SEQUENCES

A NUMERICAL SEQUENCE

A.1 FINDING THE VALUE OF A SPECIFIC TERM (u_n)

Ex 1: Using the table below, find u_4.

n	1	2	3	4	5	6
u_n	**3**	**5**	**7**	**9**	**11**	**13**

$$u_4 = \boxed{9}$$

Answer: To find u_4, we look for the column where $n = 4$. The corresponding value in the u_n row is 9.
Therefore, $\boldsymbol{u_4 = 9}$.

Ex 2: Using the table below, find u_5.

n	1	2	3	4	5	6
u_n	**2**	**6**	**12**	**20**	**30**	**42**

$$u_5 = \boxed{30}$$

Answer: To find u_5, we locate the column where $n = 5$. The corresponding value in the u_n row is 30.
Therefore, $\boldsymbol{u_5 = 30}$.

Ex 3: Using the table below, find u_7.

n	1	2	3	4	5	6	7	8
u_n	**4**	**9**	**16**	**25**	**36**	**49**	**64**	**81**

$$u_7 = \boxed{64}$$

Answer: To find u_7, we locate the column where $n = 7$. The corresponding value in the u_n row is 64.
Therefore, $\boldsymbol{u_7 = 64}$.

Ex 4: Using the table below, find u_8.

n	1	2	3	4	5	6	7	8
u_n	**1**	**3**	**7**	**15**	**31**	**63**	**127**	**255**

$$u_8 = \boxed{255}$$

Answer: To find u_8, we locate the column where $n = 8$. The corresponding value in the u_n row is 255.
Therefore, $\boldsymbol{u_8 = 255}$.

B RECURSIVE DEFINITION

B.1 CALCULATING TERMS FROM A RECURSIVE RULE

Ex 5: A sequence is defined recursively by:

- $u_1 = 5$.
- $u_{n+1} = u_n + 3$.

Find the first four terms of this sequence.

- $u_1 = \boxed{5}$
- $u_2 = \boxed{8}$
- $u_3 = \boxed{11}$
- $u_4 = \boxed{14}$

Answer: Let's build the sequence step-by-step using the recursive definition.

- **1st term**: The starting term is given: $\boldsymbol{u_1 = 5}$.
- **2nd term**: Use the rule with $n = 1$. $\boldsymbol{u_2} = u_1 + 3 = 5 + 3 = \mathbf{8}$.
- **3rd term**: Use the rule with $n = 2$. $\boldsymbol{u_3} = u_2 + 3 = 8 + 3 = \mathbf{11}$.
- **4th term**: Use the rule with $n = 3$. $\boldsymbol{u_4} = u_3 + 3 = 11 + 3 = \mathbf{14}$.

Ex 6: A sequence is defined recursively by:

- $u_0 = 1$.
- $u_{n+1} = u_n + \frac{1}{2}$.

Find the first four terms of this sequence (from u_0 to u_3).

- $u_0 = \boxed{1}$
- $u_1 = \boxed{\frac{3}{2}}$
- $u_2 = \boxed{2}$
- $u_3 = \boxed{\frac{5}{2}}$

Answer: Let's build the sequence step-by-step using the recursive definition.

- **Term 0**: The starting term is given: $\boldsymbol{u_0 = 1}$.
- **Term 1**: Use the rule with $n = 0$. $\boldsymbol{u_1} = u_0 + \frac{1}{2} = 1 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$.
- **Term 2**: Use the rule with $n = 1$. $\boldsymbol{u_2} = u_1 + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2$.
- **Term 3**: Use the rule with $n = 2$. $\boldsymbol{u_3} = u_2 + \frac{1}{2} = 2 + \frac{1}{2} = \frac{4}{2} + \frac{1}{2} = \frac{5}{2}$.

Ex 7: A sequence is defined recursively by:

- $u_0 = 0$.
- $u_{n+1} = 2u_n + 1$.

Find the first four terms of this sequence (from u_0 to u_3).

- $u_0 = \boxed{0}$
- $u_1 = \boxed{1}$
- $u_2 = \boxed{3}$
- $u_3 = \boxed{7}$

Answer: Let's build the sequence step-by-step using the recursive definition.

- **Term 0**: The starting term is given: $\boldsymbol{u_0 = 0}$.

- **Term 1**: Use the rule with $n = 0$. $\boldsymbol{u_1} = 2u_0+1 = 2(0)+1 = \mathbf{1}$.
- **Term 2**: Use the rule with $n = 1$. $\boldsymbol{u_2} = 2u_1+1 = 2(1)+1 = \mathbf{3}$.
- **Term 3**: Use the rule with $n = 2$. $\boldsymbol{u_3} = 2u_2+1 = 2(3)+1 = \mathbf{7}$.

Ex 8: A sequence is defined recursively by:

- $u_0 = 3$.
- $u_{n+1} = -u_n + 1$.

Find the first four terms of this sequence (from u_0 to u_3).

- $u_0 = \boxed{3}$
- $u_1 = \boxed{-2}$
- $u_2 = \boxed{3}$
- $u_3 = \boxed{-2}$

Answer: Let's build the sequence step-by-step using the recursive definition.

- **Term 0**: The starting term is given: $\boldsymbol{u_0 = 3}$.
- **Term 1**: Use the rule with $n = 0$. $\boldsymbol{u_1} = -u_0 + 1 = -(3) + 1 = \mathbf{-2}$.
- **Term 2**: Use the rule with $n = 1$. $\boldsymbol{u_2} = -u_1+1 = -(-2)+1 = 2 + 1 = \mathbf{3}$.
- **Term 3**: Use the rule with $n = 2$. $\boldsymbol{u_3} = -u_2 + 1 = -(3) + 1 = \mathbf{-2}$.

B.2 MODELING REAL SITUATIONS WITH SEQUENCES

Ex 9: A scientist observes a culture of bacteria. Initially (at day 0), there are $u_0 = 5$ bacteria. Each day, the number of bacteria doubles. Let u_n be the number of bacteria at day n.
Part A: Define the Sequence Recursively

- The initial term is $u_0 = \boxed{5}$
- The recursive rule is $u_{n+1} = \boxed{2} \times u_n$.

Part B: Calculate the Terms for the First Five Days

- $u_1 = \boxed{10}$ bacteria
- $u_2 = \boxed{20}$ bacteria
- $u_3 = \boxed{40}$ bacteria
- $u_4 = \boxed{80}$ bacteria
- $u_5 = \boxed{160}$ bacteria

Answer: The recursive rule is $u_{n+1} = 2 \times u_n$, with an initial term of $u_0 = 5$.

- $u_1 = 2 \times u_0 = 2 \times 5 = 10$
- $u_2 = 2 \times u_1 = 2 \times 10 = 20$
- $u_3 = 2 \times u_2 = 2 \times 20 = 40$
- $u_4 = 2 \times u_3 = 2 \times 40 = 80$
- $u_5 = 2 \times u_4 = 2 \times 80 = 160$

Ex 10: Let u_n be the number of steps I walk on day n. On day 0, I walk $u_0 = 1000$ steps. Each day, I walk 500 more steps than the previous day.
Part A: Define the Sequence Recursively

- The initial term is $u_0 = \boxed{1000}$.
- The recursive rule is $u_{n+1} = u_n + \boxed{500}$.

Part B: Calculate the Number of Steps for the Next Five Days

- $u_1 = \boxed{1500}$ steps
- $u_2 = \boxed{2000}$ steps
- $u_3 = \boxed{2500}$ steps
- $u_4 = \boxed{3000}$ steps
- $u_5 = \boxed{3500}$ steps

Answer: The recursive rule is $u_{n+1} = u_n + 500$, with an initial term of $u_0 = 1000$.

- $u_1 = u_0 + 500 = 1000 + 500 = 1500$ steps
- $u_2 = u_1 + 500 = 1500 + 500 = 2000$ steps
- $u_3 = u_2 + 500 = 2000 + 500 = 2500$ steps
- $u_4 = u_3 + 500 = 2500 + 500 = 3000$ steps
- $u_5 = u_4 + 500 = 3000 + 500 = 3500$ steps

Ex 11: Let u_n be the amount of money you have at the start of week n. At the start of week 0, you have $u_0 = 20$ dollars. Each week, you receive an allowance of \$10.
Part A: Define the Sequence Recursively

- The initial term is $u_0 = \boxed{20}$.
- The recursive rule is $u_{n+1} = u_n + \boxed{10}$.

Part B: Calculate the Amount of Money for the Next Five Weeks

- $u_1 = \boxed{30}$ dollars
- $u_2 = \boxed{40}$ dollars
- $u_3 = \boxed{50}$ dollars
- $u_4 = \boxed{60}$ dollars
- $u_5 = \boxed{70}$ dollars

Answer: The recursive rule is $u_{n+1} = u_n + 10$, with an initial term of $u_0 = 20$.

- $u_1 = u_0 + 10 = 20 + 10 = 30$ dollars
- $u_2 = u_1 + 10 = 30 + 10 = 40$ dollars
- $u_3 = u_2 + 10 = 40 + 10 = 50$ dollars
- $u_4 = u_3 + 10 = 50 + 10 = 60$ dollars
- $u_5 = u_4 + 10 = 60 + 10 = 70$ dollars

B.3 IDENTIFYING THE RECURSIVE RULE

Ex 12: Given the sequence $(3, 5, 7, 9, 11, 13, \dots)$, starting with index $n = 0$. Find its recursive definition.

- The initial term is $u_0 = \boxed{3}$.
- The recursive rule is $u_{n+1} = \boxed{u_n + 2}$

Answer:

- The initial term is given as the first number in the sequence, so $\boldsymbol{u_0 = 3}$.
- This is an arithmetic sequence. We find the common difference: $d = u_1 - u_0 = 5 - 3 = 2$.
- Therefore, the recursive rule is $\boldsymbol{u_{n+1} = u_n + 2}$.

$$(\ 3 \overset{+2}{\curvearrowright} 5 \overset{+2}{\curvearrowright} 7 \overset{+2}{\curvearrowright} 9\ , \dots ,\ u_n \overset{+2}{\curvearrowright} u_{n+1}\ , \dots)$$

Ex 13: Given the sequence $(100, 90, 80, 70, 60, \dots)$, starting with index $n = 0$. Find its recursive definition.

- The initial term is $u_0 = \boxed{100}$.
- The recursive rule is $u_{n+1} = \boxed{u_n - 10}$

Answer:

- The initial term is given as the first number in the sequence, so $\boldsymbol{u_0 = 100}$.
- This is an arithmetic sequence. We find the common difference: $d = u_1 - u_0 = 90 - 100 = -10$.
- Therefore, the recursive rule is $\boldsymbol{u_{n+1} = u_n - 10}$.

$$(\ 100 \overset{-10}{\curvearrowright} 90 \overset{-10}{\curvearrowright} 80 \overset{-10}{\curvearrowright} 70\ , \dots ,\ u_n \overset{-10}{\curvearrowright} u_{n+1}\ , \dots)$$

Ex 14: Given the sequence $(2, 6, 18, 54, 162, \dots)$, starting with index $n = 0$. Find its recursive definition.

- The initial term is $u_0 = \boxed{2}$.
- The recursive rule is $u_{n+1} = \boxed{3u_n}$

Answer:

- The initial term is given as the first number in the sequence, so $\boldsymbol{u_0 = 2}$.
- This is a geometric sequence. We find the common ratio: $r = \frac{u_1}{u_0} = \frac{6}{2} = 3$.
- Therefore, the recursive rule is $\boldsymbol{u_{n+1} = 3u_n}$.

$$(\ 2 \overset{\times 3}{\curvearrowright} 6 \overset{\times 3}{\curvearrowright} 18 \overset{\times 3}{\curvearrowright} 54\ , \dots ,\ u_n \overset{\times 3}{\curvearrowright} u_{n+1}\ , \dots)$$

Ex 15: Given the sequence $(8, 4, 2, 1, 0.5, \dots)$, starting with index $n = 0$. Find its recursive definition.

- The initial term is $u_0 = \boxed{8}$.
- The recursive rule is $u_{n+1} = \boxed{0.5u_n}$

Answer:

- The initial term is given as the first number in the sequence, so $\boldsymbol{u_0 = 8}$.
- This is a geometric sequence. We find the common ratio: $r = \frac{u_1}{u_0} = \frac{4}{8} = 0.5$.
- Therefore, the recursive rule is $\boldsymbol{u_{n+1} = 0.5u_n}$ or $\boldsymbol{u_{n+1} = \frac{u_n}{2}}$.

$$(\ 8 \overset{\div 2}{\curvearrowright} 4 \overset{\div 2}{\curvearrowright} 2 \overset{\div 2}{\curvearrowright} 1 \overset{\div 2}{\curvearrowright} 0.5\ , \dots ,\ u_n \overset{\div 2}{\curvearrowright} u_{n+1}\ , \dots)$$

B.4 MODELING WITH ARITHMETICO-GEOMETRIC SEQUENCES

Ex 16: A company has 200 employees in 2025. Each year, 10% of the employees leave the company, and the company hires 30 new employees.
Let (u_n) be the sequence corresponding to the number of employees in the company in $2025 + n$.

