SAMPLING AND CONFIDENCE INTERVALS

A STATISTICAL MODELING

In most real-world statistical problems, we are interested in understanding the properties of a large population. However, it is often impossible or impractical to collect data from every single member of that population. As a result, we typically do not know the true population parameters, such as the population mean (μ) or the population standard deviation (σ). Instead, we collect data from a smaller subset of the population, called a **sample**. We calculate statistics from this sample (like the sample mean $\bar{x}$) and use them to estimate the unknown population parameters. This process is known as **statistical inference**. In this section, we will define the key concepts used in sampling and estimation.

Definition Sample

A **sample** of size n consists of n independent random variables $X_1, X_2, \ldots, X_n$ that follow the same probability distribution as the population.

Definition Observed Value

An **observed value**, denoted by a lowercase x, is a specific realization of a random variable X. It is the actual number obtained after performing an experiment or collecting data.

Definition Sample Mean Estimator

The **sample mean** is a statistic used to estimate the population mean.
Let $X_1, X_2, \ldots, X_n$ be a random sample. The sample mean is a random variable denoted by $\overline{X}_n$:

$$\overline{X}_n = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

For a specific set of observed values $x_1, x_2, \ldots, x_n$, the calculated sample mean is denoted by $\bar{x}$:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Proposition Unbiased Estimator of the Mean

The sample mean is an **unbiased estimator** of the population mean μ. This means that the expected value of the sample mean is equal to the true population mean:

$$E[\overline{X}_n] = \mu$$

Definition Sample Standard Deviation

The **sample standard deviation**, denoted by s_n (or simply s), is an estimator of the population standard deviation σ. For observed values $x_1, x_2, \ldots, x_n$:

$$s_n = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}}$$

Note the division by $n-1$ (Bessel's correction), which ensures that the associated sample variance s_n^2 is an unbiased estimator of the population variance.

Proposition Unbiased Estimator of the Variance

The sample variance $S_n^2 = \frac{\sum (X_i - \overline{X}_n)^2}{n-1}$ is an unbiased estimator of the population variance σ^2:

$$E[S_n^2] = \sigma^2$$

Ex: Survey: Do you like Mathematics? For an education survey, 10 students rate how much they like mathematics on a scale of 0 to 10.Let $X_1, X_2, \ldots, X_{10}$ be the random variables representing the rating of each student.The observed values are: $x = \{2, 4, 0, 9, 10, 3, 7, 2, 8, 9\}$.

- **Sample Mean:**

$$\bar{x} = \frac{2+4+0+9+10+3+7+2+8+9}{10} = \frac{54}{10} = 5.4$$

- **Sample Standard Deviation:** Using a calculator (List statistics):

$$s_n \approx 3.50$$

B CONFIDENCE INTERVALS FOR MEANS WITH KNOWN VARIANCE

A point estimate, like the sample mean $\bar{x}$, provides a single value as an estimate of the population parameter. However, it does not tell us how precise this estimate is. Due to sampling variability, $\bar{x}$ is rarely exactly equal to the true mean μ.
To address this, we use a **confidence interval**. A confidence interval provides a range of values within which we expect the true population parameter to lie, with a certain level of confidence (probability). In this section, we assume that the population variance σ^2 is known, which allows us to use the standard normal distribution (Z).

Proposition Probability Interval

Assume we take a sample of size n from a population with mean μ and standard deviation σ, and that n is sufficiently large (typically $n \geq 30$ so that the Central Limit Theorem applies). Then

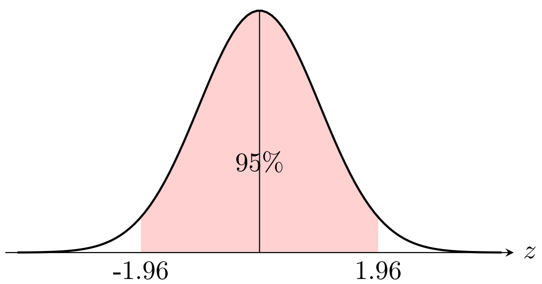
$$P\left(\overline{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95.$$

Proof

Since n is large, the Central Limit Theorem applies. The sampling distribution of the mean $\overline{X}_n$ is approximately normal with mean μ and standard deviation $\frac{\sigma}{\sqrt{n}}$.
The standardized variable Z follows a standard normal distribution:

$$Z = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

For a 95% confidence level, we seek the critical value z such that $P(-z \leqslant Z \leqslant z) = 0.95$.



Using a calculator (Inverse Normal), we find $z \approx 1.96$.

$$P\left(-1.96 \leqslant \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \leqslant 1.96\right) = 0.95$$
$$P\left(-1.96\frac{\sigma}{\sqrt{n}} \leqslant \overline{X}_n - \mu \leqslant 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95$$
$$P\left(\overline{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \leqslant \mu \leqslant \overline{X}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95.$$

This is the stated probability interval.

