REASONING AND PROOF

A LOGICAL CONNECTIVES AND PROPOSITIONS

A.1 PROPOSITION

A.1.1 DETERMINING TRUTH VALUES

MCQ 1: State the truth value of the following proposition:

$$1 = 0$$

☐ True

☐ False

MCQ 2: State the truth value of the following proposition:

$$5 - 2 = 3$$

☐ True

☐ False

MCQ 3: State the truth value of the following proposition:

"9 is a prime number."

☐ True

☐ False

MCQ 4: State the truth value of the following proposition:

"The derivative of x^3 is $3x^2$."

☐ True

☐ False

MCQ 5: State the truth value of the following proposition:

$$(x+y)^2 = x^2 + y^2 \quad \text{for all } x, y \in \mathbb{R}$$

☐ True

☐ False

MCQ 6: State the truth value of the following proposition:

$$\mathbb{N} \subset \mathbb{Z}$$

(The set of natural numbers is a subset of the set of integers.)

☐ True

☐ False

MCQ 7: State the truth value of the following proposition:

$$\int \ln(x)\,dx = \frac{1}{x} + C$$

☐ True

☐ False

A.1.2 DEDUCING TRUTH VALUES

MCQ 8: Given that the proposition "$x > 5$" is True, state the truth value of the proposition "$x > 2$".

☐ True

☐ False

☐ Cannot be determined

MCQ 9: Given that the proposition "n is a prime number" is True, state the truth value of the proposition "n is an odd number".

☐ True

☐ False

☐ Cannot be determined

MCQ 10: Given that the proposition "$ABCD$ is a rhombus" is True, state the truth value of the proposition "$ABCD$ is a square".

☐ True

☐ False

☐ Cannot be determined

MCQ 11: Given that the proposition "x is a multiple of 4" is True, state the truth value of the proposition "x is a multiple of 2".

☐ True

☐ False

☐ Cannot be determined

A.2 NEGATION

A.2.1 FINDING THE NEGATION OF A PROPOSITION

MCQ 12: Find the negation of the proposition "$x = 3$".

☐ $x = -3$

☐ $x \neq 3$

☐ $x < 3$

☐ $x > 3$

MCQ 13: Find the negation of the proposition "$x \geq 0$".

☐ $x > 0$

☐ $x < 0$

☐ $x \leq 0$

☐ $x \neq 0$

MCQ 14: Find the negation of the proposition "$x < -2$".

☐ $x > -2$

☐ $x \leq -2$

☐ $x = -2$

☐ $x \geq -2$

MCQ 15: Find the negation of the proposition "All students in the class have black hair."

☐ "No students in the class have black hair."

☐ "All students in the class do not have black hair."

☐ "There is at least one student in the class who does not have black hair."

MCQ 16: Find the negation of the proposition "$x \in \mathbb{Q}$" (x is a rational number).

☐ "x is an integer."

☐ "x is a real number."

☐ "x is an irrational number."

☐ "x is a natural number."

A.2.2 DEDUCING TRUTH VALUES

MCQ 17: Given that the proposition "$x = 1$" is False, state the truth value of the proposition "$x \neq 1$".

☐ True

☐ False

☐ Cannot be determined

MCQ 18: Given that the proposition "All students succeeded in the exam" is False, state the truth value of the proposition "At least one student did not succeed in the exam".

☐ True

☐ False

☐ Cannot be determined

MCQ 19: Given that the proposition "$x \geq 1$" is False, state the truth value of the proposition "$x = 0$".

☐ True

☐ False

☐ Cannot be determined

A.3 COMPOUND PROPOSITIONS

A.3.1 EVALUATING COMPOUND PROPOSITIONS

MCQ 20: Let p be the proposition "$(-2)^2 = 4$" and q be the proposition "$-2 < -3$". What is the truth value of the conjunction $p \wedge q$?