1. How many employees will there be in 2026?

$$\boxed{210}$$

2. How many employees will there be in 2027?

$$\boxed{219}$$

3. For all $n \in \mathbb{N}$, express u_{n+1} in terms of u_n.

$$u_{n+1} = \boxed{0.9u_n + 30}$$

Answer:

1. In 2026, which corresponds to $n = 1$:

$$\begin{aligned} u_1 &= \overbrace{u_0}^{\text{employees from the previous year}} - \overbrace{0.1 \times u_0}^{10\%\text{who leave}} + \overbrace{30}^{\text{hired}} \\ &= 200 - 0.1 \times 200 + 30 \\ &= 200 - 20 + 30 \\ &= 210 \end{aligned}$$

There will therefore be 210 employees in 2026.

2. In 2027, which corresponds to $n = 2$:

$$\begin{aligned} u_2 &= \overbrace{u_1}^{\text{employees from the previous year}} - \overbrace{0.1 \times u_1}^{10\%\text{who leave}} + \overbrace{30}^{\text{hired}} \\ &= 210 - 0.1 \times 210 + 30 \\ &= 210 - 21 + 30 \\ &= 219 \end{aligned}$$

There will therefore be 219 employees in 2027.

3. The recursive relation is

$$\begin{aligned} u_{n+1} &= \overbrace{u_n}^{\text{employees from the previous year}} - \overbrace{0.1 \times u_n}^{10\%\text{who leave}} + \overbrace{30}^{\text{hired}} \\ &= u_n - 0.1 \times u_n + 30 \\ &= (1 - 0.1) \times u_n + 30 \\ &= 0.9u_n + 30 \end{aligned}$$

$u_{n+1} = 0.9u_n + 30$.

Ex 17: A gym has 200 members in 2025. Each year, the number of members increases by 10% through referrals, and the gym adds 20 new members from advertising.
Let (u_n) be the sequence corresponding to the number of members in the gym in $2025+n$.

1. How many members will there be in 2026?

$$\boxed{240}$$

2. How many members will there be in 2027?

$$\boxed{284}$$

3. For all $n \in \mathbb{N}$, express u_{n+1} in terms of u_n.

$$u_{n+1} = \boxed{1.1u_n + 20}$$

Answer:

1. In 2026, which corresponds to $n=1$:

$$\begin{aligned} u_1 &= \overbrace{u_0}^{\text{members from the previous year}} + \overbrace{0.1 \times u_0}^{10\%\text{through referrals}} + \overbrace{20}^{\text{from advertising}} \\ &= 200 + 0.1 \times 200 + 20 \\ &= 200 + 20 + 20 \\ &= 240 \end{aligned}$$

There will therefore be 240 members in 2026.

2. In 2027, which corresponds to $n=2$:

$$\begin{aligned} u_2 &= \overbrace{u_1}^{\text{members from the previous year}} + \overbrace{0.1 \times u_1}^{10\%\text{through referrals}} + \overbrace{20}^{\text{from advertising}} \\ &= 240 + 0.1 \times 240 + 20 \\ &= 240 + 24 + 20 \\ &= 284 \end{aligned}$$

There will therefore be 284 members in 2027.

3. The recursive relation is

$$\begin{aligned} u_{n+1} &= \overbrace{u_n}^{\text{members from the previous year}} + \overbrace{0.1 \times u_n}^{10\%\text{through referrals}} + \overbrace{20}^{\text{from advertising}} \\ &= u_n + 0.1 \times u_n + 20 \\ &= (1+0.1) \times u_n + 20 \\ &= 1.1u_n + 20 \end{aligned}$$

$u_{n+1} = 1.1u_n + 20.$

C EXPLICIT DEFINITION

C.1 CALCULATING TERMS USING AN EXPLICIT FORMULA

Ex 18: Consider the sequence defined by the explicit formula: $u_n = 3n + 2$.
Calculate u_{100}.

$$u_{100} = \boxed{302}$$

Answer: To find the value of u_{100}, we substitute $n = 100$ into the explicit formula.

$$\begin{aligned} u_{100} &= 3(100) + 2 \\ &= 300 + 2 \\ &= \mathbf{302} \end{aligned}$$

Ex 19: Consider the sequence defined by the explicit formula: $u_n = -5n + 100$.
Calculate u_{50}.

$$u_{50} = \boxed{-150}$$

Answer: To find the value of u_{50}, we substitute $n = 50$ into the explicit formula.

$$\begin{aligned} u_{50} &= -5(50) + 100 \\ &= -250 + 100 \\ &= \mathbf{-150} \end{aligned}$$

Ex 20: Consider the sequence defined by the explicit formula: $u_n = 3 \times 2^n$.
Calculate u_{10}.

$$u_{10} = \boxed{3072}$$

Answer: To find the value of u_{10}, we substitute $n = 10$ into the explicit formula.

$$\begin{aligned} u_{10} &= 3 \times 2^{10} \\ &= 3 \times 1024 \\ &= \mathbf{3072} \end{aligned}$$

Ex 21: Consider the sequence defined by the explicit formula: $u_n = n^2 + 5$.
Calculate u_{20}.

$$u_{20} = \boxed{405}$$

Answer: To find the value of u_{20}, we substitute $n = 20$ into the explicit formula.

$$\begin{aligned} u_{20} &= (20)^2 + 5 \\ &= 400 + 5 \\ &= \mathbf{405} \end{aligned}$$

Ex 22: Consider the sequence defined by the explicit formula: $u_n = \frac{n}{4} + 1$.
Calculate u_{40}.

$$u_{40} = \boxed{11}$$

Answer: To find the value of u_{40}, we substitute $n = 40$ into the explicit formula.

$$\begin{aligned} u_{40} &= \frac{40}{4} + 1 \\ &= 10 + 1 \\ &= \mathbf{11} \end{aligned}$$

C.2 FINDING THE EXPLICIT FORMULA FROM A PATTERN

Ex 23: For the sequence given in the table below, find the explicit formula for u_n.

n	0	1	2	3	4
u_n	0	3	6	9	12

$$u_n = \boxed{3n}$$

Answer: By observing the table, we can see that the value of each term u_n is three times its position n.

- For $n = 0$, $u_0 = 0 = 3 \times 0$
- For $n = 1$, $u_1 = 3 = 3 \times 1$
- For $n = 2$, $u_2 = 6 = 3 \times 2$

Therefore, the explicit formula is $\boldsymbol{u_n = 3n}$.

Ex 24: For the sequence given in the table below, find the explicit formula for u_n.

n	0	1	2	3	4	5
u_n	1	2	4	8	16	32

$$u_n = \boxed{2^n}$$

Answer: By observing the table, we can see that the value of each term u_n is 2 raised to the power of its position n.

- For $n = 0$, $u_0 = 1 = 2^0$
- For $n = 1$, $u_1 = 2 = 2^1$
- For $n = 2$, $u_2 = 4 = 2^2$

Therefore, the explicit formula is $\boldsymbol{u_n = 2^n}$.

Ex 25: For the sequence given in the table below, find the explicit formula for u_n.

n	1	2	3	4	5
u_n	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$

$$u_n = \boxed{\frac{1}{n}}$$

Answer: By observing the table, we can see that the value of each term u_n is the reciprocal of its position n.

- For $n = 1$, $u_1 = 1 = \frac{1}{1}$
- For $n = 2$, $u_2 = \frac{1}{2}$
- For $n = 3$, $u_3 = \frac{1}{3}$

Therefore, the explicit formula is $\boldsymbol{u_n = \frac{1}{n}}$.

Ex 26: For the sequence given in the table below, find the explicit formula for u_n.

n	0	1	2	3	4
u_n	1	3	5	7	9

$$u_n = \boxed{2n+1}$$

Answer: By observing the table, we can see that the value of each term u_n is two times its position n, plus one.

- For $n = 0$, $u_0 = 1 = 2(0) + 1$
- For $n = 1$, $u_1 = 3 = 2(1) + 1$
- For $n = 2$, $u_2 = 5 = 2(2) + 1$

Therefore, the explicit formula is $\boldsymbol{u_n = 2n + 1}$.

Ex 27: For the sequence given in the table below, find the explicit formula for u_n.

n	1	2	3	4	5
u_n	1	4	9	16	25

$$u_n = \boxed{n^2}$$

Answer: By observing the table, we can see that the value of each term u_n is its position n multiplied by itself (squared).

- For $n = 1$, $u_1 = 1 = 1^2$
- For $n = 2$, $u_2 = 4 = 2^2$
- For $n = 3$, $u_3 = 9 = 3^2$

Therefore, the explicit formula is $\boldsymbol{u_n = n^2}$.

C.3 FINDING EXPRESSIONS FOR ADJACENT TERMS

Ex 28: Consider the sequence defined by the explicit formula: $u_n = 3n + 2$.
Calculate and simplify u_{n+1}.

$$u_{n+1} = \boxed{3n+5}$$

Answer: To find u_{n+1}, we substitute n with $(n+1)$ into the explicit formula.

$$\begin{aligned} u_{n+1} &= 3(n+1) + 2 \\ &= 3n + 3 + 2 \\ &= 3n + 5 \end{aligned}$$

Ex 29: Consider the sequence defined by the explicit formula: $u_n = 5n - 2$.
Calculate and simplify u_{n-1}.

$$u_{n-1} = \boxed{5n-7}$$

Answer: To find u_{n-1}, we substitute n with $(n-1)$ into the explicit formula.

$$\begin{aligned} u_{n-1} &= 5(n-1) - 2 \\ &= 5n - 5 - 2 \\ &= 5n - 7 \end{aligned}$$

Ex 30: Consider the sequence defined by the explicit formula: $u_n = n^2 + 3n$.
Calculate and simplify u_{n+1}.

$$u_{n+1} = \boxed{n^2+5n+4}$$

Answer: To find u_{n+1}, we substitute n with $(n+1)$ into the explicit formula.

$$\begin{aligned} u_{n+1} &= (n+1)^2 + 3(n+1) \\ &= (n^2 + 2n + 1) + (3n + 3) \\ &= n^2 + 5n + 4 \end{aligned}$$

Ex 31: Consider the sequence defined by the explicit formula: $u_n = 10 - 4n$.
Calculate and simplify u_{n-1}.

$$u_{n-1} = \boxed{14 - 4n}$$

Answer: To find u_{n-1}, we substitute n with $(n-1)$ into the explicit formula.

$$\begin{aligned} u_{n-1} &= 10 - 4(n-1) \\ &= 10 - 4n + 4 \\ &= 14 - 4n \end{aligned}$$

D ARITHMETIC SEQUENCE

D.1 STUDYING AN ARITHMETIC SEQUENCE

Ex 32: Consider the sequence $(u_0 = 5,\ u_1 = 8,\ u_2 = 11,\ u_3 = 14,\ u_4 = 17, \ldots)$.

1. - $u_1 - u_0 = \boxed{3}$
   - $u_2 - u_1 = \boxed{3}$
   - $u_3 - u_2 = \boxed{3}$
2. Show that the sequence is arithmetic.

   The difference between consecutive terms is constant.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{u_n + 3}$$
4. What is its explicit rule?

   $$u_n = \boxed{3n + 5}$$
5. Find the 50th term of the sequence.

   $$u_{50} = \boxed{155}$$

Answer:

1. - $u_1 - u_0 = 8 - 5 = 3$
   - $u_2 - u_1 = 11 - 8 = 3$
   - $u_3 - u_2 = 14 - 11 = 3$
2. Yes. The difference between consecutive terms is constant, so the sequence is arithmetic.
3. Recursive rule: $u_{n+1} = u_n + 3$
4. Explicit rule: $u_n = 3n + 5$ (with $u_0 = 5$)
5. $u_{50} = 3 \times 50 + 5 = 155$

Ex 33: Consider the sequence $(u_0 = 4,\ u_1 = 9,\ u_2 = 14,\ u_3 = 19, \ldots)$.

1. - $u_1 - u_0 = \boxed{5}$
   - $u_2 - u_1 = \boxed{5}$
   - $u_3 - u_2 = \boxed{5}$
2. Show that the sequence is arithmetic.

   The difference between consecutive terms is constant.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{u_n + 5}$$
4. What is its explicit rule?

   $$u_n = \boxed{5n + 4}$$
5. Find the 50th term of the sequence.

   $$u_{50} = \boxed{254}$$

Answer:

1. - $u_1 - u_0 = 9 - 4 = 5$
   - $u_2 - u_1 = 14 - 9 = 5$
   - $u_3 - u_2 = 19 - 14 = 5$
2. Yes. The difference between consecutive terms is constant, so the sequence is arithmetic.
3. Recursive rule: $u_{n+1} = u_n + 5$
4. Explicit rule: $u_n = 5n + 4$ (with $u_0 = 4$)
5. $u_{50} = 5 \times 50 + 4 = 254$

Ex 34: Consider the sequence $(u_0 = 125,\ u_1 = 115,\ u_2 = 105,\ u_3 = 95, \ldots)$.

1. - $u_1 - u_0 = \boxed{-10}$
   - $u_2 - u_1 = \boxed{-10}$
   - $u_3 - u_2 = \boxed{-10}$
2. Show that the sequence is arithmetic.

