ROOTS

A SQUARE ROOTS

Definition Square root

The **square root** of a non-negative number a (that is, $a \geq 0$), written as $\sqrt{a}$, is the **non-negative number** that, when multiplied by itself, gives a.

$$\left(\sqrt{a}\right)^2 = a$$

Note

- The square root symbol $\sqrt{}$ always asks for the **positive** root. For example, $\sqrt{25} = 5$. It is a common mistake to think that $\sqrt{25}$ is ± 5.
  While it's true that both $5^2 = 25$ and $(-5)^2 = 25$, the symbol $\sqrt{25}$ refers only to the positive solution, which is 5.
- **Why can't we take the square root of a negative number (in the real numbers)?**
  Consider $\sqrt{-9}$. To find this value, we need a number that, when multiplied by itself, gives -9.
  - A positive number squared is positive ($3 \times 3 = 9$).
  - A negative number squared is also positive ($-3 \times -3 = 9$).

  No real number, when squared, can result in a negative number. Therefore, we cannot find the square root of a negative number in the set of real numbers.

Definition Perfect Squares

A **perfect square** is an integer that is the square of another integer. The square root of a perfect square is an integer.

Ex: The first few perfect squares are:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$

Their square roots are:

$$\sqrt{1} = 1, \quad \sqrt{4} = 2, \quad \sqrt{9} = 3, \quad \sqrt{16} = 4, \quad \ldots$$

Definition Simplest Radical Form

A radical is written in **simplest form** if the number under the square root sign is as small as possible.

B CALCULATING SQUARE ROOTS

While the square roots of perfect squares are easy to find, most numbers are not perfect squares. We can estimate their square roots or use a calculator for a more precise value.

Method Use a calculator

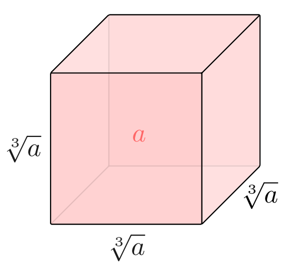
On most calculators, you can find a square root using the $\boxed{\sqrt{}}$ button.

Ex: Use a calculator to find $\sqrt{10}$, rounded to 2 decimal places.

Answer: Entering $\sqrt{10}$ into a calculator gives approximately $3.162277\ldots$
Rounded to 2 decimal places, $\sqrt{10} \approx 3.16$.

C NTH ROOTS

Just as the square root of a number is the side length of a square with a given area, the **cube root** $\sqrt[3]{a}$ is the side length of a cube with volume a.



It is the inverse operation of **cubing** a number (raising it to the power of 3).

Definition Cube Root

The **cube root** of a real number a, written $\sqrt[3]{a}$, is the **(real) number** which, when cubed, gives a:

$$\left(\sqrt[3]{a}\right)^3 = a$$

Note Unlike square roots, cube roots are defined for **all real numbers**, including negatives. For example, $\sqrt[3]{-27} = -3$ because $(-3) \times (-3) \times (-3) = -27$.

Ex: Find $\sqrt[3]{125}$.

Answer: $\sqrt[3]{125} = 5$ because $5 \times 5 \times 5 = 125$.

Definition Nth Root

The concept of a root can be generalized. For a positive integer n, the **nth root** of a real number a, written $\sqrt[n]{a}$, is the number which, when raised to the power of n, gives a:

$$\left(\sqrt[n]{a}\right)^n = a.$$

The rules for their domains depend on whether the index n is even or odd.

- **Even roots** (e.g. $\sqrt[2]{a}, \sqrt[4]{a}, \ldots$): an even root of a negative number is not a real number. For even n, $\sqrt[n]{a}$ is only defined for $\boldsymbol{a \geq 0}$.
- **Odd roots** (e.g. $\sqrt[3]{a}, \sqrt[5]{a}, \ldots$): an odd root is defined for all real numbers a.

D LAWS OF RADICALS

Expressions involving square roots are called **radical expressions** (or simply **radicals**). To simplify and manipulate these expressions, we use a set of important laws.

Proposition Multiplication Law

For any real numbers $a, b \geq 0$:

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

Ex: Show that $\sqrt{4} \times \sqrt{9} = \sqrt{36}$.

Answer:

$$\begin{aligned}\sqrt{4} \times \sqrt{9} &= \sqrt{4 \times 9}\\ &= \sqrt{36}\\ &= 6\end{aligned}$$

Proposition Square Root of a Square

For any real number $a \geq 0$:

$$\sqrt{a^2} = a$$

Ex: Find $\sqrt{25}$.

Answer:

$$\begin{aligned}\sqrt{25} &= \sqrt{5^2}\\ &= 5\end{aligned}$$

Proposition Simplifying Law

For any real numbers $a, b \geq 0$:

$$\sqrt{a^2 b} = a\sqrt{b}$$

Ex: Simplify $\sqrt{12}$.

