# SCALAR PRODUCT

The scalar product, also known as the dot product, is a fundamental operation in vector algebra. It is used in various fields such as physics (for example, when computing work or projections) and in mathematics (for finding angles between vectors and testing orthogonality).

A DEFINITION

Discover: The concept of the scalar product provides a way to multiply two vectors to obtain a scalar (a real number). This operation can be viewed from two fundamental perspectives: a geometric one, involving the angle between the vectors, and an analytical one, using their coordinates. A key result in vector algebra, derived from the Law of Cosines, is that these two perspectives are entirely consistent.

Let's explore this by considering two vectors, $\overrightarrow{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, with an angle θ between them.
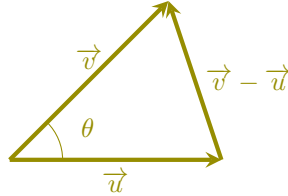
The Law of Cosines applied to the triangle formed by $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{v} - \overrightarrow{u}$ states:

$$\|\overrightarrow{v} - \overrightarrow{u}\|^2 = \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 - 2\|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$

By expanding the squared norms in terms of coordinates:

- $\|\overrightarrow{u}\|^2 = x^2 + y^2$

- $\|\overrightarrow{v}\|^2 = x'^2 + y'^2$

- $\|\overrightarrow{v} - \overrightarrow{u}\|^2 = (x' - x)^2 + (y' - y)^2$ since $\overrightarrow{v} - \overrightarrow{u} = \begin{pmatrix} x' - x \\ y' - y \end{pmatrix}$.

Substituting these into the Law of Cosines:

$$(x' - x)^2 + (y' - y)^2 = (x^2 + y^2) + (x'^2 + y'^2) - 2\|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$
$$x'^2 - 2xx' + x^2 + y'^2 - 2yy' + y^2 = x^2 + y^2 + x'^2 + y'^2 - 2\|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$
$$-2xx' - 2yy' = -2\|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$
$$xx' + yy' = \|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$
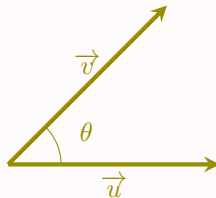
This fundamental equality shows that the analytical expression $xx' + yy'$ is equivalent to the geometric expression $\|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$. This allows us to define the scalar product in two equivalent ways.

> **Definition Geometric Scalar Product**
>
> The **scalar product** of two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as:
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> $$\overrightarrow{u} \cdot \overrightarrow{v} = \|\overrightarrow{u}\|\|\overrightarrow{v}\|\cos\theta$$
>
> where θ is the angle between the vectors.
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>

$\overrightarrow{u} \cdot \overrightarrow{v}$ is read as "$\overrightarrow{u}$ dot $\overrightarrow{v}$".

The scalar product is a real number (a scalar), not a vector. It is:

- **positive** if $0 < \theta < \frac{\pi}{2}$ (acute angle);

- **zero** if $\theta = \frac{\pi}{2}$ (orthogonal vectors), or if at least one of the vectors is the zero vector;

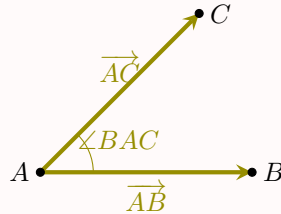- **negative** if $\frac{\pi}{2} < \theta \leq \pi$ (obtuse angle).
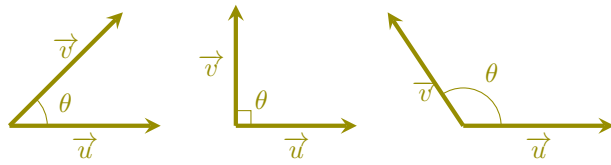
Definition Scalar Product with Point Notation

The **scalar product** of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is given by:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = AB \times AC \times \cos(\angle BAC),$$

where AB and AC denote the lengths of the segments $[AB]$ and $[AC]$.

Ex: Given $AB = 2$, $AC = 5$, and $\angle BAC = \frac{\pi}{4}$, calculate $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

Answer:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = AB \times AC \times \cos(\angle BAC)$$
$$= 2 \times 5 \times \cos\left(\frac{\pi}{4}\right)$$
$$= 10 \times \frac{\sqrt{2}}{2}$$
$$= 5\sqrt{2}$$

