SEQUENCES

A NUMERICAL SEQUENCE

In mathematics, a sequence is more than just a pattern; it's an ordered list of numbers where each number has a specific position (its place in the list). To work with sequences effectively, we use a special notation to distinguish a term's position from its value.

Definition Numerical Sequence

A **numerical sequence** is an ordered list of numbers, called **terms**. We use the notation u_n to describe a term in the sequence.

- The subscript n is the **index**, which tells us the **position** of the term (often starting from $n = 0$). The index n usually takes integer values: $0, 1, 2, \ldots$
- u_n represents the **value** of the term at that specific position.

So, u_0 is the value of the term at position 0, u_1 is the value at position 1, and so on.

Index (n)	0	1	2	$\ldots$
Term (u_n)	u_0	u_1	u_2	$\ldots$

Ex: Given the sequence defined by the table below, find the value of the term u_4.

n	0	1	2	3	4	5	$\ldots$
u_n	3	5	7	9	11	13	$\ldots$

Answer: To find u_4, we look in the table for the column where the index is $n = 4$. The value in the row below it is 11. Therefore, $u_4 = 11$.

B RECURSIVE DEFINITION

Discover: Let's consider a sequence where the first term is 2, and each term is obtained by adding 3 to the previous term. The terms are:

$$(\overset{+3}{2 \curvearrowright} \; 5 \overset{+3}{\curvearrowright} \; 8 \overset{+3}{\curvearrowright} \; 11, \ldots)$$

$$u_0 = 2, \quad u_1 = 5, \quad u_2 = 8, \quad u_3 = 11$$

Here the sequence is indexed starting from $n = 0$: $u_0 = 2$, $u_1 = 5$, $u_2 = 8$, $u_3 = 11$, etc.We can describe the relationship between the terms using formal notation:

- $5 = 2+3 \longrightarrow u_1 = u_0+3 \longrightarrow u_{0+1} = u_0 + 3$
- $8 = 5+3 \longrightarrow u_2 = u_1+3 \longrightarrow u_{1+1} = u_1 + 3$
- $11 = 8+3 \longrightarrow u_3 = u_2+3 \longrightarrow u_{2+1} = u_2 + 3$

This pattern shows that any term u_{n+1} can be found by adding 3 to the previous term u_n. We can generalize this relationship as a rule (valid for all integers $n \geq 0$):

$$u_{n+1} = u_n + 3$$

$$(u_0 \overset{+3}{\curvearrowright} u_1 \overset{+3}{\curvearrowright} u_2 \overset{+3}{\curvearrowright} u_3, \ldots, u_n \overset{+3}{\curvearrowright} u_{n+1}, \ldots)$$

Definition Recursive Definition

A **recursive definition** of a sequence includes two parts:

- The **first term** (or initial term), denoted for example by u_0 or u_1. This is the starting point.
- The **recursive rule** (or recurrence relation), which is a formula that connects the next term, u_{n+1}, to the current term, u_n, for all appropriate integers n (such as $n \geq 0$ or $n \geq 1$).

Once these two parts are known, every term in the sequence can be calculated step by step.

Ex: A sequence is defined recursively by:

- $u_1 = 5$ (The first term is 5).
- $u_{n+1} = u_n + 3$ (The rule is to add 3 to the previous term, for all integers $n \geq 1$).

Find the first five terms of this sequence.

Answer: Let's build the sequence step-by-step using the recursive definition.

- **1st term**: The starting term is given: $\boldsymbol{u_1 = 5}$.
- **2nd term**: Use the rule with $n = 1$. $\boldsymbol{u_2} = u_1 + 3 = 5 + 3 = \mathbf{8}$.
- **3rd term**: Use the rule with $n = 2$. $\boldsymbol{u_3} = u_2 + 3 = 8 + 3 = \mathbf{11}$.
- **4th term**: Use the rule with $n = 3$. $\boldsymbol{u_4} = u_3 + 3 = 11 + 3 = \mathbf{14}$.
- **5th term**: Use the rule with $n = 4$. $\boldsymbol{u_5} = u_4 + 3 = 14 + 3 = \mathbf{17}$.

$$5 \xrightarrow{+3} 8 \xrightarrow{+3} 11 \xrightarrow{+3} 14 \xrightarrow{+3} 17$$

The first five terms are: $5, 8, 11, 14, 17$.

