RIGHT-TRIANGLE TRIGONOMETRY

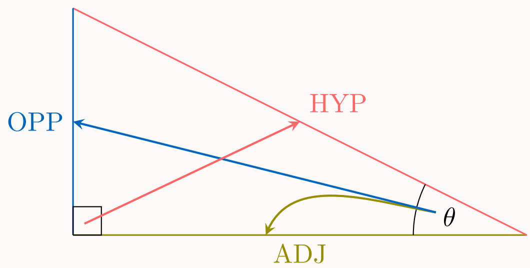
Trigonometry is a branch of mathematics that studies the relationships between the side lengths and angles of triangles, especially right-angled triangles. It is widely used in science, engineering, astronomy, architecture, and even video game development. In this chapter, we focus on three main trigonometric ratios: sine, cosine, and tangent.

A SIDES OF A RIGHT-ANGLED TRIANGLE

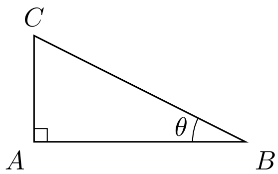
Definition Right-Angled Triangle Sides

A **right-angled triangle** has one 90° angle. We name the sides *relative to a chosen acute angle* θ (not the right angle).

- The **Hypotenuse (HYP)** is the longest side, always opposite the right angle. It does *not* depend on the choice of θ.
- The **Opposite (OPP)** side is directly across from the angle θ; it does not touch θ.
- The **Adjacent (ADJ)** side is the leg next to the angle θ; it touches θ but is not the hypotenuse.



Ex: In the triangle below, the angle θ is at vertex B. Identify the hypotenuse, the adjacent side, and the opposite side relative to angle θ.

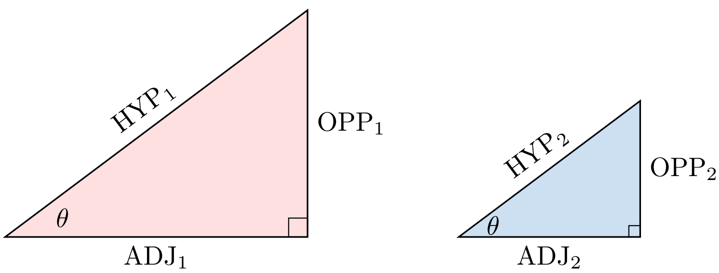


Answer: Relative to angle θ at B:

- Hypotenuse: $\overline{BC}$
- Adjacent side: $\overline{AB}$
- Opposite side: $\overline{AC}$

B TRIGONOMETRIC RATIOS

The Foundation of Trigonometric Ratios Why do trigonometric ratios work? The answer lies in the properties of similar triangles.
Consider any two right-angled triangles that share a common acute angle, θ.



Both triangles have a right angle (90°) and both share the angle θ. Because they have two corresponding angles that are equal, the triangles are **similar** by the **Angle-Angle (AA) similarity criterion**.

A fundamental property of similar triangles is that the ratios of their corresponding sides are equal. This means that for any right-angled triangle with angle θ:

$$\frac{\text{OPP}_1}{\text{HYP}_1} = \frac{\text{OPP}_2}{\text{HYP}_2}, \quad \frac{\text{ADJ}_1}{\text{HYP}_1} = \frac{\text{ADJ}_2}{\text{HYP}_2}, \quad \frac{\text{OPP}_1}{\text{ADJ}_1} = \frac{\text{OPP}_2}{\text{ADJ}_2}.$$

Since these ratios are constant for any given angle θ, regardless of the size of the triangle, we can give them special names. These are the **trigonometric ratios**.

Definition The Three Trigonometric Ratios

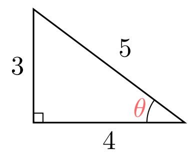
For an angle θ in a right-angled triangle, we define the three main trigonometric ratios: **sine**, **cosine**, and **tangent**.

$$\sin(\theta) = \frac{\text{Opposite}}{\text{Hypotenuse}}, \quad \cos(\theta) = \frac{\text{Adjacent}}{\text{Hypotenuse}}, \quad \tan(\theta) = \frac{\text{Opposite}}{\text{Adjacent}}.$$

A common mnemonic to remember these is **SOH-CAH-TOA**:

- **S**in = **O**pposite / **H**ypotenuse
- **C**os = **A**djacent / **H**ypotenuse
- **T**an = **O**pposite / **A**djacent

Ex: In the triangle below, find $\cos\theta$, $\sin\theta$, and $\tan\theta$.



Answer: Relative to θ at B:

- Hypotenuse: $BC = 5$
- Adjacent side: $AB = 4$
- Opposite side: $AC = 3$

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{4}{5},$$
$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{3}{5},$$
$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{3}{4}.$$

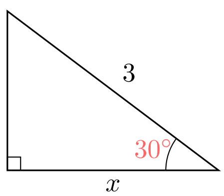
Proposition Tangent Formula

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Method Using Calculator

Trigonometric ratios for any angle can be calculated using a scientific calculator in **degree mode**. Always check that your calculator is set to "DEG" (degrees) before calculating.