   The difference between consecutive terms is constant.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{u_n - 10}$$
4. What is its explicit rule?

   $$u_n = \boxed{-10n + 125}$$
5. Find the 1000th term of the sequence.

   $$u_{1000} = \boxed{-9875}$$

Answer:

1. - $u_1 - u_0 = 115 - 125 = -10$
   - $u_2 - u_1 = 105 - 115 = -10$
   - $u_3 - u_2 = 95 - 105 = -10$
2. Yes. The difference between consecutive terms is constant, so the sequence is arithmetic.
3. Recursive rule: $u_{n+1} = u_n - 10$
4. Explicit rule: $u_n = -10n + 125$ (with $u_0 = 125$)
5. $u_{1000} = -10 \times 1000 + 125 = -9875$

D.2 MODELING REAL SITUATIONS WITH EXPLICIT FORMULAS

Ex 35: You have an initial savings of \$30. Each week, you add \$10 to your savings. Let u_n be the total amount of money you have after n weeks.

- **Part A: Write the Explicit Formula**
  The formula for the amount of money after n weeks is:
  $$u_n = \boxed{30 + 10n}$$
- **Part B: Calculate a Future Value**
  How much money will you have after 20 weeks?
  $$u_{20} = \boxed{230}$$

Answer: The initial amount is $u_0 = 30$ and the common difference is $d = 10$. The explicit formula is $u_n = 30 + 10n$.
To find the amount after 20 weeks, we substitute $n = 20$:

$$\begin{aligned} u_{20} &= 30 + 10 \times 20 \\ &= 30 + 200 \\ &= 230 \end{aligned}$$

After 20 weeks, you will have \$230.

Ex 36: You deposit \$1,500 in a savings account that pays simple interest at a rate of 4% per year. Let u_n be the total amount in the account after n years.

- **Part A: Write the Explicit Formula**
  The interest earned each year is 4% of \$1,500, which is $0.04 \times 1500 = 60$ dollars.
  The formula for the amount after n years is:
  $$u_n = \boxed{1500 + 60n}$$
- **Part B: Calculate a Future Value**
  What will your account balance be after 20 years?
  $$u_{20} = \boxed{2700} \text{ dollars}$$

Answer: The initial amount is $u_0 = 1500$ and the common difference (interest per year) is $d = 60$. The explicit formula is $u_n = 1500 + 60n$.
To find the amount at year 20, we substitute $n = 20$:

$$\begin{aligned} u_{20} &= 1500 + 60 \times 20 \\ &= 1500 + 1200 \\ &= 2700 \end{aligned}$$

Your amount at year 20 will be \$2,700.

Ex 37: You start a stamp collection with 12 stamps. Each month, you add 4 new stamps. Let u_n be the total number of stamps after n months.

- **Part A: Write the Explicit Formula**
  The formula for the number of stamps after n months is:
  $$u_n = \boxed{12 + 4n}$$
- **Part B: Calculate a Future Value**
  How many stamps will you have after 15 months?
  $$u_{15} = \boxed{72} \text{ stamps}$$

Answer: The initial number of stamps is $u_0 = 12$ and the common difference is $d = 4$. The explicit formula is $u_n = 12 + 4n$.
To find the number of stamps after 15 months, we substitute $n = 15$:

$$\begin{aligned} u_{15} &= 12 + 4 \times 15 \\ &= 12 + 60 \\ &= 72 \end{aligned}$$

After 15 months, you will have 72 stamps.

D.3 FINDING THE TERM NUMBER IN AN ARITHMETIC SEQUENCE

Ex 38: An arithmetic sequence is defined by its initial term $u_0 = 8$ and a common difference $d = 4$.
Determine the index n for which the term u_n has a value of 56.

$$n = \boxed{12}$$

Answer: Our goal is to find the index n that corresponds to the term value $u_n = 56$.
Step 1: Write the explicit formula for the sequence
We start with the general explicit formula for a sequence starting at u_0:

$$u_n = u_0 + nd$$

Substitute the given values, $u_0 = 8$ and $d = 4$, to get the specific formula for this sequence:

$$u_n = 8 + 4n$$

Step 2: Set up and solve the equation
Now, we set the formula for u_n equal to the desired value, 56, and solve for n:

$$\begin{aligned} 8 + 4n &= 56 \\ 4n &= 56 - 8 \\ 4n &= 48 \\ n &= \frac{48}{4} \\ n &= \mathbf{12} \end{aligned}$$

Therefore, the term with a value of 56 is at position $n = 12$.

Ex 39: An arithmetic sequence is defined by its first term $u_1 = 10$ and a common difference $d = 5$.
Determine the index n for which the term u_n has a value of 105.

$$n = \boxed{20}$$

Answer: Our goal is to find the index n that corresponds to the term value $u_n = 105$.
Step 1: Write the explicit formula for the sequence
We start with the general explicit formula for a sequence starting at u_1:

$$u_n = u_1 + (n-1)d$$

Substitute the given values, $u_1 = 10$ and $d = 5$, to get the specific formula for this sequence:

$$u_n = 10 + (n-1) \times 5$$

Step 2: Set up and solve the equation
Now, we set the formula for u_n equal to the desired value, 105, and solve for n:

$$\begin{aligned}10 + (n-1) \times 5 &= 105\\ 5(n-1) &= 105 - 10\\ 5(n-1) &= 95\\ n-1 &= \frac{95}{5}\\ n-1 &= 19\\ n &= 19+1\\ n &= \mathbf{20}\end{aligned}$$

Therefore, the term with a value of 105 is at position $n = 20$.

Ex 40: An arithmetic sequence is defined by its first term $u_1 = 50$ and a common difference $d = -4$.
Determine the index n for which the term u_n has a value of 10.

$$n = \boxed{11}$$

Answer: Our goal is to find the index n that corresponds to the term value $u_n = 10$.
Step 1: Write the explicit formula for the sequence
We start with the general explicit formula for a sequence starting at u_1:

$$u_n = u_1 + (n-1)d$$

Substitute the given values, $u_1 = 50$ and $d = -4$, to get the specific formula for this sequence:

$$u_n = 50 + (n-1)(-4)$$

Step 2: Set up and solve the equation
Now, we set the formula for u_n equal to the desired value, 10, and solve for n:

$$\begin{aligned}50 + (n-1)(-4) &= 10\\ -4(n-1) &= 10 - 50\\ -4(n-1) &= -40\\ n-1 &= \frac{-40}{-4}\\ n-1 &= 10\\ n &= 10+1\\ n &= \mathbf{11}\end{aligned}$$

Therefore, the term with a value of 10 is at position $n = 11$.

Ex 41: An arithmetic sequence is defined by its initial term $u_0 = 1$ and a common difference $d = 2$.
Determine the index n for which the term u_n has a value of 21.

$$n = \boxed{10}$$

Answer: Our goal is to find the index n that corresponds to the term value $u_n = 21$.
Step 1: Write the explicit formula for the sequence
We start with the general explicit formula for a sequence starting at u_0:

$$u_n = u_0 + nd$$

Substitute the given values, $u_0 = 1$ and $d = 2$, to get the specific formula for this sequence:

$$u_n = 1 + 2n$$

Step 2: Set up and solve the equation
Now, we set the formula for u_n equal to the desired value, 21, and solve for n:

$$\begin{aligned}1 + 2n &= 21\\ 2n &= 21 - 1\\ 2n &= 20\\ n &= \frac{20}{2}\\ n &= \mathbf{10}\end{aligned}$$

Therefore, the term with a value of 21 is at position $n = 10$.

D.4 FINDING THE EXPLICIT FORMULA FROM TWO TERMS

Ex 42: An arithmetic sequence is given by two of its terms: $u_2 = 11$ and $u_6 = 31$.
Determine the explicit formula for u_n.

$$u_n = \boxed{5n+1}$$

Answer: To find the explicit formula $u_n = u_0 + nd$, we need to find the initial term u_0 and the common difference d. We can set up a system of two linear equations using the two terms provided:

$$\begin{cases} u_2 = u_0 + 2d = 11 & (1)\\ u_6 = u_0 + 6d = 31 & (2)\end{cases}$$

Step 1: Find the common difference (d)
Subtract equation (1) from equation (2) to eliminate u_0:

$$\begin{aligned}(u_0 + 6d) - (u_0 + 2d) &= 31 - 11\\ 4d &= 20\\ d &= \mathbf{5}\end{aligned}$$

Step 2: Find the initial term (u_0)
Substitute the value $d = 5$ back into equation (1):

$$\begin{aligned}u_0 + 2(5) &= 11\\ u_0 + 10 &= 11\\ u_0 &= 11 - 10\\ u_0 &= \mathbf{1}\end{aligned}$$

Step 3: Write the explicit formula
With $u_0 = 1$ and $d = 5$, the explicit formula is:

$$\boldsymbol{u_n = 1 + 5n}$$

Ex 43: An arithmetic sequence is given by two of its terms: $u_5 = 20$ and $u_{10} = 5$.
Determine the explicit formula for u_n.

$$u_n = \boxed{-3n+35}$$

Answer: To find the explicit formula $u_n = u_0 + nd$, we set up a system of two linear equations using the given terms:

$$\begin{cases} u_5 = u_0 + 5d = 20 & (1)\\ u_{10} = u_0 + 10d = 5 & (2)\end{cases}$$

Step 1: Find the common difference (d)
Subtract equation (1) from equation (2) to eliminate u_0:

$$\begin{aligned}(u_0 + 10d) - (u_0 + 5d) &= 5 - 20\\ 5d &= -15\\ d &= \mathbf{-3}\end{aligned}$$

Step 2: Find the initial term (u_0)
Substitute the value $d = -3$ back into equation (1):

$$\begin{aligned} u_0 + 5(-3) &= 20 \\ u_0 - 15 &= 20 \\ u_0 &= 20 + 15 \\ u_0 &= \mathbf{35} \end{aligned}$$

Step 3: Write the explicit formula
With $u_0 = 35$ and $d = -3$, the explicit formula is:

$$\boldsymbol{u_n = 35 - 3n}$$

Ex 44: An arithmetic sequence is given by two of its terms: $u_3 = 4$ and $u_7 = 6$.
Determine the explicit formula for u_n.

$$u_n = \boxed{0.5n + 2.5}$$

Answer: To find the explicit formula $u_n = u_0 + nd$, we set up a system of two linear equations:

$$\begin{cases} u_3 = u_0 + 3d = 4 & (1) \\ u_7 = u_0 + 7d = 6 & (2) \end{cases}$$

Step 1: Find the common difference (d)
Subtract equation (1) from equation (2) to eliminate u_0:

$$\begin{aligned} (u_0 + 7d) - (u_0 + 3d) &= 6 - 4 \\ 4d &= 2 \\ d &= \frac{2}{4} = \mathbf{0.5} \end{aligned}$$

Step 2: Find the initial term (u_0)
Substitute the value $d = 0.5$ back into equation (1):

$$\begin{aligned} u_0 + 3(0.5) &= 4 \\ u_0 + 1.5 &= 4 \\ u_0 &= 4 - 1.5 \\ u_0 &= \mathbf{2.5} \end{aligned}$$

Step 3: Write the explicit formula
With $u_0 = 2.5$ and $d = 0.5$, the explicit formula is:

$$\boldsymbol{u_n = 2.5 + 0.5n}$$

E GEOMETRIC SEQUENCE

E.1 STUDYING A GEOMETRIC SEQUENCE

Ex 45: Consider the sequence ($u_0 = 3$, $u_1 = 6$, $u_2 = 12$, $u_3 = 24$, ...).

1. - $u_1 \div u_0 = \boxed{2}$
   - $u_2 \div u_1 = \boxed{2}$
   - $u_3 \div u_2 = \boxed{2}$
2. Show that the sequence is geometric.
   The ratio between consecutive terms is constant. The terms are all even.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{2u_n}$$
4. What is its explicit rule?

   $$u_n = \boxed{3 \times 2^n}$$
5. Find the 10th term of the sequence.

   $$u_{10} = \boxed{3072}$$

Answer:

1. - $u_1 \div u_0 = 6 \div 3 = 2$
   - $u_2 \div u_1 = 12 \div 6 = 2$
   - $u_3 \div u_2 = 24 \div 12 = 2$
2. Yes. The ratio between consecutive terms is constant, so the sequence is geometric.
3. Recursive rule: $u_{n+1} = 2u_n$ (with ratio $r = 2$)
4. Explicit rule: $u_n = 3 \times 2^n$ (with $u_0 = 3$)
5. $u_{10} = 3 \times 2^{10} = 3 \times 1024 = 3072$

Ex 46: Consider the sequence ($u_0 = 1$, $u_1 = -1$, $u_2 = 1$, $u_3 = -1$, $u_4 = 1$, ...).