The proposition above gives us a probability statement about the *random* sample mean $\overline{X}_n$ and the random interval built from it. The parameter μ is fixed (but unknown); what is random is the interval itself.
To calculate a specific confidence interval in practice, we estimate this probability interval by replacing the random variable $\overline{X}_n$ with the observed sample mean $\bar{x}$. Thus, the confidence interval we compute is an **estimate** of the theoretical interval.

Method Calculating the Confidence Interval (Mean)

1. **Identify the statistics:** Find the sample mean $\bar{x}$, the known population standard deviation σ, and the sample size n.
2. **Find the z-score:** Determine z based on the confidence level (1.645 for 90%, 1.96 for 95%, 2.576 for 99%).
3. **Calculate the margin of error:** $E = z\frac{\sigma}{\sqrt{n}}$.
4. **Write the interval:** $[\bar{x} - E, \bar{x} + E]$.

Ex: A sample of 60 rabbits was taken from a forest. The sample mean weight of the rabbits was 950 grams. Assume the population standard deviation is known to be $\sigma = 200$ grams.
Find the 95% confidence interval for the population mean weight.

Answer:

1. **Statistics:** $n = 60$, $\bar{x} = 950$, and $\sigma = 200$.
2. **z-score:** For 95%, $z = 1.96$.
3. **Margin of error:**
$$E = 1.96 \times \frac{200}{\sqrt{60}} \approx 1.96 \times 25.82 \approx 50.6.$$
4. **Interval:**
$$[950 - 50.6,\ 950 + 50.6] = [899.4, 1000.6].$$

We are 95% confident that the true mean weight is between 899.4g and 1000.6g.

C CONFIDENCE INTERVALS FOR MEANS WITH UNKNOWN VARIANCE

In most practical situations, the population standard deviation σ is unknown. We cannot use the normal distribution (Z) with a known σ, because we must estimate σ using the sample standard deviation s_n.
In theory, when the population is normal and σ is unknown, the exact sampling distribution of

$$\frac{\overline{X}_n - \mu}{S_n/\sqrt{n}}$$

is a **Student's t-distribution** with $n-1$ degrees of freedom. For large samples (say $n \geq 30$), the t-distribution is very close to the standard normal distribution. In this course, for large n, we will approximate by using the same z-values as before and replacing σ with s_n.

Method Calculating the Confidence Interval (Unknown σ, Large n)

1. **Identify the statistics:** Find the sample mean $\bar{x}$, the sample standard deviation s_n (as an estimate for σ), and the sample size n.
2. **Find the z-score:** Determine z based on the confidence level (1.645 for 90%, 1.96 for 95%, 2.576 for 99%), provided n is large.
3. **Calculate the margin of error:** $E = z\frac{s_n}{\sqrt{n}}$.
4. **Write the interval:** $[\bar{x} - E, \bar{x} + E]$.

Ex: An economist studying fuel costs wants to estimate the mean price of gasoline in her state. She takes a random sample of 40 gas stations and finds a sample mean price of $\bar{x} = \$1.29$ with a sample standard deviation of $s_n = \$0.10$.
Find the 95% confidence interval for the population mean.

Answer:

1. **Statistics:** $n = 40$, $\bar{x} = 1.29$, and $s_n = 0.10$. (Since $n \geq 30$, we approximate $\sigma \approx s_n$.)
2. **z-score:** For 95%, $z = 1.96$.
3. **Margin of error:**
$$E = 1.96 \times \frac{0.10}{\sqrt{40}} \approx 1.96 \times 0.0158 \approx 0.031.$$
4. **Interval:**
$$[1.29 - 0.031,\ 1.29 + 0.031] = [1.259, 1.321].$$

We are 95% confident that the true mean price is between \$1.259 and \$1.321.

D CONFIDENCE INTERVALS FOR PROPORTIONS

When dealing with categorical data (like voting for Candidate A vs Candidate B), we are interested in the **population proportion** p (the true percentage of votes). Since we cannot ask everyone, we estimate p using the **sample proportion** $\hat{p}$.
Each individual response (success/failure) is a Bernoulli random variable with variance $p(1-p)$ and standard deviation $\sigma = \sqrt{p(1-p)}$.
Since the sample proportion $\hat{p}$ is the mean of n Bernoulli variables, its standard deviation is $\frac{\sigma}{\sqrt{n}}$.
Therefore, the standard error for a proportion is

$$\sqrt{\frac{p(1-p)}{n}},$$

which we estimate using the sample data as

$$\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}.$$

This normal approximation works well when the sample size is large, typically when $n\hat{p} \geq 5$ and $n(1-\hat{p}) \geq 5$.

Method Constructing the Interval

1. **Calculate the sample proportion:** $\hat{p} = \frac{\text{Successes}}{\text{Total Sample}}$.
2. **Find the z-score:** Determine z based on the confidence level (1.645 for 90%, 1.96 for 95%, 2.576 for 99%).
3. **Calculate the margin of error:** $E = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$.
4. **Write the interval:** $[\hat{p} - E, \hat{p} + E]$.