☐ True

☐ False

☐ Cannot be determined

MCQ 21: Let p be the proposition "$\sqrt{9} = 3$" and q be the proposition "A square has 5 sides". What is the truth value of the disjunction $p \vee q$?

☐ True

☐ False

☐ Cannot be determined

MCQ 22: Let p be the proposition "π is a rational number" and q be the proposition "$5 < 4$". What is the truth value of the conjunction $p \wedge q$?

☐ True

☐ False

☐ Cannot be determined

MCQ 23: Let p be "A triangle has three sides" and q be "$-1 > 0$". What is the truth value of the disjunction $p \vee q$?

☐ True

☐ False

☐ Cannot be determined

A.3.2 NEGATING CONJUNCTIONS AND DISJUNCTIONS

MCQ 24: Which of the following is the negation of the proposition "$x > 0$ and x is an integer"?

☐ "$x < 0$ and x is not an integer"

☐ "$x > 0$ or x is not an integer"

☐ "$x \leq 0$ or x is not an integer"

☐ "$x \leq 0$ and x is not an integer"

MCQ 25: Which of the following is the negation of the proposition "The shape is a circle or the shape is red"?

☐ "The shape is not a circle or the shape is not red"

☐ "The shape is a circle and the shape is not red"

☐ "The shape is not a circle and the shape is red"

☐ "The shape is not a circle and the shape is not red"

MCQ 26: Which of the following is the negation of the proposition "$-2 \leq x < 3$"?

☐ "$x < -2$ and $x \geq 3$"

☐ "$x < -2$ or $x \geq 3$"

☐ "$x \leq -2$ or $x > 3$"

☐ "$x > -2$ and $x < 3$"

MCQ 27: Which of the following is the negation of the proposition "$x < 0$ or $x \geq 2$"?

☐ "$x > 0$ or $x \leq 2$"

☐ "$x \geq 0$ or $x < 2$"

☐ "$x \geq 0$ and $x < 2$"

☐ "$x > 0$ and $x \leq 2$"

A.4 IMPLICATION AND EQUIVALENCE

A.4.1 IDENTIFYING RELATED IMPLICATIONS

MCQ 28: What is the converse of the proposition "If a shape is a square, then it is a rectangle"?

- ☐ "If a shape is not a square, then it is not a rectangle."
- ☐ "If a shape is not a rectangle, then it is not a square."
- ☐ "If a shape is a rectangle, then it is a square."
- ☐ "A shape is a square if and only if it is a rectangle."

MCQ 29: Which of the following propositions is the contrapositive of "If x is a multiple of 6, then x is an even number"?

- ☐ "If x is an even number, then x is a multiple of 6."
- ☐ "If x is not an even number, then x is not a multiple of 6."
- ☐ "If x is not a multiple of 6, then x is not an even number."
- ☐ "x is a multiple of 6 and x is not an even number."

MCQ 30: What is the inverse of the proposition "If $x = 2$, then $x^2 = 4$"?

- ☐ "If $x^2 = 4$, then $x = 2$."
- ☐ "If $x^2 \neq 4$, then $x \neq 2$."
- ☐ "If $x \neq 2$, then $x^2 \neq 4$."
- ☐ "$x = 2$ and $x^2 \neq 4$."

MCQ 31: What is the contrapositive of the proposition "If a function is differentiable, then it is continuous"?

- ☐ "If a function is continuous, then it is differentiable."
- ☐ "If a function is not differentiable, then it is not continuous."
- ☐ "A function is differentiable and it is not continuous."
- ☐ "If a function is not continuous, then it is not differentiable."

MCQ 32: What is the inverse of the proposition "If an integer is a multiple of 10, then it is a multiple of 5"?

- ☐ "If an integer is a multiple of 5, then it is a multiple of 10."
- ☐ "If an integer is not a multiple of 10, then it is not a multiple of 5."
- ☐ "If an integer is not a multiple of 5, then it is not a multiple of 10."
- ☐ "An integer is a multiple of 10 and it is not a multiple of 5."

