 5$ and apply the rule repeatedly.

- $u_1 = \mathbf{5}$ (given)
- $u_2 = u_1 + 3 = 5 + 3 = \mathbf{8}$
- $u_3 = u_2 + 3 = 8 + 3 = \mathbf{11}$
- $u_4 = u_3 + 3 = 11 + 3 = \mathbf{14}$
- $u_5 = u_4 + 3 = 14 + 3 = \mathbf{17}$

The first five terms are: $5, 8, 11, 14, 17$.

E GEOMETRIC SEQUENCE

A geometric sequence is another fundamental type of sequence that follows a recursive rule. It is defined by a starting term and a constant multiplicative factor: to get from one term to the next, you always multiply by the same number.

Definition Geometric Sequence

A **geometric sequence** is a sequence where each term after the first is found by multiplying the previous term by a constant non-zero value. This constant is called the **common ratio**, denoted by r.

- **Recursive Definition:** The rule for finding the next term from the previous one is:

$$u_{n+1} = r \times u_n$$

- **Explicit Formula:** We can also find any term directly using its position, n.
  - If the sequence starts with u_1:

$$u_n = u_1 \times r^{n-1}$$

  - If the sequence starts with u_0:

$$u_n = u_0 \times r^n$$

Ex: A geometric sequence has a first term $u_1 = 2$ and a common ratio $r = 2$.
Find the first five terms of this sequence.

Answer: The recursive rule is $u_{n+1} = u_n \times 2$. We start with $u_1 = 2$ and apply the rule repeatedly.

- $u_1 = \mathbf{2}$ (given)
- $u_2 = 2 \times u_1 = 2 \times 2 = \mathbf{4}$
- $u_3 = 2 \times u_2 = 2 \times 4 = \mathbf{8}$
- $u_4 = 2 \times u_3 = 2 \times 8 = \mathbf{16}$
- $u_5 = 2 \times u_4 = 2 \times 16 = \mathbf{32}$

The first five terms are: $2, 4, 8, 16, 32$.

F SERIES

While a sequence is a list of numbers, a **series** is what you get when you add those numbers together. Every sequence has a corresponding series. We are often interested in the **partial sum** of a sequence, which is the sum of a specific number of its terms, starting from the beginning.

Definition Series and Partial Sum

A **series** is the sum of the terms in a sequence. The **partial sum**, denoted $\boldsymbol{S_n}$, is the sum of the terms of a sequence up to a specific index n.

- If a sequence starts at u_0, the partial sum S_n is the sum of the first $(n+1)$ terms:

$$S_n = u_0 + u_1 + u_2 + \ldots + u_n = \sum_{i=0}^{n} u_i$$

- If a sequence starts at u_1, the partial sum S_n is the sum of the first n terms:

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{i=1}^{n} u_i$$

G SUM OF AN ARITHMETIC SEQUENCE

Proposition Sum of an Arithmetic Sequence

The sum of the first terms of an arithmetic sequence is given by the formula:

$$S_n = \text{Number of terms} \times \frac{(\text{First term} + \text{Last term})}{2}$$

- If a sequence starts at u_0 and ends at u_n, there are $\boldsymbol{n+1}$ terms. The formula is:

$$S_n = \frac{n+1}{2}(u_0 + u_n)$$

- If a sequence starts at u_1 and ends at u_n, there are $\boldsymbol{n}$ terms. The formula is:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

H SUM OF A GEOMETRIC SEQUENCE

Proposition Sum of a Geometric Sequence

The sum of the first terms of a geometric sequence is given by the formula:

$$S_n = \text{First term} \times \frac{1 - (\text{common ratio})^{\text{Number of terms}}}{1 - \text{common ratio}}$$

- If a sequence starts at u_0 and has a common ratio r, the sum of the first $\boldsymbol{n+1}$ terms is:

$$S_n = u_0 \left(\frac{1 - r^{n+1}}{1 - r} \right)$$

- If a sequence starts at u_1 and has a common ratio r, the sum of the first $\boldsymbol{n}$ terms is:

$$S_n = u_1 \left(\frac{1 - r^{n}}{1 - r} \right)$$

These formulas are valid for any common ratio $r \neq 1$.

I SUM OF AN INFINITE GEOMETRIC SERIES

We have seen how to calculate the sum of a finite number of terms. But what happens if we continue adding the terms of a geometric sequence forever? This concept, known as an infinite series, does not always yield a finite sum. However, under a specific condition, the sum can **converge** to a single, finite value.

Proposition Convergence of a Geometric Series

An infinite geometric series converges to a finite sum if and only if its common ratio r is between -1 and 1, i.e., $\boldsymbol{|r| < 1}$.
If the series converges, its sum to infinity, denoted S_∞, is given by the formula:

$$\boldsymbol{S_\infty = \frac{u_1}{1 - r}}$$

where u_1 is the first term and $|r| < 1$.If the sequence starts at u_0, the formula is:

$$\boldsymbol{S_\infty = \frac{u_0}{1 - r}}$$