Definition Analytic Scalar Product

For vectors in $\mathbb{R}^2$ with coordinates $\overrightarrow{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the **scalar product** is defined by:

$$\overrightarrow{u} \cdot \overrightarrow{v} = xx' + yy'.$$

For vectors in $\mathbb{R}^3$ with coordinates $\overrightarrow{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\overrightarrow{v} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$, this generalises to:

$$\overrightarrow{u} \cdot \overrightarrow{v} = xx' + yy' + zz'.$$

B PROPERTIES

Proposition Square of a vector

$$\overrightarrow{u} \cdot \overrightarrow{u} = \|\overrightarrow{u}\|^2$$

Proof

Let $\overrightarrow{u} = \begin{pmatrix} x \\ y \end{pmatrix}$. Using the analytical definition:

$$\overrightarrow{u} \cdot \overrightarrow{u} = x \cdot x + y \cdot y$$
$$= x^2 + y^2$$
$$= (\sqrt{x^2 + y^2})^2$$
$$= \|\overrightarrow{u}\|^2$$

The scalar product is commutative:
$$\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}.$$

Proof

Let $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} x' \\ y' \end{pmatrix}$. Using the analytical definition and commutativity of real multiplication:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= xx' + yy' \\ &= x'x + y'y \\ &= \vec{v} \cdot \vec{u}. \end{aligned}$$

Proposition **Bilinearity**

The scalar product is linear with respect to each of its vector arguments.

- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (distributivity in the second argument);
- $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ (distributivity in the first argument);
- $\vec{u} \cdot (k\vec{v}) = k(\vec{u} \cdot \vec{v})$ and $(k\vec{u}) \cdot \vec{v} = k(\vec{u} \cdot \vec{v})$, where k is a real number.

Proposition **Notable Identities**

- $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\,\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$.
- $\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\,\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$.
- $(\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = \|\vec{u}\|^2 - \|\vec{v}\|^2$.

Proof

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$$\begin{aligned} \|\vec{u} + \vec{v}\|^2 &= (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v}) \\ &= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \quad \text{(distributivity)} \\ &= \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 \qquad \text{(symmetry and } \vec{u} \cdot \vec{u} = \|\vec{u}\|^2) \end{aligned}$$

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$$\begin{aligned} \|\vec{u} - \vec{v}\|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 - 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 \end{aligned}$$

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$$\begin{aligned} (\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} - \vec{v} \cdot \vec{v} \\ &= \|\vec{u}\|^2 - \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{v} - \|\vec{v}\|^2 \quad \text{(symmetry)} \\ &= \|\vec{u}\|^2 - \|\vec{v}\|^2 \end{aligned}$$

Proposition **Polarization Identity**

$$\vec{u} \cdot \vec{v} = \frac{1}{2} \left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2 \right).$$

Proof

This identity is derived directly from the notable identities. Starting from the expansion of $\|\vec{u} + \vec{v}\|^2$:

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2.$$

We can rearrange the equation to solve for $\vec{u} \cdot \vec{v}$:

$$2(\vec{u} \cdot \vec{v}) = \|\vec{u} + \vec{v}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2.$$

Dividing by 2 gives the identity:

$$\vec{u} \cdot \vec{v} = \frac{1}{2} \left(\|\vec{u} + \vec{v}\|^2 - \|\vec{u}\|^2 - \|\vec{v}\|^2 \right).$$

Equivalently, using $\|\vec{u} - \vec{v}\|^2$, one can also show that

$$\vec{u} \cdot \vec{v} = \frac{1}{2} \left(\|\vec{u}\|^2 + \|\vec{v}\|^2 - \|\vec{u} - \vec{v}\|^2 \right).$$

C GEOMETRICAL INTERPRETATIONS

Proposition **Collinearity Condition**

Let $\vec{u}$ and $\vec{v}$ be two non-zero vectors.

- If $\vec{u}$ and $\vec{v}$ are collinear and point in the same direction, then $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|$.
- If $\vec{u}$ and $\vec{v}$ are collinear and point in opposite directions, then $\vec{u} \cdot \vec{v} = -\|\vec{u}\|\|\vec{v}\|$.

Proof

This follows directly from the geometric definition $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$.

- If the vectors are collinear and have the same direction, the angle θ between them is 0. Since $\cos(0) = 1$, the result is $\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\|$.
- If they are collinear and have opposite directions, the angle θ is π. Since $\cos(\pi) = -1$, the result is $\vec{u} \cdot \vec{v} = -\|\vec{u}\|\|\vec{v}\|$.