1. - $u_1 \div u_0 = \boxed{-1}$
   - $u_2 \div u_1 = \boxed{-1}$
   - $u_3 \div u_2 = \boxed{-1}$
2. Show that the sequence is geometric.
   The ratio between consecutive terms is constant. Every term is positive.
3. What is its recursive rule?

   $$u_{n+1} = \boxed{-1 \times u_n}$$
4. What is its explicit rule?

   $$u_n = \boxed{(-1)^n}$$
5. Find the 10th term of the sequence.

   $$u_{10} = \boxed{1}$$

Answer:

1. - $u_1 \div u_0 = -1 \div 1 = -1$
   - $u_2 \div u_1 = 1 \div (-1) = -1$
   - $u_3 \div u_2 = -1 \div 1 = -1$
2. Yes. The ratio between consecutive terms is constant, so the sequence is geometric.
3. Recursive rule: $u_{n+1} = -1 \times u_n$ (with $r = -1$)
4. Explicit rule: $u_n = (-1)^n$ (with $u_0 = 1$)
5. $u_{10} = (-1)^{10} = 1$

Ex 47: Consider the sequence ($u_0 = 4,\ u_1 = 2,\ u_2 = 1,\ u_3 = 0.5,\ u_4 = 0.25, \ldots$).

1.
   - $u_1 \div u_0 = \boxed{0.5}$
   - $u_2 \div u_1 = \boxed{0.5}$
   - $u_3 \div u_2 = \boxed{0.5}$
2. Show that the sequence is geometric.
   The ratio between consecutive terms is constant. The terms are increasing.
3. What is its recursive rule?
   $$u_{n+1} = \boxed{0.5\,u_n}$$
4. What is its explicit rule?
   $$u_n = \boxed{4 \times (0.5)^n}$$
5. Find the 10th term of the sequence.
   $$u_{10} = \boxed{0.00390625}$$

Answer:

1.
   - $u_1 \div u_0 = 2 \div 4 = 0.5$
   - $u_2 \div u_1 = 1 \div 2 = 0.5$
   - $u_3 \div u_2 = 0.5 \div 1 = 0.5$
2. Yes. The ratio between consecutive terms is constant, so the sequence is geometric.
3. Recursive rule: $u_{n+1} = 0.5\,u_n$ (with $r = \frac{1}{2}$)
4. Explicit rule: $u_n = 4 \times (0.5)^n$ (with $u_0 = 4$)
5. $u_{10} = 4 \times (0.5)^{10} = 4 \times \frac{1}{1024} = 0.00390625$

E.2 MODELING REAL SITUATIONS WITH EXPLICIT FORMULAS

Ex 48: A scientist observes a culture of bacteria. Initially (at hour 0), there are $u_0 = 50$ bacteria. Each hour, the number of bacteria doubles. Let u_n be the number of bacteria after n hours.

- **Part A: Write the Explicit Formula**
  The formula for the number of bacteria after n hours is:
  $$u_n = \boxed{50 \times 2^n}$$
- **Part B: Calculate a Future Value**
  How many bacteria will there be after 6 hours?
  $$u_6 = \boxed{3200} \text{ bacteria}$$

Answer: The initial amount is $u_0 = 50$ and the common ratio is $r = 2$. The explicit formula is $u_n = 50 \times 2^n$.
To find the amount after 6 hours, we substitute $n = 6$:

$$\begin{aligned} u_6 &= 50 \times 2^6 \\ &= 50 \times 64 \\ &= 3200 \end{aligned}$$

After 6 hours, there will be 3,200 bacteria.

Ex 49: You invest \$2,000 in an account with compound interest that grows by 5% each year. Let u_n be the total amount in the account after n years.

- **Part A: Write the Explicit Formula**
  To increase by 5%, we multiply by 1.05. The formula for the amount after n years is:
  $$u_n = \boxed{2000 \times (1.05)^n}$$
- **Part B: Calculate a Future Value**
  What will the balance be after 10 years? (Round to two decimal places)
  $$u_{10} = \boxed{3257.79}$$

Answer: The initial amount is $u_0 = 2000$ and the common ratio is $r = 1.05$. The explicit formula is $u_n = 2000 \times (1.05)^n$.
To find the amount after 10 years, we substitute $n = 10$:

$$\begin{aligned} u_{10} &= 2000 \times (1.05)^{10} \\ &\approx 2000 \times 1.62889 \\ &\approx 3257.79 \end{aligned}$$

After 10 years, the balance will be \$3,257.79.

Ex 50: A radioactive substance has an initial mass of 1000 grams. Its half-life is one year, meaning it loses half of its mass every year through nuclear decay. Let u_n be the mass of the substance after n years.

- **Part A: Write the Explicit Formula**
  The formula for the mass remaining after n years is:
  $$u_n = \boxed{1000 \times (0.5)^n}$$
- **Part B: Calculate a Future Value**
  How much of the substance will remain after 5 years? (Round to two decimal places)
  $$u_5 = \boxed{31.25} \text{ grams}$$

Answer: The initial mass is $u_0 = 1000$ and the common ratio is $r = 0.5$ (since the mass is halved each year). The explicit formula is $u_n = 1000 \times (0.5)^n$.
To find the mass after 5 years, we substitute $n = 5$:

$$\begin{aligned} u_5 &= 1000 \times (0.5)^5 \\ &= 1000 \times 0.03125 \\ &= 31.25 \end{aligned}$$

After 5 years, 31.25 grams of the substance will remain.

E.3 FINDING THE TERM NUMBER IN A GEOMETRIC SEQUENCE

Ex 51: A geometric sequence is defined by its initial term $u_0 = 10$ and a common ratio $r = 2$.
Determine the index n for which the term u_n has a value of 160.

$$n = \boxed{4}$$

Answer: Our goal is to find the index n such that $u_n = 160$.

- **Step 1: Write the explicit formula for the sequence**
  We use the general explicit formula for a sequence starting at u_0:
  $$u_n = u_0 \times r^n$$
  Substitute the given values, $u_0 = 10$ and $r = 2$:
  $$u_n = 10 \times 2^n$$
- **Step 2: Set up and solve the equation**
  We set the formula for u_n equal to 160:
  $$10 \times 2^n = 160$$
  First, isolate the exponential term by dividing by 10:
  $$2^n = 16$$
- **Step 3: solve for n**
  - **Method 1: By Inspection**
    We can ask ourselves: "2 raised to what power equals 16?" By testing values:
    $$2^1 = 2, 2^2 = 4, 2^3 = 8, \mathbf{2^4 = 16}$$
    This shows that $\boldsymbol{n = 4}$.
  - **Method 2: By Analytical Method**
    To solve for an unknown exponent algebraically, we use logarithms. We take the logarithm of both sides of the equation $2^n = 16$:
    $$\begin{aligned} \log(2^n) &= \log(16) \\ n \cdot \log(2) &= \log(16) \quad \text{(log power rule)} \\ n &= \frac{\log(16)}{\log(2)} \\ n &= \mathbf{4} \end{aligned}$$

  Both methods confirm that the term with a value of 160 is at position $n = 4$.

Ex 52: A geometric sequence is defined by its initial term $u_0 = 1000$ and a common ratio $r = 0.5$.
Determine the index n for which the term u_n has a value of 125.

$$n = \boxed{3}$$

Answer: Our goal is to find the index n such that $u_n = 125$.

- **Step 1: Write the explicit formula for the sequence**
  We use the general explicit formula for a sequence starting at u_0:
  $$u_n = u_0 \times r^n$$
  Substitute the given values, $u_0 = 1000$ and $r = 0.5$:
  $$u_n = 1000 \times (0.5)^n$$
- **Step 2: Set up and solve the equation**
  We set the formula for u_n equal to 125:
  $$1000 \times (0.5)^n = 125$$
  First, isolate the exponential term by dividing by 1000:
  $$(0.5)^n = \frac{125}{1000} = 0.125$$
- **Step 3: solve for n**
  - **Method 1: By Inspection**
    We can ask: "0.5 to what power equals 0.125?" Or, "$\frac{1}{2}$ to what power equals $\frac{1}{8}$?"
    $$\left(\frac{1}{2}\right)^1 = 0.5, \left(\frac{1}{2}\right)^2 = 0.25, \boldsymbol{\left(\frac{1}{2}\right)^3 = 0.125}$$
    This shows that $\boldsymbol{n = 3}$.
  - **Method 2: Using Logarithms**
    Take the logarithm of both sides of the equation $(0.5)^n = 0.125$:
    $$\begin{aligned} \log((0.5)^n) &= \log(0.125) \\ n \cdot \log(0.5) &= \log(0.125) \\ n &= \frac{\log(0.125)}{\log(0.5)} \\ n &= \mathbf{3} \end{aligned}$$

  Both methods confirm that the term with a value of 125 is at position $n = 3$.

Ex 53: A geometric sequence is defined by its first term $u_1 = 5$ and a common ratio $r = 3$.
Determine the index n for which the term u_n has a value of 3645.

$$n = \boxed{7}$$

Answer: Our goal is to find the index n such that $u_n = 3645$.

- **Step 1: Write the explicit formula for the sequence**
  We use the general explicit formula for a sequence starting at u_1:
  $$u_n = u_1 \times r^{n-1}$$
  Substitute the given values, $u_1 = 5$ and $r = 3$:
  $$u_n = 5 \times 3^{n-1}$$
- **Step 2: Set up and solve the equation**
  We set the formula for u_n equal to 3645:
  $$5 \times 3^{n-1} = 3645$$
  First, isolate the exponential term by dividing by 5:
  $$3^{n-1} = \frac{3645}{5} = 729$$
- **Step 3: solve for n**
  - **Method 1: By Inspection**
    We can ask: "3 raised to what power equals 729?"
    $$3^1 = 3, 3^2 = 9, 3^3 = 27, 3^4 = 81, 3^5 = 243, \mathbf{3^6 = 729}$$
    This shows that $n - 1 = 6$, so $\boldsymbol{n = 7}$.
  - **Method 2: Using Logarithms**
    Take the logarithm of both sides of the equation $3^{n-1} = 729$:
    $$\begin{aligned} \log(3^{n-1}) &= \log(729) \\ (n-1) \cdot \log(3) &= \log(729) \\ n - 1 &= \frac{\log(729)}{\log(3)} \\ n - 1 &= 6 \\ n &= \mathbf{7} \end{aligned}$$

  Both methods confirm that the term with a value of 3645 is at position $n = 7$.

E.4 FINDING THE EXPLICIT FORMULA FROM TWO TERMS

Ex 54: A geometric sequence is given by two of its terms: $u_2 = 4$ and $u_6 = 64$.Assuming the common ratio is positive, determine the explicit formula for u_n.

$$u_n = \boxed{2^n}$$

Answer: To find the explicit formula $u_n = u_0 \times r^n$, we need to find the initial term u_0 and the common ratio r. We can set up a system of two equations using the two terms provided:

$$\begin{cases} u_2 = u_0 \times r^2 = 4 & (1) \\ u_6 = u_0 \times r^6 = 64 & (2) \end{cases}$$

Step 1: Find the common ratio (r)
Divide equation (2) by equation (1) to eliminate u_0:

$$\begin{aligned} \frac{u_6}{u_2} &= \frac{u_0 \times r^6}{u_0 \times r^2} \\ \frac{64}{4} &= r^{6-2} \\ 16 &= r^4 \\ r &= (16)^{\frac{1}{4}} \quad \text{(since we assume } r > 0) \\ r &= \mathbf{2} \end{aligned}$$

Step 2: Find the initial term (u_0)
Substitute the value $r = 2$ back into equation (1):

$$\begin{aligned} u_0 \times (2)^2 &= 4 \\ u_0 \times 4 &= 4 \\ u_0 &= \frac{4}{4} \\ u_0 &= \mathbf{1} \end{aligned}$$

Step 3: Write the explicit formula
With $u_0 = 1$ and $r = 2$, the explicit formula is:

$$u_n = 1 \times 2^n$$

$$\boldsymbol{u_n = 2^n}$$

Ex 55: A geometric sequence is given by two of its terms: $u_3 = 20$ and $u_5 = 5$.Assuming the common ratio is positive, determine the explicit formula for u_n.

$$u_n = \boxed{160 \times (0.5)^n}$$

Answer: To find the explicit formula $u_n = u_0 \times r^n$, we set up a system of two equations:

$$\begin{cases} u_3 = u_0 \times r^3 = 20 & (1) \\ u_5 = u_0 \times r^5 = 5 & (2) \end{cases}$$

Step 1: Find the common ratio (r)
Divide equation (2) by equation (1) to eliminate u_0:

$$\begin{aligned} \frac{u_5}{u_3} &= \frac{u_0 \times r^5}{u_0 \times r^3} \\ \frac{5}{20} &= r^{5-3} \\ 0.25 &= r^2 \\ r &= \sqrt{0.25} \\ r &= \mathbf{0.5} \quad \text{(since we assume } r > 0) \end{aligned}$$

Step 2: Find the initial term (u_0)
Substitute the value $r = 0.5$ back into equation (1):

$$\begin{aligned} u_0 \times (0.5)^3 &= 20 \\ u_0 \times 0.125 &= 20 \\ u_0 &= \frac{20}{0.125} \\ u_0 &= \mathbf{160} \end{aligned}$$

