SCALAR PRODUCT

The scalar product, also known as the dot product, is a fundamental operation in vector algebra. It is used in various fields such as physics (for example, when computing work or projections) and in mathematics (for finding angles between vectors and testing orthogonality).

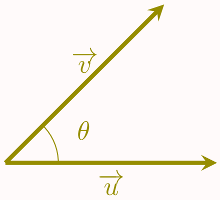
A DEFINITION

Definition Geometric Scalar Product

The **scalar product** of two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as:

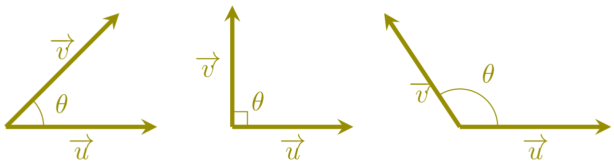
$$\overrightarrow{u} \cdot \overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\| \cos\theta$$

where θ is the angle between the vectors.



$\overrightarrow{u} \cdot \overrightarrow{v}$ is read as "$\overrightarrow{u}$ dot $\overrightarrow{v}$".
The scalar product is a real number (a scalar), not a vector. It is:

- **positive** if $0 < \theta < \frac{\pi}{2}$ (acute angle);
- **zero** if $\theta = \frac{\pi}{2}$ (orthogonal vectors), or if at least one of the vectors is the zero vector;
- **negative** if $\frac{\pi}{2} < \theta \leq \pi$ (obtuse angle).

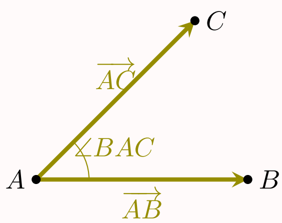


Definition Scalar Product with Point Notation

The **scalar product** of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = AB \times AC \times \cos(\angle BAC),$$

where AB and AC denote the lengths of the segments $[AB]$ and $[AC]$.



Ex: Given $AB = 2$, $AC = 5$, and $\angle BAC = \frac{\pi}{4}$, calculate $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

Answer:

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{AC} &= AB \times AC \times \cos(\angle BAC) \\ &= 2 \times 5 \times \cos\left(\frac{\pi}{4}\right) \\ &= 10 \times \frac{\sqrt{2}}{2} \\ &= 5\sqrt{2}\end{aligned}$$

Definition Analytic Scalar Product

For vectors in $\mathbb{R}^2$ with coordinates $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the **scalar product** is defined by:

$$\vec{u} \cdot \vec{v} = xx' + yy'.$$

For vectors in $\mathbb{R}^3$ with coordinates $\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$, this generalises to:

$$\vec{u} \cdot \vec{v} = xx' + yy' + zz'.$$

B PROPERTIES

Proposition Square of a vector

$$\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$$

Proposition Symmetry

The scalar product is commutative:

$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}.$$

Proposition Bilinearity

The scalar product is linear with respect to each of its vector arguments.

- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributivity in the second argument);
- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ (distributivity in the first argument);
- $\vec{u} \cdot (k\vec{v}) = k(\vec{u} \cdot \vec{v})$ and $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$, where k is a real number.

Proposition Notable Identities

- $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$.
- $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$.
- $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

Proposition Polarization Identity

$$\vec{u} \cdot \vec{v} = \frac{1}{2}\left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2\right).$$

C GEOMETRICAL INTERPRETATIONS

Proposition Collinearity Condition

Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors.

- If $\vec{u}$ and $\vec{v}$ are collinear and point in the same direction, then $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|$.
- If $\vec{u}$ and $\vec{v}$ are collinear and point in opposite directions, then $\vec{u} \cdot \vec{v} = -\|\vec{u}\|\|\vec{v}\|$.

Definition Orthogonal Vectors

Two vectors $\vec{u}$ and $\vec{v}$ are said to be **orthogonal** if their scalar product is zero:

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0.$$

Proposition Perpendicular Lines

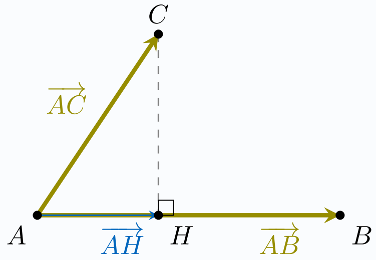
Two lines (AB) and (CD) are perpendicular if and only if the scalar product of their direction vectors is zero:

$$(AB) \perp (CD) \iff \overrightarrow{AB} \cdot \overrightarrow{CD} = 0.$$

Proposition Orthogonal Projection

Let A, B and C be three points. Let H be the orthogonal projection of point C onto the line $\overleftrightarrow{AB}$. Then:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AB} \cdot \overrightarrow{AH}.$$



This implies:

- If $\overrightarrow{AB}$ and $\overrightarrow{AH}$ have the same direction, $\overrightarrow{AB} \cdot \overrightarrow{AC} = AB \times AH$.
- If $\overrightarrow{AB}$ and $\overrightarrow{AH}$ have opposite directions, $\overrightarrow{AB} \cdot \overrightarrow{AC} = -AB \times AH$.

D MIDPOINT THEOREM AND APPLICATIONS

Proposition Midpoint Theorem

Given two points A and B and their midpoint I (i.e. I is the midpoint of $\overline{AB}$), for any point M in the plane, we have:

$$\overrightarrow{MA} \cdot \overrightarrow{MB} = MI^2 - \frac{1}{4}AB^2.$$

Proposition Circle defined by Orthogonality

The set of all points M in the plane such that $\overrightarrow{MA} \cdot \overrightarrow{MB} = 0$ is the circle with diameter $\overline{AB}$.