Ex: In the triangle below, find x.



Answer: Relative to the angle $30°$ at B, the side of length 3 is the hypotenuse and the side x is adjacent to the angle.

$$\begin{aligned}\cos\theta &= \frac{\text{ADJ}}{\text{HYP}}\\ \cos(30°) &= \frac{x}{3}\\ x &= 3\times\cos(30°)\\ x &\approx 3\times 0.866\\ x &\approx 2.6\,\text{cm}\end{aligned}$$

C INVERSE TRIGONOMETRIC FUNCTIONS

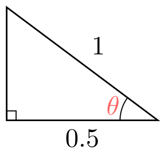
Trigonometric ratios can be used to find unknown angles in right-angled triangles when at least two side lengths are known.

Definition Inverse Trigonometric Functions

In a right-angled triangle with an angle θ:

$$\theta = \cos^{-1}\left(\frac{\text{ADJ}}{\text{HYP}}\right),\quad \theta = \sin^{-1}\left(\frac{\text{OPP}}{\text{HYP}}\right),\quad \theta = \tan^{-1}\left(\frac{\text{OPP}}{\text{ADJ}}\right).$$

Ex: In the triangle below, find the angle θ.



Answer: We know the lengths of the adjacent side ($AB = 0.5$) and the hypotenuse ($BC = 1$) relative to θ. We can use the inverse cosine function:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\text{ADJ}}{\text{HYP}}\right)\\ &= \cos^{-1}\left(\frac{0.5}{1}\right)\\ &= 60°.\end{aligned}$$

D SOLVING REAL-WORLD TRIGONOMETRY PROBLEMS

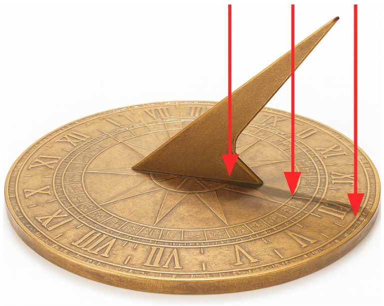
Trigonometric ratios are powerful tools for solving a wide range of problems involving right-angled triangles, especially in real-world contexts. To solve these problems effectively, follow the structured steps below:

Method Solving Real-World Trigonometry Problems

- **Draw a clear diagram** representing the situation described in the problem.
- **Label the unknown** (side or angle) you need to find. Use x for a side and θ for an angle if possible.
- **Identify a right-angled triangle** within your diagram.
- **Write an equation** relating an angle and two sides of the triangle using the appropriate trigonometric ratio.
- **Solve the equation** to find the unknown value.
- **State your answer clearly**, including appropriate units, in a complete sentence.

E ANGLE BETWEEN A LINE AND A PLANE

When the sun shines on the **gnomon** of a sundial, it casts a shadow onto the dial beneath it. If the sun's rays are perpendicular to the dial, the shadow formed is the **projection** of the gnomon onto the dial. This concept of projection is key when defining the angle between a line and a plane in three dimensions.



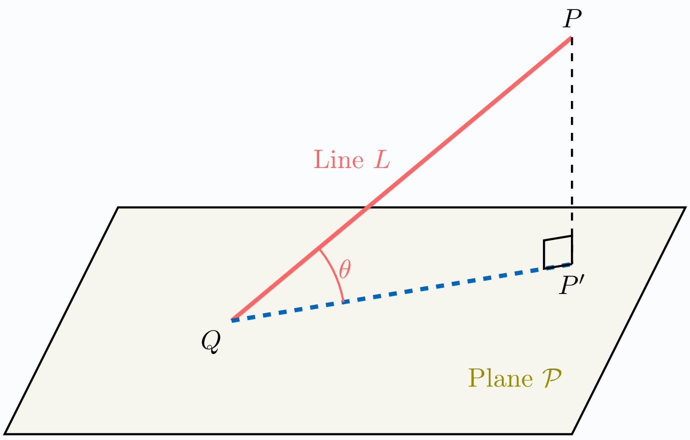
Definition Angle Between a Line and a Plane

The **angle between a line and a plane** is defined as the *acute* angle (between $0°$ and $90°$) between the line and its orthogonal projection on the plane. If the line lies in the plane, this angle is $0°$.

Method Finding the Angle Between a Line and a Plane

To find the angle θ between a line L and a plane $\mathcal{P}$, proceed as follows (assuming L meets $\mathcal{P}$ at a point Q):

1. **Find the projection** of the line onto the plane.Choose a point P on the line L that is not on the plane. Drop a perpendicular from P to the plane, and call the foot of this perpendicular P'. The line passing through Q and P' is the projection of L onto the plane $\mathcal{P}$.
2. **Form a right-angled triangle** using the original line segment QP, its projection QP', and the perpendicular segment PP'. The right angle is at P'.
3. **Use trigonometry (SOH-CAH-TOA)** to calculate the acute angle θ at Q between the original line L (segment QP) and its projection in the plane (segment QP').



In the diagram, the angle between the line L and the plane $\mathcal{P}$ is the acute angle θ at Q.