Step 3: Write the explicit formula
With $u_0 = 160$ and $r = 0.5$, the explicit formula is:

$$\boldsymbol{u_n = 160 \times (0.5)^n}$$

Ex 56: A geometric sequence is given by two of its terms: $u_2 = 12$ and $u_5 = 96$.Assuming the common ratio is positive, determine the explicit formula for u_n (starting from u_1).

$$u_n = \boxed{6 \times 2^{n-1}}$$

Answer: To find the explicit formula $u_n = u_1 \times r^{n-1}$, we set up a system of two equations:

$$\begin{cases} u_2 = u_1 \times r^{2-1} = u_1 \times r = 12 & (1) \\ u_5 = u_1 \times r^{5-1} = u_1 \times r^4 = 96 & (2) \end{cases}$$

Step 1: Find the common ratio (r)
Divide equation (2) by equation (1) to eliminate u_1:

$$\begin{aligned} \frac{u_5}{u_2} &= \frac{u_1 \times r^4}{u_1 \times r} \\ \frac{96}{12} &= r^{4-1} \\ 8 &= r^3 \\ r &= \sqrt[3]{8} \\ r &= (8)^{\frac{1}{3}} \\ r &= \mathbf{2} \end{aligned}$$

Step 2: Find the first term (u_1)
Substitute the value $r = 2$ back into equation (1):

$$\begin{aligned} u_1 \times 2 &= 12 \\ u_1 &= \frac{12}{2} \\ u_1 &= \mathbf{6} \end{aligned}$$

Step 3: Write the explicit formula
With $u_1 = 6$ and $r = 2$, the explicit formula is:

$$\boldsymbol{u_n = 6 \times 2^{n-1}}$$

F SERIES

F.1 CALCULATING TERMS AND PARTIAL SUMS

Ex 57: Consider the sequence $(u_n) = (2, 5, 8, 11, \dots)$, where the first term is u_0. Find:

1. $u_0 = \boxed{2}$
2. $u_1 = \boxed{5}$

3. $u_2 = \boxed{8}$

4. $S_0 = \boxed{2}$

5. $S_1 = \boxed{7}$

6. $S_2 = \boxed{15}$

Answer: Since the list starts with u_0:

1. $u_0 = 2$
2. $u_1 = 5$
3. $u_2 = 8$

Now we calculate the partial sums:

1. $S_0 = u_0$
   $= 2$
2. $S_1 = u_0 + u_1$
   $= 2 + 5$
   $= 7$
3. $S_2 = u_0 + u_1 + u_2$
   $= 2 + 5 + 8$
   $= 15$

Ex 58: Consider the sequence $(u_n) = (10, 20, 40, 80, \dots)$, where the first term is u_1. Find:

1. $u_1 = \boxed{10}$
2. $u_2 = \boxed{20}$
3. $u_3 = \boxed{40}$
4. $S_1 = \boxed{10}$
5. $S_2 = \boxed{30}$
6. $S_3 = \boxed{70}$

Answer: Since the list starts with u_1:

1. $u_1 = 10$
2. $u_2 = 20$
3. $u_3 = 40$

Now we calculate the partial sums:

1. $S_1 = u_1$
   $= 10$
2. $S_2 = u_1 + u_2$
   $= 10 + 20$
   $= 30$
3. $S_3 = u_1 + u_2 + u_3$
   $= 10 + 20 + 40$
   $= 70$

Ex 59: Consider the sequence $(u_n) = (100, 95, 90, 85, \dots)$, where the first term is u_1. Find:

1. $u_1 = \boxed{100}$
2. $u_2 = \boxed{95}$
3. $u_3 = \boxed{90}$
4. $S_1 = \boxed{100}$
5. $S_2 = \boxed{195}$
6. $S_3 = \boxed{285}$

Answer: Since the list starts with u_1:

1. $u_1 = 100$
2. $u_2 = 95$
3. $u_3 = 90$

Now we calculate the partial sums:

1. $S_1 = u_1$
   $= 100$
2. $S_2 = u_1 + u_2$
   $= 100 + 95$
   $= 195$
3. $S_3 = u_1 + u_2 + u_3$
   $= 100 + 95 + 90$
   $= 285$

Ex 60: Consider the sequence $(u_n) = (64, 16, 4, 1, \dots)$, where the first term is u_0. Find:

1. $u_0 = \boxed{64}$
2. $u_1 = \boxed{16}$
3. $u_2 = \boxed{4}$
4. $S_0 = \boxed{64}$
5. $S_1 = \boxed{80}$
6. $S_2 = \boxed{84}$

Answer: Since the list starts with u_0:

1. $u_0 = 64$
2. $u_1 = 16$
3. $u_2 = 4$

Now we calculate the partial sums:

1. $S_0 = u_0$
   $= 64$
2. $S_1 = u_0 + u_1$
   $= 64 + 16$
   $= 80$
3. $S_2 = u_0 + u_1 + u_2$
   $= 64 + 16 + 4$
   $= 84$

F.2 CALCULATING PARTIAL SUMS FROM AN EXPLICIT FORMULA

Ex 61: Consider the sequence (u_n) defined by the explicit formula $u_n = 2n + 1$, starting from $n = 1$.
Calculate the partial sum S_4.

$$S_4 = \boxed{24}$$

Answer: To calculate the partial sum S_4, we first need to find the values of the terms from u_1 to u_4.
Step 1: Calculate the required terms
Using the explicit formula $u_n = 2n + 1$:

- $u_1 = 2(1) + 1 = 3$
- $u_2 = 2(2) + 1 = 5$
- $u_3 = 2(3) + 1 = 7$
- $u_4 = 2(4) + 1 = 9$

Step 2: Calculate the partial sum S_4
The partial sum S_4 is the sum of all terms from u_1 to u_4.

$$\begin{aligned} S_4 &= u_1 + u_2 + u_3 + u_4 \\ &= 3 + 5 + 7 + 9 \\ &= \mathbf{24} \end{aligned}$$

Ex 62: Consider the sequence (u_n) defined by the explicit formula $u_n = 2^n$, starting from $n = 0$.
Calculate the partial sum S_4.

$$S_4 = \boxed{31}$$

Answer: To calculate the partial sum S_4, we first need to find the values of the terms from u_0 to u_4.
Step 1: Calculate the required terms
Using the explicit formula $u_n = 2^n$:

- $u_0 = 2^0 = 1$
- $u_1 = 2^1 = 2$
- $u_2 = 2^2 = 4$
- $u_3 = 2^3 = 8$
- $u_4 = 2^4 = 16$

Step 2: Calculate the partial sum S_4
The partial sum S_4 is the sum of all terms from u_0 to u_4.

$$\begin{aligned} S_4 &= u_0 + u_1 + u_2 + u_3 + u_4 \\ &= 1 + 2 + 4 + 8 + 16 \\ &= \mathbf{31} \end{aligned}$$

Ex 63: Consider the sequence (u_n) defined by the explicit formula $u_n = 15 - 10n$, starting from $n = 0$.
Calculate the partial sum S_3.

$$S_3 = \boxed{0}$$

Answer: To calculate the partial sum S_3, we first need to find the values of the terms from u_0 to u_3.
Step 1: Calculate the required terms
Using the explicit formula $u_n = 15 - 10n$:

- $u_0 = 15 - 10(0) = 15$
- $u_1 = 15 - 10(1) = 5$
- $u_2 = 15 - 10(2) = -5$
- $u_3 = 15 - 10(3) = -15$

Step 2: Calculate the partial sum S_3
The partial sum S_3 is the sum of all terms from u_0 to u_3.

$$\begin{aligned} S_3 &= u_0 + u_1 + u_2 + u_3 \\ &= 15 + 5 + (-5) + (-15) \\ &= \mathbf{0} \end{aligned}$$

Ex 64: Consider the sequence (u_n) defined by the explicit formula $u_n = n^2$, starting from $n = 0$.
Calculate the partial sum S_3.

$$S_3 = \boxed{14}$$

Answer: To calculate the partial sum S_3, we first need to find the values of the terms from u_0 to u_3.
Step 1: Calculate the required terms
Using the explicit formula $u_n = n^2$:

- $u_0 = 0^2 = 0$
- $u_1 = 1^2 = 1$
- $u_2 = 2^2 = 4$
- $u_3 = 3^2 = 9$

Step 2: Calculate the partial sum S_3
The partial sum S_3 is the sum of all terms from u_0 to u_3.

$$\begin{aligned} S_3 &= u_0 + u_1 + u_2 + u_3 \\ &= 0 + 1 + 4 + 9 \\ &= \mathbf{14} \end{aligned}$$

F.3 EVALUATING SUMS IN SIGMA NOTATION

Ex 65: Calculate the sum:

$$\sum_{i=1}^{7} i = \boxed{28}$$

Answer: The notation $\sum_{i=1}^{7} i$ represents the sum of all integers i from the starting value $i = 1$ up to the ending value $i = 7$.

$$\begin{aligned} \sum_{i=1}^{7} i &= 1 + 2 + 3 + 4 + 5 + 6 + 7 \\ &= 28 \end{aligned}$$

Ex 66: Calculate the sum:

$$\sum_{k=0}^{3} k^2 = \boxed{14}$$

Answer: The notation $\sum_{k=0}^{3} k^2$ represents the sum of the term k^2 for all integer values of k from the starting value $k = 0$ up to the ending value $k = 3$.

$$\begin{aligned} \sum_{k=0}^{3} k^2 &= 0^2 + 1^2 + 2^2 + 3^2 \\ &= 0 + 1 + 4 + 9 \\ &= 14 \end{aligned}$$

Ex 67: Calculate the sum:

$$\sum_{k=1}^{3} \frac{1}{k} = \boxed{\frac{11}{6}}$$

Answer: The notation $\sum_{k=1}^{3} \frac{1}{k}$ represents the sum of the term $\frac{1}{k}$ for all integer values of k from the starting value $k = 1$ up to the ending value $k = 3$.

$$\begin{aligned}\sum_{k=1}^{3} \frac{1}{k} &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \\ &= \frac{6}{6} + \frac{3}{6} + \frac{2}{6} \\ &= \frac{6+3+2}{6} \\ &= \frac{11}{6}\end{aligned}$$

Ex 68: Calculate the sum:

$$\sum_{i=0}^{2} 4\left(\frac{3}{2}\right)^i = \boxed{19}$$

Answer: The notation $\sum_{i=0}^{2} 4\left(\frac{3}{2}\right)^i$ represents the sum of the term $4\left(\frac{3}{2}\right)^i$ for all integer values of i from the starting value $i = 0$ up to the ending value $i = 2$.

$$\begin{aligned}\sum_{i=0}^{2} 4\left(\frac{3}{2}\right)^i &= 4\left(\frac{3}{2}\right)^0 + 4\left(\frac{3}{2}\right)^1 + 4\left(\frac{3}{2}\right)^2 \\ &= 4(1) + 4\left(\frac{3}{2}\right) + 4\left(\frac{9}{4}\right) \\ &= 4 + 6 + 9 \\ &= 19\end{aligned}$$

Ex 69: Calculate the sum:

$$\sum_{i=0}^{3} (2i - 1) = \boxed{8}$$

Answer: The notation $\sum_{i=0}^{3}(2i-1)$ represents the sum of the term $(2i - 1)$ for all integer values of i from the starting value $i = 0$ up to the ending value $i = 3$.

$$\begin{aligned}\sum_{i=0}^{3} (2i - 1) &= (2(0) - 1) + (2(1) - 1) + (2(2) - 1) + (2(3) - 1) \\ &= (-1) + (1) + (3) + (5) \\ &= 8\end{aligned}$$

G SUM OF AN ARITHMETIC SEQUENCE

G.1 CALCULATING THE SUM OF AN ARITHMETIC SERIES: LEVEL 1

Ex 70: Calculate the sum of the first 7 positive integers.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 = \boxed{28}$$

Answer: We are asked to calculate the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the terms one by one:

  $$1 + 2 + 3 + 4 + 5 + 6 + 7 = \mathbf{28}$$

- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 7$ terms, a first term $u_1 = 1$, and a last term $u_7 = 7$. We can use the formula for the sum of an arithmetic sequence:

  $$S_n = \frac{n}{2}(u_1 + u_n)$$

  Substituting the values:

  $$\begin{aligned}S_7 &= \frac{7}{2}(1 + 7) \\ &= \frac{7}{2}(8) \\ &= 7 \times 4 \\ &= \mathbf{28}\end{aligned}$$

Ex 71: Calculate the sum of the first 7 positive even integers.

$$2 + 4 + 6 + 8 + 10 + 12 + 14 = \boxed{56}$$

Answer: We are asked to calculate the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the terms one by one:

  $$2 + 4 + 6 + 8 + 10 + 12 + 14 = \mathbf{56}$$

- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 7$ terms, a first term $u_1 = 2$, and a last term $u_7 = 14$. We can use the formula for the sum of an arithmetic sequence:

  $$S_n = \frac{n}{2}(u_1 + u_n)$$

  Substituting the values:

  $$\begin{aligned}S_7 &= \frac{7}{2}(2 + 14) \\ &= \frac{7}{2}(16) \\ &= 7 \times 8 \\ &= \mathbf{56}\end{aligned}$$

Ex 72: Calculate the sum of the following arithmetic sequence.