A.4.2 WRITING THE CONVERSE AND CONTRAPOSITIVE

Ex 33: Consider the proposition: "If a shape is a square, then it is a rectangle."

1. Write the converse of the proposition.
2. Write the contrapositive of the proposition.

Ex 34: Consider the proposition: "If x is a multiple of 6, then x is an even number."

1. Write the converse of the proposition.
2. Write the contrapositive of the proposition.

Ex 35: Consider the proposition: "If $x = 2$, then $x^2 = 4$."

1. Write the converse of the proposition.
2. Write the contrapositive of the proposition.

A.4.3 TRANSLATING STATEMENTS INTO IMPLICATIONS

Ex 36: Rewrite the following statement in the form of a logical implication by completing the sentence below.

"The sum of two even integers is an even integer."

If a and b are even integers, then ....

Ex 37: Rewrite the following statement in the form of a logical implication by completing the sentence below.

"All prime numbers greater than 2 are odd."

Complete the sentence:

If a number n is a prime number and $n > 2$, then ....

Ex 38: Rewrite the following statement in the form of a logical implication by completing the sentence below.

"The square of any real number is non-negative."

Complete the sentence:

If x is a real number, then ....

A.5 QUANTIFIERS

A.5.1 EVALUATING QUANTIFIED STATEMENTS

MCQ 39: Let $A = \{2, 4, 6, 8\}$. Which of the following propositions is true?

- ☐ $\forall x \in A$, x is a multiple of 4.
- ☐ $\exists x \in A$, x is an odd number.
- ☐ $\forall x \in A$, x is an even number.
- ☐ $\exists x \in A$, $x > 10$.

MCQ 40: Let $\mathbb{Z}$ be the set of integers. Which of the following propositions is true?

- ☐ $\forall n \in \mathbb{Z}$, $n^2 > 0$.
- ☐ $\exists n \in \mathbb{Z}$, $n + 1 = n$.
- ☐ $\forall n \in \mathbb{Z}$, $\sqrt{n}$ is a real number.
- ☐ $\exists n \in \mathbb{Z}$, $n^2 = n$.

MCQ 41: Let P be the set of all prime numbers. Which of the following propositions is true?

- ☐ $\forall x \in P$, $x + 1$ is a composite number.
- ☐ $\forall x \in P$, x is an odd number.
- ☐ $\exists x \in P$, x is an even number.
- ☐ $\forall x \in P$, $x + 2$ is a prime number.

A.5.2 NEGATING QUANTIFIED STATEMENTS

MCQ 42: Which of the following is the negation of the proposition "$\exists x \in \mathbb{R}, x^2 = -1$"?

- ☐ $\exists x \in \mathbb{R}, x^2 \neq -1$
- ☐ $\exists x \in \mathbb{R}, x^2 > -1$
- ☐ $\forall x \in \mathbb{R}, x^2 = -1$
- ☐ $\forall x \in \mathbb{R}, x^2 \neq -1$

MCQ 43: Which of the following is the negation of the proposition "$\forall n \in \mathbb{N}, n \geq 0$"?

- ☐ $\forall n \in \mathbb{N}, n < 0$
- ☐ $\exists n \in \mathbb{N}, n < 0$
- ☐ $\exists n \in \mathbb{N}, n \leq 0$
- ☐ $\exists n \in \mathbb{N}, n > 0$

MCQ 44: Which of the following is the negation of the proposition "Some prime numbers are even"?

- ☐ "Some prime numbers are odd."
- ☐ "All prime numbers are even."
- ☐ "All prime numbers are odd."

A.5.3 TRANSLATING STATEMENTS INTO QUANTIFIED FORM

Ex 45: Write the following statement using a quantifier: "The square of any real number is non-negative."

Ex 46: Write the following statement using a quantifier: "There exists an integer that is a multiple of both 2 and 3."