Definition **Orthogonal Vectors**

Two vectors $\vec{u}$ and $\vec{v}$ are said to be **orthogonal** if their scalar product is zero:

$$\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = 0.$$

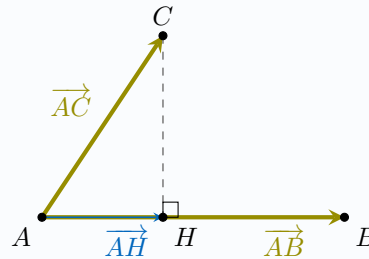### Proposition **Perpendicular Lines**

Two lines (AB) and (CD) are perpendicular if and only if the scalar product of their direction vectors is zero:

$$(AB) \perp (CD) \iff \overrightarrow{AB} \cdot \overrightarrow{CD} = 0.$$

Proposition **Orthogonal Projection**

Let A, B and C be three points. Let H be the orthogonal projection of point C onto the line $\overleftrightarrow{AB}$. Then:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AB} \cdot \overrightarrow{AH}.$$

This implies:

- If $\overrightarrow{AB}$ and $\overrightarrow{AH}$ have the same direction, $\overrightarrow{AB} \cdot \overrightarrow{AC} = AB \times AH$.
- If $\overrightarrow{AB}$ and $\overrightarrow{AH}$ have opposite directions, $\overrightarrow{AB} \cdot \overrightarrow{AC} = -AB \times AH$.

Proof

Using Chasles's relation, we can write $\overrightarrow{AC} = \overrightarrow{AH} + \overrightarrow{HC}$:

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AB} \cdot (\overrightarrow{AH} + \overrightarrow{HC})$$
$$= \overrightarrow{AB} \cdot \overrightarrow{AH} + \overrightarrow{AB} \cdot \overrightarrow{HC}.$$

Since H is the orthogonal projection of C on (AB), the vectors $\overrightarrow{AB}$ and $\overrightarrow{HC}$ are orthogonal. Therefore, their scalar product is zero: $\overrightarrow{AB} \cdot \overrightarrow{HC} = 0$. This leaves us with

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \overrightarrow{AB} \cdot \overrightarrow{AH}.$$

The second part of the proposition follows from the collinearity condition applied to $\overrightarrow{AB}$ and $\overrightarrow{AH}$.

D MIDPOINT THEOREM AND APPLICATIONS

> **Proposition Midpoint Theorem**
>
> Given two points A and B and their midpoint I (i.e. I is the midpoint of $\overline{AB}$), for any point M in the plane, we have:
> $$\overrightarrow{MA} \cdot \overrightarrow{MB} = MI^2 - \frac{1}{4}AB^2.$$
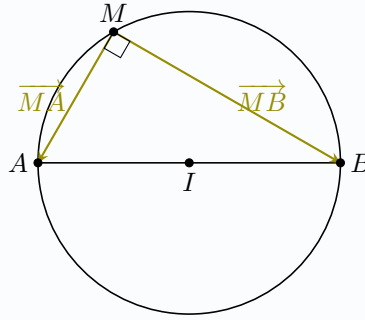
Proof

Using Chasles's relation, we introduce the midpoint I:

$$\begin{aligned}
\overrightarrow{MA} \cdot \overrightarrow{MB} &= (\overrightarrow{MI} + \overrightarrow{IA}) \cdot (\overrightarrow{MI} + \overrightarrow{IB}) \\
&= (\overrightarrow{MI} + \overrightarrow{IA}) \cdot (\overrightarrow{MI} - \overrightarrow{IA}) \quad \text{(since I is the midpoint, $\overrightarrow{IB} = -\overrightarrow{IA}$)} \\
&= MI^2 - IA^2 \quad \text{(difference of squares identity)} \\
&= MI^2 - \left(\frac{1}{2}AB\right)^2 \\
&= MI^2 - \frac{1}{4}AB^2 \quad \text{(since $IA = \frac{1}{2}AB$)}
\end{aligned}$$

> **Proposition Circle defined by Orthogonality**
>
> The set of all points M in the plane such that $\overrightarrow{MA} \cdot \overrightarrow{MB} = 0$ is the circle with diameter $\overline{AB}$.
>
>

Proof

We apply the Midpoint Theorem, which states $\overrightarrow{MA} \cdot \overrightarrow{MB} = MI^2 - \frac{1}{4}AB^2$. The condition for orthogonality is $\overrightarrow{MA} \cdot \overrightarrow{MB} = 0$. By substituting, we get:

$$\begin{aligned}
MI^2 - \frac{1}{4}AB^2 &= 0 \\
MI^2 &= \frac{1}{4}AB^2 \\
MI &= \sqrt{\frac{1}{4}AB^2} \\
MI &= \frac{1}{2}AB.
\end{aligned}$$

This result means the distance from any point M to the midpoint I is constant and equal to half the length of the segment $\overline{AB}$. This is the definition of the circle with center I and diameter $\overline{AB}$.