$$11 + 16 + 21 + 26 + 31 + 36 + 41 + 46 + 51 = \boxed{279}$$

Answer: We are asked to calculate the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the terms one by one:

  $$11 + 16 + 21 + 26 + 31 + 36 + 41 + 46 + 51 = \mathbf{279}$$

- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 9$ terms, a first term

$u_1 = 11$, and a last term $u_9 = 51$. We can use the formula for the sum of an arithmetic sequence:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

Substituting the values:

$$\begin{aligned} S_9 &= \frac{9}{2}(11+51) \\ &= \frac{9}{2}(62) \\ &= 9 \times 31 \\ &= \mathbf{279} \end{aligned}$$

Ex 73: Calculate the sum of the following arithmetic sequence.

$$60 + 55 + 50 + 45 + 40 + 35 + 30 + 25 + 20 = \boxed{360}$$

Answer: We are asked to calculate the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the terms one by one:
  $$60 + 55 + 50 + 45 + 40 + 35 + 30 + 25 + 20 = \mathbf{360}$$
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 9$ terms, a first term $u_1 = 60$, and a last term $u_9 = 20$. We can use the formula for the sum of an arithmetic sequence:
  $$S_n = \frac{n}{2}(u_1 + u_n)$$
  Substituting the values:
  $$\begin{aligned} S_9 &= \frac{9}{2}(60+20) \\ &= \frac{9}{2}(80) \\ &= 9 \times 40 \\ &= \mathbf{360} \end{aligned}$$

G.2 CALCULATING THE SUM OF AN ARITHMETIC SERIES: LEVEL 2

Ex 74: Calculate the sum of the first 100 positive integers.

$$1 + 2 + 3 + \cdots + 100 = \boxed{5050}$$

Answer: We are asked to calculate the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  Adding all the integers from 1 to 100 one by one would be very time-consuming.
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 100$ terms, a first term $u_1 = 1$, and a last term $u_{100} = 100$.
  Using the formula for the sum is much more efficient:
  $$S_n = \frac{n}{2}(u_1 + u_n)$$
  Substituting the values:
  $$\begin{aligned} S_{100} &= \frac{100}{2}(1+100) \\ &= 50(101) \\ &= \mathbf{5050} \end{aligned}$$

Ex 75: Calculate the sum of the arithmetic sequence:

$$3 + 6 + 9 + 12 + \cdots + 252 = \boxed{10710}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms (n)**
  The sequence is an arithmetic sequence with a first term $u_1 = 3$ and a common difference $d = 6 - 3 = 3$. The last term is $u_n = 252$.
  We use the explicit formula $u_n = u_1 + (n-1)d$ and solve for n:
  $$\begin{aligned} 252 &= 3 + (n-1) \times 3 \\ 252 - 3 &= 3(n-1) \\ 249 &= 3(n-1) \\ \frac{249}{3} &= n - 1 \\ 83 &= n - 1 \\ n &= 84 \end{aligned}$$
  So, there are $\boldsymbol{n = 84}$ terms in this sequence.
- **Step 2: Apply the Arithmetic Series Formula**
  Now we use the formula for the sum with $n = 84$, $u_1 = 3$, and $u_{84} = 252$:
  $$S_n = \frac{n}{2}(u_1 + u_n)$$
  Substituting the values:
  $$\begin{aligned} S_{84} &= \frac{84}{2}(3+252) \\ &= 42(255) \\ &= \mathbf{10710} \end{aligned}$$
  The sum of the sequence is 10,710.

Ex 76: Calculate the sum of the arithmetic sequence:

$$100 + 90 + 80 + \cdots + 10 = \boxed{550}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms (n)**
  The sequence is an arithmetic sequence with a first term $u_1 = 100$ and a common difference $d = 90 - 100 = -10$. The last term is $u_n = 10$.
  We use the explicit formula $u_n = u_1 + (n-1)d$ and solve for n:
  $$\begin{aligned} 10 &= 100 + (n-1)(-10) \\ 10 - 100 &= -10(n-1) \\ -90 &= -10(n-1) \\ \frac{-90}{-10} &= n - 1 \\ 9 &= n - 1 \\ n &= 10 \end{aligned}$$
  So, there are $\boldsymbol{n = 10}$ terms in this sequence.

- **Step 2: Apply the Arithmetic Series Formula**
Now we use the formula for the sum with $n = 10$, $u_1 = 100$, and $u_{10} = 10$:
$$S_n = \frac{n}{2}(u_1 + u_n)$$
Substituting the values:
$$\begin{aligned} S_{10} &= \frac{10}{2}(100 + 10) \\ &= 5(110) \\ &= \mathbf{550} \end{aligned}$$
The sum of the sequence is 550.

Ex 77: Calculate the sum of the arithmetic sequence:

$$5 + 7 + 9 + \cdots + 43 = \boxed{480}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms**
The sequence is an arithmetic sequence with an initial term $u_0 = 5$ and a common difference $d = 7 - 5 = 2$. The last term is $u_n = 43$.
We use the explicit formula $u_n = u_0 + nd$ and solve for n:
$$\begin{aligned} 43 &= 5 + n \times 2 \\ 43 - 5 &= 2n \\ 38 &= 2n \\ n &= \frac{38}{2} \\ n &= 19 \end{aligned}$$
Since the sequence starts at $n = 0$ and ends at $n = 19$, there are $\mathbf{19 + 1 = 20}$ terms in total.
- **Step 2: Apply the Arithmetic Series Formula**
Now we use the formula for the sum with 20 terms, a first term $u_0 = 5$, and a last term $u_{19} = 43$:
$$S_n = \frac{\text{Number of terms}}{2}(u_0 + u_n)$$
Substituting the values:
$$\begin{aligned} S_{19} &= \frac{20}{2}(5 + 43) \\ &= 10(48) \\ &= \mathbf{480} \end{aligned}$$
The sum of the sequence is 480.

Ex 78: Calculate the sum of the arithmetic sequence:

$$(-8) + (-4) + 0 + 4 + \cdots + 40 = \boxed{208}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms (n)**
The sequence is an arithmetic sequence with a first term $u_1 = -8$ and a common difference $d = -4 - (-8) = 4$. The last term is $u_n = 40$.
We use the explicit formula $u_n = u_1 + (n-1)d$ and solve for n:
$$\begin{aligned} 40 &= -8 + (n-1) \times 4 \\ 40 + 8 &= 4(n-1) \\ 48 &= 4(n-1) \\ \frac{48}{4} &= n - 1 \\ 12 &= n - 1 \\ n &= 13 \end{aligned}$$
So, there are $\boldsymbol{n = 13}$ terms in this sequence.
- **Step 2: Apply the Arithmetic Series Formula**
Now we use the formula for the sum with $n = 13$, $u_1 = -8$, and $u_{13} = 40$:
$$S_n = \frac{n}{2}(u_1 + u_n)$$
Substituting the values:
$$\begin{aligned} S_{13} &= \frac{13}{2}(-8 + 40) \\ &= \frac{13}{2}(32) \\ &= 13 \times 16 \\ &= \mathbf{208} \end{aligned}$$
The sum of the sequence is 208.

G.3 CALCULATING THE SUM OF AN ARITHMETIC SERIES IN SIGMA NOTATION: LEVEL 1

Ex 79: Calculate the sum:

$$\sum_{i=1}^{7} i = \boxed{28}$$

Answer: First, let's expand the sum represented by the sigma notation:
$$\sum_{i=1}^{7} i = 1 + 2 + 3 + 4 + 5 + 6 + 7$$
This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
We can add the terms one by one:
$$1 + 2 + 3 + 4 + 5 + 6 + 7 = \mathbf{28}$$
- **Method 2: Using the Arithmetic Series Formula**
This is an arithmetic sequence with $n = 7$ terms, a first term $u_1 = 1$, and a last term $u_7 = 7$. We can use the formula for the sum of an arithmetic sequence:
$$S_n = \frac{n}{2}(u_1 + u_n)$$
Substituting the values:
$$\begin{aligned} S_7 &= \frac{7}{2}(1 + 7) \\ &= \frac{7}{2}(8) \\ &= 7 \times 4 \\ &= \mathbf{28} \end{aligned}$$

Ex 80: Calculate the sum:

$$\sum_{i=1}^{6}(2i+1) = \boxed{48}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\begin{aligned}\sum_{i=1}^{6}(2i+1) &= (2(1)+1)+(2(2)+1)+(2(3)+1)+(2(4)+1)+(2(5)+1)+(2(6)+1)\\ &= 3+5+7+9+11+13\end{aligned}$$

This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the expanded terms one by one:
  $$3+5+7+9+11+13 = \mathbf{48}$$
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 6$ terms, a first term $u_1 = 3$, and a last term $u_6 = 13$. We use the formula for the sum of an arithmetic sequence:
  $$S_n = \frac{n}{2}(u_1 + u_n)$$
  Substituting the values:
  $$\begin{aligned}S_6 &= \frac{6}{2}(3+13)\\ &= 3(16)\\ &= \mathbf{48}\end{aligned}$$

Ex 81: Calculate the sum:

$$\sum_{i=2}^{4}3i = \boxed{27}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\begin{aligned}\sum_{i=2}^{4}3i &= (3\times 2)+(3\times 3)+(3\times 4)\\ &= 6+9+12\end{aligned}$$

This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the expanded terms one by one:
  $$6+9+12 = \mathbf{27}$$
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 3$ terms (from i=2 to i=4), a first term of 6, and a last term of 12. We can use the formula for the sum of an arithmetic sequence:
  $$S = \frac{\text{Number of terms}}{2}(\text{First term} + \text{Last term})$$
  Substituting the values:
  $$\begin{aligned}S &= \frac{3}{2}(6+12)\\ &= \frac{3}{2}(18)\\ &= 3\times 9\\ &= \mathbf{27}\end{aligned}$$

Ex 82: Calculate the sum:

$$\sum_{i=1}^{5}(12-2i) = \boxed{30}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\begin{aligned}\sum_{i=1}^{5}(12-2i) &= (12-2(1))+(12-2(2))+(12-2(3))+(12-2(4))+(12-2(5))\\ &= 10+8+6+4+2\end{aligned}$$

This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  We can add the expanded terms one by one:
  $$10+8+6+4+2 = \mathbf{30}$$
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with $n = 5$ terms, a first term $u_1 = 10$, and a last term $u_5 = 2$. We can use the formula for the sum of an arithmetic sequence:
  $$S_n = \frac{n}{2}(u_1 + u_n)$$
  Substituting the values:
  $$\begin{aligned}S_5 &= \frac{5}{2}(10+2)\\ &= \frac{5}{2}(12)\\ &= 5\times 6\\ &= \mathbf{30}\end{aligned}$$

G.4 CALCULATING THE SUM OF AN ARITHMETIC SERIES IN SIGMA NOTATION: LEVEL 2

Ex 83: Calculate the sum:

$$\sum_{i=1}^{84}3i = \boxed{10710}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\sum_{i=1}^{84}3i = 3+6+9+\cdots+252$$

This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  Adding all 84 terms by hand would be very time-consuming. Using a formula is more efficient.
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with the following properties:
  - Number of terms: **84**
  - First term ($i = 1$): $\boldsymbol{u_1 = 3(1) = 3}$
  - Last term ($i = 84$): $\boldsymbol{u_{84} = 3(84) = 252}$

We use the formula for the sum of an arithmetic sequence:

$$S_n = \frac{n}{2}(u_1 + u_n)$$

Substituting the values:

$$\begin{aligned} S_{84} &= \frac{84}{2}(3 + 252) \\ &= 42(255) \\ &= \mathbf{10710} \end{aligned}$$

Ex 84: Calculate the sum:

$$\sum_{i=0}^{20}(100 - 5i) = \boxed{1050}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\sum_{i=0}^{20}(100 - 5i) = 100 + 95 + 90 + \cdots + 0$$

This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  Adding all 21 terms by hand would be very time-consuming. Using a formula is more efficient.
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with the following properties:
  - Number of terms: $\mathbf{20 - 0 + 1 = 21}$
  - First term ($i = 0$): $\boldsymbol{u_0 = 100 - 5(0) = 100}$
  - Last term ($i = 20$): $\boldsymbol{u_{20} = 100 - 5(20) = 0}$

  We use the formula for the sum of an arithmetic sequence starting at u_0:

  $$S_n = \frac{n+1}{2}(u_0 + u_n)$$

  Substituting the values:

  $$\begin{aligned} S_{20} &= \frac{21}{2}(100 + 0) \\ &= \frac{21}{2}(100) \\ &= 21 \times 50 \\ &= \mathbf{1050} \end{aligned}$$

Ex 85: Calculate the sum:

$$\sum_{i=3}^{24}(5i + 2) = \boxed{1529}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\sum_{i=3}^{24}(5i + 2) = (5(3) + 2) + (5(4) + 2) + \cdots + (5(24) + 2)$$

$$= 17 + 22 + \cdots + 122$$

This is the sum of an arithmetic sequence.