C EXPLICIT DEFINITION

While a recursive rule tells you how to get from one term to the next, it is not very efficient if you want to find a term far into the sequence (like the 100^{th} term), because you would have to calculate all the terms before it.
An alternative and often more powerful way to define a sequence is with an **explicit rule**. An explicit rule is a formula that gives the value of u_n directly when you know the position n in the sequence. You choose n, and the rule gives you u_n immediately.

Definition Explicit Rule

An **explicit rule** (or **explicit formula**) defines the nth term of a sequence, u_n, directly as a function of its position n.

$$u_n = f(n)$$

Note The key advantage is that it allows us to calculate any term in the sequence directly, without first having to find all the previous terms one by one.

Ex: Consider the sequence defined by the explicit formula:

$$u_n = 3n + 2$$

Calculate u_{100}.

Answer: To find the value of u_{100}, we substitute $n = 100$ into the explicit formula:

$$\begin{aligned} u_{100} &= 3(100) + 2 \\ &= 300 + 2 \\ &= \mathbf{302} \end{aligned}$$

We did not need to know any of the previous terms in the sequence.

D ARITHMETIC SEQUENCE

An arithmetic sequence is the most common type of sequence that follows a recursive rule. It is defined by a starting term and a constant change between consecutive terms.

Discover: Let's consider an arithmetic sequence with a first term $u_1 = 5$ and a common difference $d = 3$.
We can write out the first few terms by repeatedly adding 3:

- $u_1 = 5$
- $u_2 = 5 + 3$
- $u_3 = 5 + 3 + 3$
- $u_4 = 5 + 3 + 3 + 3$

Let's rewrite this using multiplication to see the pattern:

- $u_1 = 5 + 0 \times 3$
- $u_2 = 5 + 1 \times 3$
- $u_3 = 5 + 2 \times 3$
- $u_4 = 5 + 3 \times 3$

The pattern is: to find the nth term (u_n), we start with u_1 and add the common difference d exactly $(n-1)$ times. This gives us the **explicit formula**:

$$u_n = u_1 + (n-1)d$$

Now we can find u_{10} directly:

$$u_{10} = u_1 + (10-1)d = 5 + (9 \times 3) = 5 + 27 = 32$$

This is much faster than calculating each term one by one!

Definition Arithmetic Sequence

An **arithmetic sequence** is a sequence where each term after the first is found by adding a constant value to the previous term. This constant is called the **common difference**, denoted by d.

- **Recursive Definition:** The rule for finding the next term from the previous one is:

$$u_{n+1} = u_n + d$$

- **Explicit Formula:** We can also find any term directly using its position, n.
  - If the sequence starts with u_1:

$$u_n = u_1 + (n-1)d$$

  - If the sequence starts with u_0:

$$u_n = u_0 + nd$$

Ex: An arithmetic sequence has a first term $u_1 = 5$ and a common difference $d = 3$.
Find the first five terms of this sequence.

Answer: The recursive rule is $u_{n+1} = u_n + 3$. We start with $u_1 = 5$ and apply the rule repeatedly.

- $u_1 = \mathbf{5}$ (given)
- $u_2 = u_1 + 3 = 5 + 3 = \mathbf{8}$
- $u_3 = u_2 + 3 = 8 + 3 = \mathbf{11}$
- $u_4 = u_3 + 3 = 11 + 3 = \mathbf{14}$
- $u_5 = u_4 + 3 = 14 + 3 = \mathbf{17}$

The first five terms are: $5, 8, 11, 14, 17$.

E GEOMETRIC SEQUENCE

A geometric sequence is another fundamental type of sequence that follows a recursive rule. It is defined by a starting term and a constant multiplicative factor: to get from one term to the next, you always multiply by the same number.

Discover: Let's consider a geometric sequence with a first term $u_1 = 2$ and a common ratio $r = 3$.
We can write out the first few terms by repeatedly multiplying by 3:

- $u_1 = 2$
- $u_2 = 2 \times 3$
- $u_3 = 2 \times 3 \times 3$
- $u_4 = 2 \times 3 \times 3 \times 3$

Let's rewrite this using exponents to see the pattern:

- $u_1 = 2 \times 3^0$

- $u_2 = 2 \times 3^1$
- $u_3 = 2 \times 3^2$
- $u_4 = 2 \times 3^3$

The pattern is: to find the nth term (u_n), we start with u_1 and multiply by the common ratio r exactly $(n-1)$ times. This gives us the **explicit formula**:

$$u_n = u_1 \times r^{n-1}$$

Now we can find u_{10} directly:

$$u_{10} = u_1 \times r^{10-1} = 2 \times 3^9 = 2 \times 19683 = 39366$$

This is much faster than calculating each term one by one!