Ex 47: Write the following statement using quantifiers and a logical connector: "For every real number x, if $x > 1$ then $x^2 > 1$."

B WRITTEN PROOF

B.1 STRUCTURE FOR WRITTEN PROOFS

B.1.1 ANALYZING PROOF STRUCTURES

MCQ 48: In a direct proof of the implication "If n is an odd integer, then $n + 1$ is an even integer", what is the correct first step?

- ☐ Assume that $n + 1$ is an odd integer.
- ☐ Assume that n is an even integer.
- ☐ Assume that n is an odd integer.
- ☐ Assume that $n + 1$ is an even integer.

Ex 49: A student was asked to prove the statement: "If n is an odd integer, then $n+1$ is an even integer."Below is the **Student's Proof Attempt**.

1. Assume n is an even integer.

2. By definition, there exists an integer k such that $n = 2k$.
3. Then $n + 1 = 2k + 1 = 2(k + 1)$.
4. Therefore, $n + 1$ is an even integer.

Identify the errors in the student's reasoning and write a correct version.

Ex 50: A student was asked to prove the statement: "If n is an even integer, then n^2 is an even integer."
Below is the **Student's Proof Attempt**.

1. Assume n is an odd integer.
2. By definition, there exists an integer k such that $n = 2k+1$.
3. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.
4. Therefore, n^2 is an even integer.

Identify the errors in the student's reasoning and write a correct version.

B.2 INTRODUCING A VARIABLE

B.2.1 STRUCTURING A PROOF

MCQ 51: Which of the following is the correct first sentence for a direct proof of the statement "For all real numbers x, $x^2 \geq 0$"?

- ☐ "Let $x^2 \geq 0$."
- ☐ "Assume x is a positive number."
- ☐ "Let x be a real number."
- ☐ "For some real number x."

MCQ 52: A student begins a proof with the sentence: "Let k be an integer such that $k > 10$." Which statement is the student most likely trying to prove?

- ☐ For all integers k, if $k > 10$, then $k^2 > 100$.
- ☐ There exists an integer k such that $k > 10$.
- ☐ For all integers k, $k^2 > 100$.
- ☐ If $k^2 > 100$, then $k > 10$.

Ex 53: A student was asked to prove the statement: "If n is an even integer, then n^2 is divisible by 4."
Below is the **Student's Proof Attempt**.

1. Let $n = 4$.
2. Then $n^2 = 4^2 = 16$.
3. Since 16 is divisible by 4, the statement is true.

Identify the errors in the student's reasoning and write a correct version.

C METHODS OF PROOF

C.1 DIRECT PROOF (PROOF BY DEDUCTION)

C.1.1 WRITING DIRECT PROOFS IN ARITHMETIC

Ex 54: Use a direct proof to show that the product of two odd integers is an odd integer.

Ex 55: Use a direct proof to show that if an integer is divisible by 6, then it is divisible by 3.

Ex 56: Use a direct proof to show that the sum of an even integer and an odd integer is an odd integer.

Ex 57: Use a direct proof to show that if an integer a divides both integers b and c, then a divides their difference $b - c$.

C.1.2 CONSTRUCTING DIRECT PROOFS IN VARIOUS CONTEXTS

Ex 58: Use a direct proof to show that if a triangle is equilateral, then it is isosceles.

Ex 59: Use a direct proof to show that if a function is linear, then its square is a quadratic function.

Ex 60: Use a direct proof to show that if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Ex 61: Use a direct proof to show that if a function $f(x)$ is odd, then the function $g(x) = [f(x)]^2$ is even.

C.1.3 CONSTRUCTING DIRECT PROOFS: PROVING A STATEMENT IS TRUE

Ex 62: Use a direct proof to show that $0.111\cdots \in \mathbb{Q}$

Ex 63: Use a direct proof to show that $0.151515\cdots \in \mathbb{Q}$.