- **Method 1: Direct Addition**
  Adding all the terms by hand would be very time-consuming and prone to error. Using the formula is much more efficient.
- **Method 2: Using the Arithmetic Series Formula**
  This is an arithmetic sequence with the following properties:
  - Number of terms: $\mathbf{24 - 3 + 1 = 22}$
  - First term (for $i = 3$): $\mathbf{5(3) + 2 = 17}$
  - Last term (for $i = 24$): $\mathbf{5(24) + 2 = 122}$

  We use the formula for the sum of an arithmetic sequence:

  $$S = \frac{\text{Number of terms}}{2}(\text{First term} + \text{Last term})$$

  Substituting the values:

  $$\begin{aligned} S &= \frac{22}{2}(17 + 122) \\ &= 11(139) \\ &= \mathbf{1529} \end{aligned}$$

G.5 PROVING ARITHMETIC SERIES FORMULAS

Ex 86: Use the formula for the sum of an arithmetic sequence to prove that the sum of the first n positive integers is:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Answer:

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

This is an arithmetic sequence with the following properties:

- The number of terms is $\boldsymbol{n}$.
- The first term is $\boldsymbol{u_1 = 1}$.
- The last term is $\boldsymbol{u_n = n}$.

We apply the formula for the sum of an arithmetic sequence:

$$\begin{aligned} \sum_{i=1}^{n} i &= \frac{\text{Number of terms}}{2}(\text{First term} + \text{Last term}) \\ &= \frac{n}{2}(1 + n) \\ &= \frac{n(n+1)}{2} \end{aligned}$$

Ex 87: Use the formula for the sum of an arithmetic sequence to prove that the sum of the first n positive even integers is:

$$\sum_{i=1}^{n} 2i = n(n+1)$$

Answer:

$$\sum_{i=1}^{n} 2i = 2 + 4 + 6 + \cdots + 2n$$

This is an arithmetic sequence with the following properties:

- The number of terms is $\boldsymbol{n}$.
- The first term is $\boldsymbol{u_1 = 2}$.

- The last term is $\boldsymbol{u_n = 2n}$.

We apply the formula for the sum of an arithmetic sequence:

$$\begin{aligned}\sum_{i=1}^{n} 2i &= \frac{\text{Number of terms}}{2}(\text{First term} + \text{Last term}) \\ &= \frac{n}{2}(2+2n) \\ &= \frac{n\cdot 2(1+n)}{2} \\ &= n(n+1)\end{aligned}$$

Ex 88: Let (u_n) be an arithmetic sequence with initial term u_0 and common difference d.
Use the formula for the sum of an arithmetic sequence to prove that:

$$\sum_{i=0}^{n} u_i = \frac{(n+1)(2u_0+nd)}{2}$$

Answer: The series $\sum_{i=0}^{n} u_i$ represents the sum of the terms of an arithmetic sequence from u_0 to u_n.
This sequence has the following properties:

- The number of terms is $\boldsymbol{n+1}$ (from $i=0$ to $i=n$).
- The first term is $\boldsymbol{u_0}$.
- The last term is $\boldsymbol{u_n}$.

The explicit formula for the last term is $u_n = u_0 + nd$.

$$\begin{aligned}\sum_{i=0}^{n} u_i &= \frac{\text{Number of terms}}{2}(\text{First term} + \text{Last term}) \\ &= \frac{n+1}{2}(u_0+u_n) \\ &= \frac{n+1}{2}(u_0+(u_0+nd)) \quad \text{(substituting the expression for } u_n\text{)} \\ &= \frac{n+1}{2}(2u_0+nd) \\ &= \frac{(n+1)(2u_0+nd)}{2}\end{aligned}$$

H SUM OF A GEOMETRIC SEQUENCE

H.1 CALCULATING THE SUM OF A GEOMETRIC SERIES: LEVEL 1

Ex 89: Calculate the sum of the following geometric sequence.

$$1+2+4+8+16+32 = \boxed{63}$$

Answer: We are asked to calculate the sum of a geometric sequence.

- **Method 1: Direct Addition**
  We can add the terms one by one:
  $$1+2+4+8+16+32 = \mathbf{63}$$
- **Method 2: Using the Geometric Series Formula**
  $$1+2+4+8+16+32 = 2^0+2^1+2^2+2^3+2^4+2^5$$
  This is a geometric sequence starting at $u_0 = 1$ with a common ratio $r = 2$. There are $n+1 = 6$ terms in total (from $n=0$ to $n=5$). We use the formula for the sum of a geometric sequence:
  $$S_n = u_0\frac{1-r^{n+1}}{1-r}$$
  Substituting the values:
  $$\begin{aligned}S_5 &= 1\times\frac{1-2^{5+1}}{1-2} \\ &= \frac{1-2^6}{-1} \\ &= \frac{1-64}{-1} \\ &= \frac{-63}{-1} \\ &= \mathbf{63}\end{aligned}$$

Ex 90: Calculate the sum of the following geometric sequence.

$$S = 6+12+24+48+96+192 = \boxed{378}$$

Answer: We are asked to calculate the sum of a geometric sequence.

- **Method 1: Direct Addition**
  We can add the terms one by one:
  $$6+12+24+48+96+192 = \mathbf{378}$$
- **Method 2: Using the Geometric Series Formula**
  $$6+12+24+48+96+192 = 6\cdot2^0+6\cdot2^1+6\cdot2^2+6\cdot2^3+6\cdot2^4+6\cdot2^5$$
  This is a geometric sequence starting at $u_0 = 6$ with a common ratio $r = 2$. There are $n+1 = 6$ terms in total (from $n=0$ to $n=5$). We use the formula for the sum of a geometric sequence:
  $$S_n = u_0\frac{1-r^{n+1}}{1-r}$$
  Substituting the values:
  $$\begin{aligned}S_5 &= 6\times\frac{1-2^{5+1}}{1-2} \\ &= 6\times\frac{1-2^6}{-1} \\ &= 6\times\frac{1-64}{-1} \\ &= 6\times\frac{-63}{-1} \\ &= 6\times 63 \\ &= \mathbf{378}\end{aligned}$$

Ex 91: Calculate the sum of the following geometric sequence.

$$S = 32+16+8+4+2 = \boxed{62}$$

Answer: We are asked to calculate the sum of a geometric sequence.

- **Method 1: Direct Addition**
We can add the terms one by one:

$$32 + 16 + 8 + 4 + 2 = \mathbf{62}$$

- **Method 2: Using the Geometric Series Formula**
First, we can express each term using the first term (32) and the common ratio (0.5):

$$32+16+8+4+2 = 32\cdot(0.5)^0+32\cdot(0.5)^1+32\cdot(0.5)^2+32\cdot(0.5)^3+32\cdot(0.5)^4$$

This is a geometric sequence starting at $u_0 = 32$ with a common ratio $r = 0.5$. There are $n + 1 = 5$ terms in total (from $n = 0$ to $n = 4$). We use the formula for the sum of a geometric sequence:

$$S_n = u_0 \frac{1 - r^{n+1}}{1 - r}$$

Substituting the values:

$$\begin{aligned} S_4 &= 32 \times \frac{1 - (0.5)^{4+1}}{1 - 0.5} \\ &= 32 \times \frac{1 - (0.5)^5}{0.5} \\ &= 32 \times \frac{1 - 0.03125}{0.5} \\ &= 32 \times \frac{0.96875}{0.5} \\ &= 32 \times 1.9375 \\ &= \mathbf{62} \end{aligned}$$

H.2 CALCULATING THE SUM OF A GEOMETRIC SERIES: LEVEL 2

Ex 92: Calculate the sum of the geometric sequence:

$$1 + 2 + 4 + 8 + \cdots + 2048 = \boxed{4095}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms**
The sequence is a geometric sequence with an initial term $u_0 = 1$ and a common ratio $r = 2/1 = 2$. The last term is $u_n = 2048$.
We use the explicit formula $u_n = u_0 \times r^n$ and solve for n:

$$\begin{aligned} 2048 &= 1 \times 2^n \\ 2048 &= 2^n \\ \log(2048) &= \log(2^n) \\ \log(2048) &= n \log(2) \\ n &= \frac{\log(2048)}{\log(2)} \\ n &= 11 \end{aligned}$$

Since the sequence starts at $n = 0$ and ends at $n = 11$, there are $\mathbf{11 + 1 = 12}$ terms in total.

- **Step 2: Apply the Geometric Series Formula**
Now we use the formula for the sum with $u_0 = 1$, $r = 2$, and $n + 1 = 12$ terms:

$$S_n = u_0 \frac{1 - r^{n+1}}{1 - r}$$

Substituting the values:

$$\begin{aligned} S_{11} &= 1 \times \frac{1 - 2^{12}}{1 - 2} \\ &= \frac{1 - 4096}{-1} \\ &= \frac{-4095}{-1} \\ &= \mathbf{4095} \end{aligned}$$

The sum of the sequence is 4,095.

Ex 93: Calculate the sum of the geometric sequence:

$$5 + 15 + 45 + \cdots + 3645 = \boxed{5465}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms (n)**
The sequence is a geometric sequence with a first term $u_1 = 5$ and a common ratio $r = 15/5 = 3$. The last term is $u_n = 3645$.We use the explicit formula $u_n = u_1 \times r^{n-1}$ and solve for n:

$$\begin{aligned} 3645 &= 5 \times 3^{n-1} \\ \frac{3645}{5} &= 3^{n-1} \\ 729 &= 3^{n-1} \\ \log(729) &= \log(3^{n-1}) \\ \log(729) &= (n-1)\log(3) \\ n - 1 &= \frac{\log(729)}{\log(3)} \\ n - 1 &= 6 \\ n &= 7 \end{aligned}$$

So, there are $\boldsymbol{n = 7}$ terms in this sequence.

- **Step 2: Apply the Geometric Series Formula**
Now we use the formula for the sum with $u_1 = 5$, $r = 3$, and $n = 7$ terms:

$$S_n = u_1 \frac{1 - r^n}{1 - r}$$

Substituting the values:

$$\begin{aligned} S_7 &= 5 \times \frac{1 - 3^7}{1 - 3} \\ &= 5 \times \frac{1 - 2187}{-2} \\ &= 5 \times \frac{-2186}{-2} \\ &= 5 \times 1093 \\ &= \mathbf{5465} \end{aligned}$$

The sum of the sequence is 5,465.

Ex 94: Calculate the sum of the geometric sequence:

$$100 + 50 + 25 + \cdots + 3.125 = \boxed{196.875}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms**
  The sequence is a geometric sequence with an initial term $u_0 = 100$ and a common ratio $r = 50/100 = 0.5$. The last term is $u_n = 3.125$.We use the explicit formula $u_n = u_0 \times r^n$ and solve for n:

$$\begin{aligned} 3.125 &= 100 \times (0.5)^n \\ \frac{3.125}{100} &= (0.5)^n \\ 0.03125 &= (0.5)^n \\ \log(0.03125) &= \log((0.5)^n) \\ \log(0.03125) &= n\log(0.5) \\ n &= \frac{\log(0.03125)}{\log(0.5)} \\ n &= 5 \end{aligned}$$

  Since the sequence starts at $n = 0$ and ends at $n = 5$, there are $\mathbf{5 + 1 = 6}$ terms in total.

- **Step 2: Apply the Geometric Series Formula**
  Now we use the formula for the sum with $u_0 = 100$, $r = 0.5$, and $n + 1 = 6$ terms:

$$S_n = u_0 \frac{1 - r^{n+1}}{1 - r}$$

  Substituting the values:

$$\begin{aligned} S_5 &= 100 \times \frac{1 - (0.5)^6}{1 - 0.5} \\ &= 100 \times \frac{1 - 0.015625}{0.5} \\ &= 100 \times \frac{0.984375}{0.5} \\ &= 100 \times 1.96875 \\ &= \mathbf{196.875} \end{aligned}$$

  The sum of the sequence is 196.875.

Ex 95: Calculate the sum of the geometric sequence:

$$10 + 20 + 40 + \cdots + 1280 = \boxed{2550}$$

Answer: To calculate the sum, we first need to determine the number of terms in the sequence.

- **Step 1: Find the number of terms (n)**
  The sequence is a geometric sequence with a first term $u_1 = 10$ and a common ratio $r = 20/10 = 2$. The last term is $u_n = 1280$.We use the explicit formula $u_n = u_1 \times r^{n-1}$ and solve for n:

$$\begin{aligned} 1280 &= 10 \times 2^{n-1} \\ \frac{1280}{10} &= 2^{n-1} \\ 128 &= 2^{n-1} \\ \log(128) &= \log(2^{n-1}) \\ \log(128) &= (n-1)\log(2) \\ n - 1 &= \frac{\log(128)}{\log(2)} \\ n - 1 &= 7 \\ n &= 8 \end{aligned}$$

  So, there are $\boldsymbol{n = 8}$ terms in this sequence.

- **Step 2: Apply the Geometric Series Formula**
  Now we use the formula for the sum with $u_1 = 10$, $r = 2$, and $n = 8$ terms:

$$S_n = u_1 \frac{1 - r^n}{1 - r}$$

  Substituting the values:

$$\begin{aligned} S_8 &= 10 \times \frac{1 - 2^8}{1 - 2} \\ &= 10 \times \frac{1 - 256}{-1} \\ &= 10 \times \frac{-255}{-1} \\ &= 10 \times 255 \\ &= \mathbf{2550} \end{aligned}$$

  The sum of the sequence is 2,550.