Definition Geometric Sequence

A **geometric sequence** is a sequence where each term after the first is found by multiplying the previous term by a constant non-zero value. This constant is called the **common ratio**, denoted by r.

- **Recursive Definition:** The rule for finding the next term from the previous one is:

$$u_{n+1} = r \times u_n$$

- **Explicit Formula:** We can also find any term directly using its position, n.
  - If the sequence starts with u_1:

$$u_n = u_1 \times r^{n-1}$$

  - If the sequence starts with u_0:

$$u_n = u_0 \times r^n$$

Ex: A geometric sequence has a first term $u_1 = 2$ and a common ratio $r = 2$.
Find the first five terms of this sequence.

Answer: The recursive rule is $u_{n+1} = u_n \times 2$. We start with $u_1 = 2$ and apply the rule repeatedly.

- $u_1 = \mathbf{2}$ (given)
- $u_2 = 2 \times u_1 = 2 \times 2 = \mathbf{4}$
- $u_3 = 2 \times u_2 = 2 \times 4 = \mathbf{8}$
- $u_4 = 2 \times u_3 = 2 \times 8 = \mathbf{16}$
- $u_5 = 2 \times u_4 = 2 \times 16 = \mathbf{32}$

The first five terms are: $2, 4, 8, 16, 32$.

F SERIES

While a sequence is a list of numbers, a **series** is what you get when you add those numbers together. Every sequence has a corresponding series. We are often interested in the **partial sum** of a sequence, which is the sum of a specific number of its terms, starting from the beginning.

Definition Series and Partial Sum

A **series** is the sum of the terms in a sequence. The **partial sum**, denoted $\boldsymbol{S_n}$, is the sum of the terms of a sequence up to a specific index n.

- If a sequence starts at u_0, the partial sum S_n is the sum of the first $(n+1)$ terms:

$$S_n = u_0 + u_1 + u_2 + \ldots + u_n = \sum_{i=0}^{n} u_i$$

- If a sequence starts at u_1, the partial sum S_n is the sum of the first n terms:

$$S_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{i=1}^{n} u_i$$

G SUM OF AN ARITHMETIC SEQUENCE

Discover: How can we efficiently calculate the sum of the first 20 terms of an arithmetic sequence starting with $u_0 = 5$ and a common difference of $d = 5$?

$$S_{19} = \overbrace{5 + 10 + 15 + \ldots + 90 + 95 + 100}^{\text{20 terms}}$$

The trick, famously discovered by Gauss as a child, is to write the sum twice, once in forward order and once in reverse order, and then add the two lines together term by term.

$$\begin{array}{cccccccccc} S_{19} & = & 5 & + & 10 & + & \ldots & + & 100 \\ S_{19} & = & 100 & + & 95 & + & \ldots & + & 5 \\ \hline 2S_{19} & = & 105 & + & 105 & + & \ldots & + & 105 \end{array}$$

Each pair of terms adds up to the same value ($5 + 100 = 105$, $10 + 95 = 105$, etc.). There are 20 identical sums in $2S_{19}$.

$$2S_{19} = 20 \times (5 + 100)$$

Therefore, the sum is:

$$S_{19} = 20 \times \frac{5 + 100}{2} = 1050$$

This method reveals the general formula:

$$S_n = \frac{\text{Number of terms}}{2} \times (\text{First term} + \text{Last term})$$

Proposition Sum of an Arithmetic Sequence

The sum of the first terms of an arithmetic sequence is given by the formula:

$$S_n = \text{Number of terms} \times \frac{(\text{First term} + \text{Last term})}{2}$$

- If a sequence starts at u_0 and ends at u_n, there are $\boldsymbol{n+1}$ terms. The formula is:

$$S_n = \frac{n+1}{2}(u_0 + u_n)$$

- If a sequence starts at u_1 and ends at u_n, there are $\boldsymbol{n}$ terms. The formula is:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