Ex 64: Use a direct proof to show that for all real numbers a and b:

$$\frac{a^2+b^2}{2} \geq ab$$

C.2 PROOF BY CONTRAPOSITIVE

C.2.1 CONSTRUCTING PROOFS BY CONTRAPOSITIVE

Ex 65: Use proof by contrapositive to show that if n^2 is an even integer, then n is an even integer.

Ex 66: Use proof by contrapositive to show that for integers x and y, if xy is even, then at least one of x or y is even.

Ex 67: Use proof by contrapositive to show that if the product of two real numbers, a and b, is irrational, then at least one of a or b must be irrational.

C.3 PROOF BY EXHAUSTION (CASES)

C.3.1 CONSTRUCTING PROOFS BY EXHAUSTION

Ex 68: Use proof by exhaustion to show that for any proposition p, the statement "$p \vee (\neg p)$" is always true. (This is known as the Law of the Excluded Middle).

Ex 69: Use proof by exhaustion to show that for any set A, $A \cap A' = \emptyset$. (A' denotes the complement of A).

Ex 70: Use proof by exhaustion to show that for any integer n, $n(n+1)$ is an even number.

Ex 71: Use proof by exhaustion to show that for any integer n, $n^3 - n$ is divisible by 3.

C.4 DISPROOF BY COUNTEREXAMPLE

C.4.1 DISPROVING STATEMENTS BY COUNTEREXAMPLE

Ex 72: Disprove the following statement by finding a counterexample:

"For all real numbers a and b, $\sqrt{a^2+b^2} = a + b$."

Ex 73: Disprove the following statement by finding a counterexample:

"For all real numbers x, $|x| = x$."

Ex 74: Disprove the following statement by finding a counterexample:

"For any integer n, if n is divisible by 2, then n is divisible by 4."

C.5 PROOF BY EQUIVALENCE

C.5.1 CONSTRUCTING PROOFS OF EQUIVALENCE

Ex 75: Prove that for an integer n, the following equivalence is true:

"n is odd if and only if $n + 1$ is even."

Ex 76: Prove that for any real number x, the following equivalence is true:

"$x^2 - 4x + 3 = 0$ if and only if $x = 1$ or $x = 3$."

Ex 77: Prove that for an integer n, the following equivalence is true:

"n^2 is a multiple of 4 if and only if n is a multiple of 2 (i.e., even)."

Ex 78: Prove that for any real number x where $x \neq 0$, the following equivalence is true:

"$x + \frac{1}{x} = 2$ if and only if $x = 1$."

C.5.2 CONSTRUCTING AND ANALYZING PROOFS BY EQUIVALENCE FOR IDENTITIES

Ex 79: Prove that

$$\forall a, b \in \mathbb{R} : (a+b)^2 - (a-b)^2 = 4ab$$

Ex 80: Prove that$\forall a, b \in \mathbb{R}:\ (a+b)^2 + (a-b)^2 = 2(a^2+b^2)$.

Ex 81: The following "proof" by equivalence leads to the absurd conclusion that $2 = 1$. Identify the incorrect step and explain the error in the reasoning.

$$\begin{aligned} & a = b \\ \Leftrightarrow\quad & a^2 = ab \\ \Leftrightarrow\quad & a^2 - b^2 = ab - b^2 \\ \Leftrightarrow\quad & (a-b)(a+b) = b(a-b) \\ \Leftrightarrow\quad & a+b = b \\ \Leftrightarrow\quad & 2a = a \\ \Leftrightarrow\quad & 2 = 1 \end{aligned}$$

Ex 82: The following "proof" by equivalence leads to the absurd conclusion that $1 = 0$. Identify the incorrect step and explain the error in the reasoning.