H.3 CALCULATING THE SUM OF A GEOMETRIC SERIES IN SIGMA NOTATION: LEVEL 1

Ex 96: Calculate the sum:

$$\sum_{i=0}^{5} 2^i = \boxed{63}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\sum_{i=0}^{5} 2^i = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 1 + 2 + 4 + 8 + 16 + 32$$

This is the sum of a geometric sequence.

- **Method 1: Direct Addition**
  We can add the expanded terms one by one:

$$1 + 2 + 4 + 8 + 16 + 32 = \mathbf{63}$$

- **Method 2: Using the Geometric Series Formula**
  This is a geometric sequence with $n + 1 = 6$ terms, a first term $u_0 = 1$, and a common ratio $r = 2$. We use the formula:

$$S_n = u_0 \frac{1 - r^{n+1}}{1 - r}$$

  Substituting the values:

$$\begin{aligned} S_5 &= 1 \times \frac{1 - 2^6}{1 - 2} \\ &= \frac{1 - 64}{-1} \\ &= \mathbf{63} \end{aligned}$$

Ex 97: Calculate the sum:

$$\sum_{i=1}^{4} 3 \times (-2)^{i-1} = \boxed{-15}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\begin{aligned} \sum_{i=1}^{4} 3 \times (-2)^{i-1} &= 3(-2)^0 + 3(-2)^1 + 3(-2)^2 + 3(-2)^3 \\ &= 3(1) + 3(-2) + 3(4) + 3(-8) \\ &= 3 - 6 + 12 - 24 \end{aligned}$$

This is the sum of a geometric sequence.

- **Method 1: Direct Addition**
  We can add the expanded terms one by one:

$$3 - 6 + 12 - 24 = \mathbf{-15}$$

- **Method 2: Using the Geometric Series Formula**
  This is a geometric sequence with $n = 4$ terms, a first term $u_1 = 3$, and a common ratio $r = -2$. We use the formula:

$$S_n = u_1 \frac{1 - r^n}{1 - r}$$

  Substituting the values:

$$\begin{aligned} S_4 &= 3 \times \frac{1 - (-2)^4}{1 - (-2)} \\ &= 3 \times \frac{1 - 16}{3} \\ &= 1 - 16 \\ &= \mathbf{-15} \end{aligned}$$

Ex 98: Calculate the sum:

$$\sum_{i=0}^{3} 32 \left(\frac{1}{2}\right)^i = \boxed{60}$$

Answer: First, let's expand the sum represented by the sigma notation:

$$\begin{aligned} \sum_{i=0}^{3} 32 \left(\frac{1}{2}\right)^i &= 32(\frac{1}{2})^0 + 32(\frac{1}{2})^1 + 32(\frac{1}{2})^2 + 32(\frac{1}{2})^3 \\ &= 32(1) + 32(0.5) + 32(0.25) + 32(0.125) \\ &= 32 + 16 + 8 + 4 \end{aligned}$$

This is the sum of a geometric sequence.

- **Method 1: Direct Addition**
  We can add the expanded terms one by one:

$$32 + 16 + 8 + 4 = \mathbf{60}$$

- **Method 2: Using the Geometric Series Formula**
  This is a geometric sequence with $n + 1 = 4$ terms, a first term $u_0 = 32$, and a common ratio $r = 0.5$. We use the formula:

$$S_n = u_0 \frac{1 - r^{n+1}}{1 - r}$$

  Substituting the values:

$$\begin{aligned} S_3 &= 32 \times \frac{1 - (0.5)^4}{1 - 0.5} \\ &= 32 \times \frac{1 - 0.0625}{0.5} \\ &= 32 \times \frac{0.9375}{0.5} \\ &= 32 \times 1.875 \\ &= \mathbf{60} \end{aligned}$$

H.4 CALCULATING THE SUM OF A GEOMETRIC SERIES IN SIGMA NOTATION: LEVEL 2

Ex 99: Calculate the sum:

$$\sum_{i=1}^{15} 3 \times 2^{i-1} = \boxed{98301}$$

Answer: First, let's identify the properties of the sum from the sigma notation:

$$\begin{aligned} \sum_{i=1}^{15} 3 \times 2^{i-1} &= 3 \times 2^0 + 3 \times 2^1 + \cdots + 3 \times 2^{14} \\ &= 3 + 6 + 12 + \cdots + 49152 \end{aligned}$$

This is the sum of a geometric sequence.

- **Method 1: Direct Addition**
  There are 15 terms in this series, making direct addition very time-consuming and prone to error. Using the formula is the only efficient method.
- **Method 2: Using the Geometric Series Formula**
  This is a geometric sequence with the following properties:
  - Number of terms: $\boldsymbol{n = 15}$
  - First term ($i = 1$): $\boldsymbol{u_1 = 3 \times 2^{1-1} = 3 \times 2^0 = 3}$
  - Common ratio: $\boldsymbol{r = 2}$

  We use the formula for the sum of a geometric sequence starting at u_1:

$$S_n = u_1 \frac{1 - r^n}{1 - r}$$

  Substituting the values:

$$\begin{aligned} S_{15} &= 3 \times \frac{1 - 2^{15}}{1 - 2} \\ &= 3 \times \frac{1 - 32768}{-1} \\ &= 3 \times \frac{-32767}{-1} \\ &= 3 \times 32767 \\ &= \mathbf{98301} \end{aligned}$$

Ex 100: Calculate the sum (round to two decimal places):

$$\sum_{i=0}^{10} 100 \times (0.8)^i \approx \boxed{457.05}$$

Answer: First, let's identify the properties of the sum from the sigma notation:

$$\begin{aligned} \sum_{i=0}^{10} 100 \times (0.8)^i &= 100(0.8)^0 + 100(0.8)^1 + \cdots + 100(0.8)^{10} \\ &= 100 + 80 + 64 + \ldots \end{aligned}$$

This is the sum of a geometric sequence.

- **Method 1: Direct Addition**
  There are 11 terms in this series. While possible, direct addition is time-consuming. Using the formula is more efficient.

- **Method 2: Using the Geometric Series Formula**
This is a geometric sequence with the following properties:
  - Number of terms: $\boldsymbol{n+1=10-0+1=11}$
  - First term ($i=0$): $\boldsymbol{u_0=100\times(0.8)^0=100}$
  - Common ratio: $\boldsymbol{r=0.8}$

  We use the formula for the sum of a geometric sequence starting at u_0:
$$S_n=u_0\frac{1-r^{n+1}}{1-r}$$
  Substituting the values:
$$\begin{aligned}S_{10}&=100\times\frac{1-(0.8)^{11}}{1-0.8}\\&=100\times\frac{1-0.085899...}{0.2}\\&=100\times\frac{0.914100...}{0.2}\\&=100\times4.570502...\\&\approx\mathbf{457.05}\end{aligned}$$

Ex 101: Calculate the sum:
$$\sum_{i=1}^{12}4\times(-2)^{i-1}=\boxed{-5460}$$

Answer: First, let's identify the properties of the sum from the sigma notation:
$$\begin{aligned}\sum_{i=1}^{12}4\times(-2)^{i-1}&=4(-2)^0+4(-2)^1+4(-2)^2+\cdots+4(-2)^{11}\\&=4-8+16-\dots\end{aligned}$$
This is the sum of a geometric sequence.

- **Method 1: Direct Addition**
There are 12 terms in this series, making direct addition tedious. Using the formula is more efficient.
- **Method 2: Using the Geometric Series Formula**
This is a geometric sequence with the following properties:
  - Number of terms: $\boldsymbol{n=12}$
  - First term ($i=1$): $\boldsymbol{u_1=4\times(-2)^{1-1}=4\times(-2)^0=4}$
  - Common ratio: $\boldsymbol{r=-2}$

  We use the formula for the sum of a geometric sequence starting at u_1:
$$S_n=u_1\frac{1-r^n}{1-r}$$
  Substituting the values:
$$\begin{aligned}S_{12}&=4\times\frac{1-(-2)^{12}}{1-(-2)}\\&=4\times\frac{1-4096}{3}\\&=4\times\frac{-4095}{3}\\&=4\times(-1365)\\&=\mathbf{-5460}\end{aligned}$$

I SUM OF AN INFINITE GEOMETRIC SERIES

I.1 DETERMINING CONVERGENCE OF GEOMETRIC SERIES

MCQ 102: Does the following infinite geometric series converge or diverge?
$$S=2+6+18+54+\dots$$
☐ Converges

☒ Diverges

Answer: The common ratio is $r=\frac{6}{2}=3$.
For a series to converge, the condition $|r|<1$ must be met.
Since $|3|\geq1$, this series **diverges**.

MCQ 103: Does the following infinite geometric series converge or diverge?
$$S=100-50+25-12.5+\dots$$
☒ Converges

☐ Diverges

Answer: The common ratio is $r=\frac{-50}{100}=-0.5$.
For a series to converge, the condition $|r|<1$ must be met.
Since $|-0.5|=0.5<1$, this series **converges**.

MCQ 104: Does the following infinite geometric series converge or diverge?
$$S=1-2+4-8+\dots$$
☐ Converges

☒ Diverges

Answer: The common ratio is $r=\frac{-2}{1}=-2$.
For a series to converge, the condition $|r|<1$ must be met.
Since $|-2|=2\geq1$, this series **diverges**.

MCQ 105: Does the following infinite geometric series converge or diverge?
$$S=7+7+7+7+\dots$$
☐ Converges

☒ Diverges

Answer: The common ratio is $r=\frac{7}{7}=1$.
For a series to converge, the condition $|r|<1$ must be met.
Since $|1|\geq1$, this series **diverges**.

I.2 CALCULATING THE SUM OF AN INFINITE GEOMETRIC SERIES

Ex 106: Find the sum:
$$16+8+4+2+\cdots=\boxed{32}$$

Answer: To find the sum of an infinite geometric series, we first need to identify the first term and the common ratio.

- The first term is $\boldsymbol{u_1 = 16}$.
- The common ratio is $\boldsymbol{r} = \frac{u_2}{u_1} = \frac{8}{16} = \mathbf{0.5}$.

An infinite geometric series has a finite sum if and only if $|r| < 1$.
Since $|0.5| < 1$, the series converges.
We use the formula for the sum to infinity:

$$S_\infty = \frac{u_1}{1-r}$$

Substituting the values:

$$\begin{aligned} S_\infty &= \frac{16}{1-0.5} \\ &= \frac{16}{0.5} \\ &= \mathbf{32} \end{aligned}$$

Ex 107: Find the sum of the following infinite geometric series:

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots = \boxed{1}$$

Answer: To find the sum of an infinite geometric series, we first need to identify the first term and the common ratio.

- The first term is $\boldsymbol{u_1 = \frac{1}{2}}$.
- The common ratio is $\boldsymbol{r} = \frac{u_2}{u_1} = \frac{1/4}{1/2} = \boldsymbol{\frac{1}{2}}$.

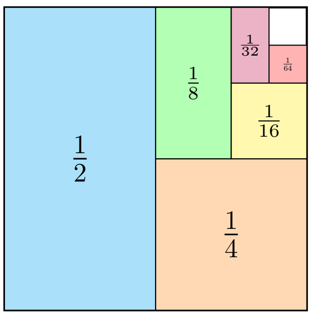
An infinite geometric series has a finite sum if and only if $|r| < 1$.
Since $|\frac{1}{2}| < 1$, the series converges.
We use the formula for the sum to infinity:

$$S_\infty = \frac{u_1}{1-r}$$

Substituting the values:

$$\begin{aligned} S_\infty &= \frac{\frac{1}{2}}{1-\frac{1}{2}} \\ &= \frac{\frac{1}{2}}{\frac{1}{2}} \\ &= \mathbf{1} \end{aligned}$$

This result can be visualized by dividing a square of area 1. Each term in the series fills exactly half of the remaining area, and the sum of all the terms perfectly fills the entire square.



Ex 108: Find the sum of the following infinite geometric series:

$$27 - 9 + 3 - 1 + \tfrac{1}{3} - \cdots = \boxed{\frac{81}{4}}$$

Answer: To find the sum of an infinite geometric series, we first need to identify the first term and the common ratio.

- The first term is $\boldsymbol{u_1 = 27}$.
- The common ratio is $\boldsymbol{r} = \frac{u_2}{u_1} = \frac{-9}{27} = \boldsymbol{-\frac{1}{3}}$.

An infinite geometric series has a finite sum if and only if $|r| < 1$.
Since $|-\frac{1}{3}| = \frac{1}{3} < 1$, the series converges.
We use the formula for the sum to infinity:

$$S_\infty = \frac{u_1}{1-r}$$

Substituting the values:

$$\begin{aligned} S_\infty &= \frac{27}{1-(-\frac{1}{3})} \\ &= \frac{27}{1+\frac{1}{3}} \\ &= \frac{27}{\frac{4}{3}} \\ &= 27 \times \frac{3}{4} \\ &= \mathbf{\frac{81}{4}} \quad \text{(or 20.25)} \end{aligned}$$