H SUM OF A GEOMETRIC SEQUENCE

Discover: How can we efficiently calculate the sum of the first 10 terms of a geometric sequence starting with $u_0 = 3$ and a common ratio of $r = 2$?

$$S_9 = 3 + 6 + 12 + 24 + \ldots + 768 + 1536$$

The trick is to multiply the entire sum by the common ratio, $r = 2$, and then subtract the original sum from this new one. Let's write the original sum:

$$S_9 = 3 + 6 + 12 + \ldots + 768 + 1536$$

Now, let's multiply every term by the common ratio, 2:

$$2S_9 = 6 + 12 + 24 + \ldots + 1536 + 3072$$

Notice that most of the terms in S_9 and $2S_9$ are identical. If we subtract the first equation from the second, these terms will cancel out:

$$\begin{array}{ccccccccccc} 2S_9 & = & & & 6 & + & 12 & +\ldots+ & 1536 & + & 3072 \\ -S_9 & = & -(3 & + & 6 & + & 12 & +\ldots+ & 1536) & & \\ \hline (2-1)S_9 & = & -3 & + & 0 & + & 0 & +\ldots+ & 0 & + & 3072 \end{array}$$

This method reveals the general formula for the sum of any geometric sequence.

$$S_n - rS_n = u_0 - u_0 \times r^{n+1}$$
$$S_n(1 - r) = u_0(1 - r^{n+1})$$
$$S_n = u_0 \left(\frac{1 - r^{n+1}}{1 - r} \right)$$

Proposition Sum of a Geometric Sequence

The sum of the first terms of a geometric sequence is given by the formula:

$$S_n = \text{First term} \times \frac{1 - (\text{common ratio})^{\text{Number of terms}}}{1 - \text{common ratio}}$$

- If a sequence starts at u_0 and has a common ratio r, the sum of the first $\boldsymbol{n+1}$ terms is:

$$S_n = u_0 \left(\frac{1 - r^{n+1}}{1 - r} \right)$$

- If a sequence starts at u_1 and has a common ratio r, the sum of the first $\boldsymbol{n}$ terms is:

$$S_n = u_1 \left(\frac{1 - r^n}{1 - r} \right)$$

These formulas are valid for any common ratio $r \neq 1$.

I SUM OF AN INFINITE GEOMETRIC SERIES

We have seen how to calculate the sum of a finite number of terms. But what happens if we continue adding the terms of a geometric sequence forever? This concept, known as an infinite series, does not always yield a finite sum. However, under a specific condition, the sum can **converge** to a single, finite value.

Proposition Convergence of a Geometric Series

An infinite geometric series converges to a finite sum if and only if its common ratio r is between -1 and 1, i.e., $\boldsymbol{|r| < 1}$.
If the series converges, its sum to infinity, denoted S_∞, is given by the formula:

$$\boldsymbol{S_\infty = \frac{u_1}{1 - r}}$$

where u_1 is the first term and $|r| < 1$.If the sequence starts at u_0, the formula is:

$$\boldsymbol{S_\infty = \frac{u_0}{1 - r}}$$

Proof

The sum of an infinite geometric series is defined as the limit of its partial sum, S_n, as n approaches infinity. The formula for the partial sum is:

$$S_n = u_1 \left(\frac{1 - r^n}{1 - r} \right)$$

To find the sum to infinity, we evaluate the limit:

$$S_\infty = \lim_{n \to \infty} u_1 \left(\frac{1 - r^n}{1 - r} \right)$$

The value of this limit depends entirely on the behavior of the term $\boldsymbol{r^n}$ as $n \to \infty$.

- **Convergent Case:**
If $\boldsymbol{|r| < 1}$, then the term r^n approaches 0 as n becomes infinitely large. That is, $\lim_{n\to\infty} r^n = 0$.Substituting this result into the limit expression gives:

$$S_\infty = u_1 \left(\frac{1 - 0}{1 - r} \right) = \frac{u_1}{1 - r}$$

- **Divergent Case:**
If $\boldsymbol{|r| \geq 1}$, the term r^n does not approach a finite value as $n \to \infty$ (it either grows infinitely or oscillates). Consequently, the partial sum S_n does not converge to a finite limit, and the series is said to **diverge**.

Therefore, a finite sum exists if and only if $|r| < 1$.