$$\begin{aligned} & a = -\frac{1}{2} \\ \Leftrightarrow\quad & 2a = -1 \\ \Leftrightarrow\quad & 2a + 1 = 0 \\ \Leftrightarrow\quad & a^2 + 2a + 1 = a^2 \\ \Leftrightarrow\quad & (a+1)^2 = a^2 \\ \Leftrightarrow\quad & a + 1 = a \\ \Leftrightarrow\quad & 1 = 0 \end{aligned}$$

C.6 PROOF BY CONTRADICTION

C.6.1 ANALYZING THE STRUCTURE OF PROOF BY CONTRADICTION

MCQ 83: When proving by contradiction the statement "$\sqrt{2}$ is irrational", what is the correct initial assumption?

- ☐ Assume that $\sqrt{2}$ is a real number.
- ☐ Assume that $\sqrt{2}$ is a rational number.
- ☐ Assume that $\sqrt{2}$ is an irrational number.
- ☐ Assume that $x^2 = 2$ has no solution.

MCQ 84: When proving by contradiction the statement "$\sqrt{2}$ is irrational", what is the final conclusion of the proof?

- ☐ "The initial assumption that $\sqrt{2}$ is rational must be true."
- ☐ "The final contradiction proves that $\sqrt{2}$ is rational."
- ☐ "The argument contains a flaw, so no conclusion can be made."
- ☐ "The initial assumption that $\sqrt{2}$ is rational must be false."

C.6.2 CONSTRUCTING PROOFS BY CONTRADICTION

Ex 85: Prove that $\log_3 5$ is an irrational number.

Ex 86: Prove that $\log_2 3$ is an irrational number.

Ex 87: Prove that $\sqrt{2}$ is an irrational number.

C.7 PROOF BY MATHEMATICAL INDUCTION

C.7.1 PROVING INEQUALITIES BY INDUCTION

Ex 88: Use proof by mathematical induction to show that $2^n > n$ for all $n \in \mathbb{Z}^+$.

Ex 89: Use proof by mathematical induction to show that $n^2 \geq 2n$ for all integers $n \geq 2$.

Ex 90: Use proof by mathematical induction to show that for any real number $a \geq 0$,

$$(1+a)^n \geq 1+na \quad \text{for all integers } n \geq 0$$

C.7.2 PROVING SUMS OF POWERS BY INDUCTION

Ex 91: Prove by mathematical induction that for all $n \in \mathbb{Z}^+$, $1+2+\cdots+n = \frac{n(n+1)}{2}$.

Ex 92: Prove that for all $n \in \mathbb{Z}^+$,

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Ex 93: Prove that for all $n \in \mathbb{Z}^+$,$1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

C.7.3 PROVING SEQUENCE PROPERTIES BY INDUCTION

Ex 94: A sequence is defined by $u_0 = 2$ and the recurrence relation $u_{n+1} = \frac{u_n}{3} + 2$ for all $n \in \mathbb{N}$.
Prove that the sequence (u_n) is increasing, i.e., that $u_{n+1} \geq u_n$ for all $n \in \mathbb{N}$.

Ex 95: A sequence is defined by $u_1 = 1$ and the recurrence relation $u_{n+1} = \sqrt{u_n + 2}$ for all $n \in \mathbb{Z}^+$.
Prove that the sequence (u_n) is bounded above by 2, i.e., that $u_n \leq 2$ for all $n \in \mathbb{Z}^+$.

Ex 96: A sequence is defined by $u_0 = 5$ and the recurrence relation $u_{n+1} = 2u_n - 3$ for all $n \in \mathbb{N}$.
Prove that $u_n = 2^{n+1} + 3$ for all $n \in \mathbb{N}$.

C.7.4 PROVING DIVISIBILITY PROPERTIES BY INDUCTION

Ex 97: Prove that $3^n + 1$ is divisible by 2 for all integers $n \geq 0$.

Ex 98: Prove that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 1$.

Ex 99: Prove that $7^n - 1$ is divisible by 6 for all integers $n \geq 1$.