Ex: A polling institute surveys $n = 1000$ random voters before an election. 520 people say they will vote for Candidate A.

1. Calculate the sample proportion $\hat{p}$.
2. Construct a 95% confidence interval for the true proportion of voters supporting Candidate A.
3. Based on this interval, can Candidate A be certain of winning (obtaining more than 50% of votes)? Explain.

Answer:

1. $\hat{p} = \frac{520}{1000} = 0.52$.
2. Using the 95% confidence level ($z = 1.96$):

$$\begin{aligned} E &= 1.96\sqrt{\frac{0.52(1-0.52)}{1000}} \\ &= 1.96\sqrt{\frac{0.52 \times 0.48}{1000}} \\ &= 1.96\sqrt{0.0002496} \\ &= 1.96(0.0158) \\ &\approx 0.031. \end{aligned}$$

   The confidence interval is:

$$[0.52 - 0.031,\ 0.52 + 0.031] = [0.489, 0.551]$$

   (Or 48.9% to 55.1%).

3. No. Although the sample proportion is 52%, the interval includes values less than 0.5 (e.g., 0.489). Therefore, it is plausible that the true proportion is below 50%. The election is too close to call with certainty.

E HYPOTHESIS TESTING USING CONFIDENCE INTERVALS

Confidence intervals can be used as a tool for hypothesis testing. If someone claims that the population mean is a specific value (μ_H), we can check if this value is "plausible" by seeing whether it falls within our calculated confidence interval.
For a two-sided test at significance level α (e.g. 5%), the $(1-\alpha)$ confidence interval (e.g. 95%) gives the same decision as the corresponding hypothesis test $H_0 : \mu = \mu_H$ versus $H_1 : \mu \neq \mu_H$.

Method Hypothesis Test with CI

To test a claim that the population mean is μ_H at a significance level α (e.g., 5%):

1. Construct the corresponding $(1-\alpha)$ confidence interval (e.g., 95%) for μ based on sample data.
2. **Decision rule:**
   - If μ_H is **inside** the interval, we **do not reject** the claim (the claim is plausible).
   - If μ_H is **outside** the interval, we **reject** the claim (the result is statistically significant at level α).

Ex: A machine is set to fill juice bottles with an average of 50cl. A quality control inspector takes a sample of 36 bottles and finds an average content of 48.5cl with a standard deviation of 5cl.
Test the claim that the machine average is still 50cl at the 5% significance level, using a 95% confidence interval for μ.

Answer: Sample size $n = 36$, $\bar{x} = 48.5$, $s_n = 5$. Since n is reasonably large, we use a normal approximation with $z = 1.96$.We construct the 95% confidence interval for the true mean μ:

$$\begin{aligned} CI &= 48.5 \pm 1.96 \frac{5}{\sqrt{36}} \\ &= 48.5 \pm 1.96 \left(\frac{5}{6}\right) \\ &= 48.5 \pm 1.633 \\ &= [46.87, 50.13]. \end{aligned}$$

Conclusion: The claimed value $\mu_H = 50$ lies **inside** the confidence interval $[46.87, 50.13]$. Therefore, we **do not reject** the claim at the 5% level. There is not enough evidence to say the machine is malfunctioning based on this sample.

F DETERMINING SAMPLE SIZE

Before conducting a study, researchers often need to know how many data points to collect to achieve a desired level of precision. By manipulating the formula for the margin of error, we can solve for the required sample size n (for example, when estimating a mean with known standard deviation σ).

Proposition Sample Size Formula

To estimate a population mean within a margin of error E with a specific confidence level (corresponding to z), when σ is known, the required sample size is:

$$n = \left(\frac{z\sigma}{E}\right)^2.$$

Note: Since n must be an integer, always **round up** to the next whole number.

Proof

Starting from the margin of error definition:

$$E = z\frac{\sigma}{\sqrt{n}}$$

we rearrange:

$$\begin{aligned} \sqrt{n}E &= z\sigma \\ \sqrt{n} &= \frac{z\sigma}{E} \\ n &= \left(\frac{z\sigma}{E}\right)^2. \end{aligned}$$

Ex: A marketing firm wants to estimate the average spending of students during Spring Break. They want the estimate to be within \$120 of the true mean with 90% confidence. A pilot study suggests the standard deviation is $\sigma = \$400$. How many students should be sampled?

Answer: Given:

- Margin of error $E = 120$
- Standard deviation $\sigma = 400$
- Confidence level 90% $\implies z \approx 1.645$ (from calculator or tables)

Calculation:

$$n = \left(\frac{1.645 \times 400}{120}\right)^2 \approx 30.07.$$

Since we cannot survey 0.07 of a student, we round up.
Result: A sample of size $n = 31$ is required.