Answer: First, find the largest perfect square factor of 12, which is 4.

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3}\\ &= \sqrt{2^2 \times 3}\\ &= 2\sqrt{3}\end{aligned}$$

Proposition Division Law

For any real numbers $a \geq 0$ and $b > 0$:

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Ex: Simplify $\sqrt{\frac{9}{16}}$.

Answer:

$$\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$$

E ALGEBRAIC OPERATIONS WITH RADICALS

- We can only add or subtract radicals if they are **like radicals**, which means they have the exact same number under the root sign.
  Think of it like algebra: you can simplify $2x + 4x$ to $6x$, but you cannot simplify $2x + 4y$. In the same way, you can simplify $2\sqrt{3} + 4\sqrt{3}$, but you cannot simplify $2\sqrt{3} + 4\sqrt{5}$.
- When multiplying expressions with radicals, we use the same rules as in algebra, such as the distributive law for expanding brackets, together with the radical laws we have just seen.

Method Algebraic Operations

We can perform operations with radicals (square roots) in a similar way to ordinary numbers, provided we respect the radical laws. In particular:

- We can add and subtract **like** radicals (i.e. the same number under the root) in the same way that we add and subtract like algebraic terms:
  $$c\sqrt{a} + d\sqrt{a} = (c + d)\sqrt{a}$$
- We can use the usual rules for expanding brackets (such as the distributive law).

Ex: Simplify: $2\sqrt{3} + 4\sqrt{3}$

Answer: Just like $2x + 4x = 6x$, we can add the coefficients of the like radical:

$$2\sqrt{3} + 4\sqrt{3} = (2 + 4)\sqrt{3} = \mathbf{6\sqrt{3}}$$

Ex: Expand and simplify $\sqrt{3}(5 - \sqrt{3})$.

$$\begin{aligned}\sqrt{3}(5 - \sqrt{3}) &= (\sqrt{3} \times 5) - (\sqrt{3} \times \sqrt{3}) \\ &= 5\sqrt{3} - 3\end{aligned}$$

F RATIONALIZING THE DENOMINATOR

In mathematics, it is standard practice to write expressions without radicals in the denominator. The process of removing a radical from the denominator is called **rationalizing the denominator**. This does not change the value of the expression but converts it to a standard form which is often easier to work with.

Method Rationalizing a Monomial Denominator

To rationalize a denominator of the form $\sqrt{a}$ (where $a > 0$), multiply the numerator and the denominator by $\sqrt{a}$.

$$\frac{b}{\sqrt{a}} = \frac{b}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{b\sqrt{a}}{a}$$

This works because multiplying by $\frac{\sqrt{a}}{\sqrt{a}}$ is equivalent to multiplying by 1, which does not change the value.

Ex: Rationalize the denominator of $\frac{6}{\sqrt{2}}$.

Answer: We multiply the numerator and denominator by $\sqrt{2}$.

$$\begin{aligned}\frac{6}{\sqrt{2}} &= \frac{6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{6\sqrt{2}}{2} \\ &= \mathbf{3\sqrt{2}}\end{aligned}$$

When the denominator is a binomial containing a square root, such as $a+\sqrt{b}$, we use a special tool called the **conjugate** to rationalize it.

Definition Conjugate

The **conjugate** of a binomial expression $x+y$ is $x-y$.

Ex:

- The conjugate of $a+\sqrt{b}$ is $\boldsymbol{a-\sqrt{b}}$.
- The conjugate of $a-\sqrt{b}$ is $\boldsymbol{a+\sqrt{b}}$.

The power of the conjugate lies in its product, which follows the difference of squares identity: $(x+y)(x-y)=x^2-y^2$.

$$(a+\sqrt{b})(a-\sqrt{b}) = a^2-(\sqrt{b})^2 = a^2-b$$

The result is a rational number, which achieves our goal.

Method Rationalizing a Binomial Denominator

To rationalize a binomial denominator, multiply the numerator and the denominator by the **conjugate** of the denominator.

Ex: Rationalize the denominator of $\frac{4}{3+\sqrt{5}}$.

Answer: The denominator is $3+\sqrt{5}$. Its conjugate is $3-\sqrt{5}$. We multiply the numerator and denominator by this conjugate.

$$\begin{aligned}\frac{4}{3+\sqrt{5}} &= \frac{4}{(3+\sqrt{5})} \times \frac{(3-\sqrt{5})}{(3-\sqrt{5})} \\ &= \frac{4(3-\sqrt{5})}{3^2-(\sqrt{5})^2} && \text{(Using } (x+y)(x-y)=x^2-y^2\text{)} \\ &= \frac{12-4\sqrt{5}}{9-5} \\ &= \frac{12-4\sqrt{5}}{4} \\ &= \frac{4(3-\sqrt{5})}{4} && \text{(Factor out 4 to simplify)} \\ &= \mathbf{3-\sqrt{5}}\end{aligned}$$