REFERENCE FUNCTIONS

Introduction In mathematics, **reference functions** are fundamental building blocks that help us understand more complex relationships. This chapter explores the following functions:

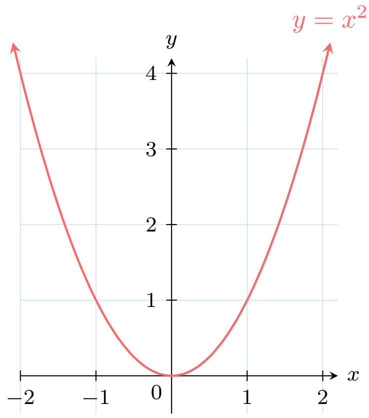
- **Square Function**: $f(x) = x^2$
- **Square Root Function**: $f(x) = \sqrt{x}$
- **Cube Function**: $f(x) = x^3$
- **Inverse Function**: $f(x) = \frac{1}{x}$

For each function, we will investigate its definition and equation, its graph and properties and Real-world examples.

A SQUARE FUNCTION

Definition square function

The **square function** is given by $f(x) = x^2$. This means that each input value x is multiplied by itself to give the output.



- **Domain**: All real numbers ($\mathbb{R}$)
- **Shape**: A **parabola** opening upwards.

Proposition Properties

- For any real number x, $x^2 \geqslant 0$.
- The square function is strictly decreasing on $(-\infty, 0]$ and strictly increasing on $[0, +\infty)$.

x	$-\infty$	0	$+\infty$
x^2	↘	0	↗

- The square function is even (its graph is symmetric with respect to the y-axis).

Proof

Let $f(x) = x^2$.

- The product of two real numbers with the same sign is positive, so $x^2 = x \times x$ is positive.
- $$\begin{aligned} f(-x) &= (-x)^2 \\ &= (-x) \times (-x) \\ &= x^2 \\ &= f(x). \end{aligned}$$

  Thus, f is even.

- – Let a and b be two real numbers such that $0 \leqslant a < b$.

$$\begin{aligned} f(b) - f(a) &= b^2 - a^2 \\ &= (b-a)(b+a) \\ &> 0 \qquad \text{since } (b-a) > 0 \text{ and } (b+a) > 0. \end{aligned}$$

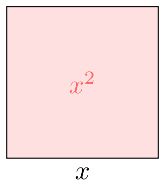
  Thus, $f(a) < f(b)$. Therefore, f is strictly increasing on $[0, +\infty)$.

  – Let a and b be two real numbers such that $a < b \leqslant 0$.

$$\begin{aligned} f(b) - f(a) &= b^2 - a^2 \\ &= (b-a)(b+a) \\ &< 0 \qquad \text{car } (b-a) > 0 \text{ and } (b+a) < 0. \end{aligned}$$

  Thus, $f(a) > f(b)$. Therefore, f is strictly decreasing on $(-\infty, 0]$.

Ex: A square has a side length x meters. Its area is given by $A(x) = x^2$.

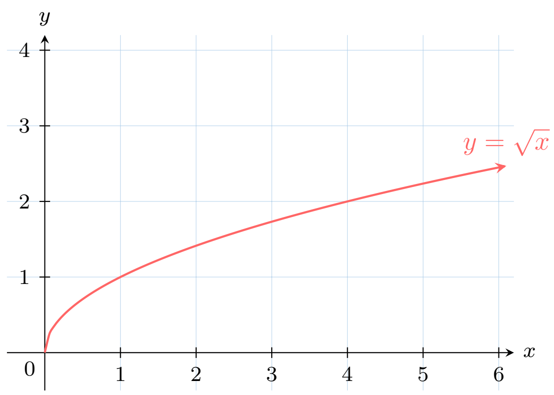


For example, for $x = 4$ meters, $A(4) = 4^2 = 16$ square meters.

B SQUARE ROOT FUNCTION

Definition Square Root Function

The **square root function** is given by $f(x) = \sqrt{x}$. It is the inverse of the square function, where the output is the non-negative value that, when squared, gives the input.



- **Domain**: $[0, +\infty)$ (non-negative real numbers)
- **Shape**: A curve that increases rapidly for small x and more slowly as x grows.

Proposition Properties

The square root function is strictly increasing on $[0, +\infty)$.

x	0	$+\infty$
$\sqrt{x}$	0 ↗	

Proof

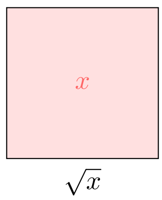
Let $f(x) = \sqrt{x}$.

Let a and b be two real numbers such that $0 \leqslant a < b$.

$$\begin{aligned} f(b) - f(a) &= \sqrt{b} - \sqrt{a} \\ &= \frac{(\sqrt{b} - \sqrt{a})(\sqrt{a} + \sqrt{b})}{\sqrt{a} + \sqrt{b}} \\ &= \frac{\left(\sqrt{b}\right)^2 - (\sqrt{a})^2}{\sqrt{a} + \sqrt{b}} \\ &= \frac{b - a}{\sqrt{a} + \sqrt{b}} \\ &> 0 \qquad \text{since } b - a > 0 \text{ and } \sqrt{a} + \sqrt{b} > 0. \end{aligned}$$

Thus, $f(a) < f(b)$.
The square root function is strictly increasing on $[0, +\infty)$.

Ex: Let a square have an area of x square meters. The length of the side of the square is $l(x) = \sqrt{x}$.

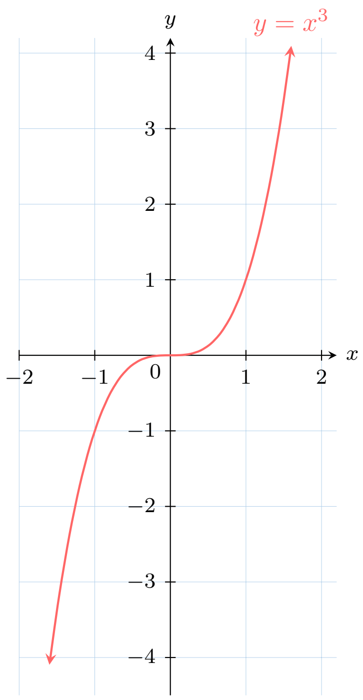


For example, for $x = 25$ square meters, $l(x) = \sqrt{25} = 5$ meters.

C CUBE FUNCTION

Definition Cube Function

The **cube function** is given by $f(x) = x^3$.



Domain: All real numbers ($\mathbb{R}$)

Proposition Properties

- The cube function is strictly increasing on $(-\infty, +\infty)$.

x	$-\infty$ $\qquad\qquad$ $+\infty$
x^3	↗

- The cube function is odd (its graph is symmetric symmetric about the origin).

Proof

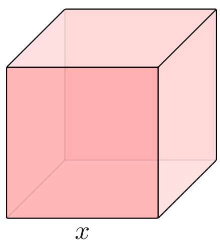
Let $f(x) = x^3$.

- $$\begin{aligned} f(-x) &= (-x)^3 \\ &= (-x) \times (-x) \times (-x) \\ &= -(x \times x \times x) \\ &= -x^3 \\ &= -f(x). \end{aligned}$$

  Thus, f is odd.

- The proof of the variation of the cube function will be done using the derivative tool.

Ex: The volume of a cube with a side length x meters is $V(x) = x^3$.

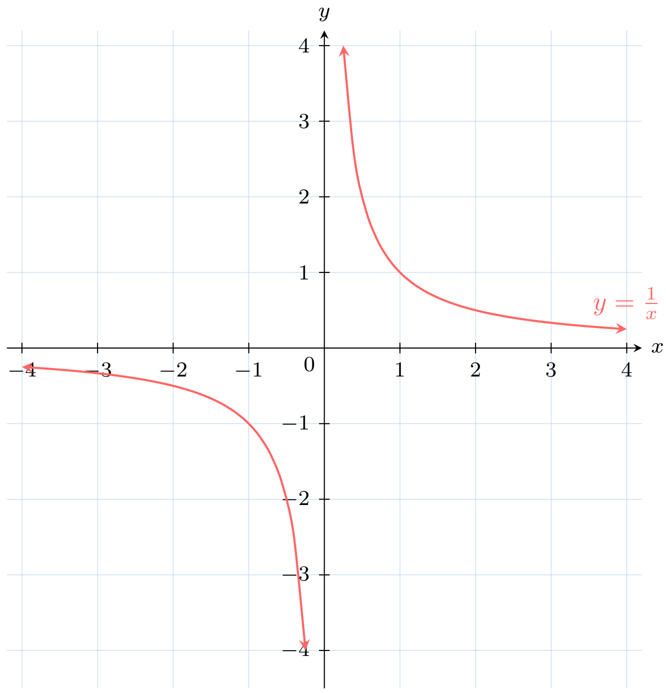


For example, for $x = 3$ meters, $V(3) = 3^3 = 27$ cubic meters.

D INVERSE FUNCTION

Definition Inverse Function

The **inverse Function** is given by $f(x) = \dfrac{1}{x}$. It represents a reciprocal relationship, where the output is the reciprocal of the input.



- **Domain**: $\mathbb{R}^*$ $(x \neq 0)$
- **Shape**: a **hyperbola**.

Proposition Properties

- The inverse function is strictly decreasing on $(-\infty, 0)$ and strictly decreasing on $(0, +\infty)$.

x	$-\infty$	0	$+\infty$
$\dfrac{1}{x}$	↘	‖	↘

- The inverse function is odd (its graph is symmetric symmetric about the origin).

Proof

Let $f(x) = \frac{1}{x}$

- $$\begin{aligned} f(-x) &= \frac{1}{-x} \\ &= -\frac{1}{x} \\ &= -f(x). \end{aligned}$$

  Thus, f is odd.

- – Let a and b be two real numbers such that $0 < a < b$.

$$\begin{aligned} f(b) - f(a) &= \frac{1}{b} - \frac{1}{a} \\ &= \frac{a-b}{ba} \\ &< 0 \qquad \text{since } a - b < 0 \text{ and } ba > 0. \end{aligned}$$

  Thus, $f(a) > f(b)$.
  Therefore, f is strictly decreasing on $(0, +\infty)$.

  – Similarly, the proof is identical to show that f is strictly decreasing on $(-\infty, 0)$.