# REAL POLYNOMIALS

## A DEFINITIONS

### A.1 IDENTIFYING POLYNOMIAL PROPERTIES

**Ex 1:** For the polynomial $P(x) = 3x^5 - 7x^3 + 2x - 10$, state:

1. the degree is $\boxed{5}$
2. the leading coefficient is $\boxed{3}$
3. the constant term is $\boxed{-10}$

*Answer:*

1. The **degree** is the highest power of $x$, which is 5.
2. The **leading coefficient** is the coefficient of the term with the highest power ($3x^5$), which is 3.
3. The **constant term** is the term without a variable, which is -10.

**Ex 2:** For the polynomial $P(x) = 7x - 4x^2 + 1$, state:

1. the degree is $\boxed{2}$
2. the leading coefficient is $\boxed{-4}$
3. the constant term is $\boxed{1}$

*Answer:* First, it is helpful to write the polynomial in standard form (descending powers of $x$): $P(x) = -4x^2 + 7x + 1$.

1. The **degree** is the highest power of $x$, which is 2.
2. The **leading coefficient** is the coefficient of the term with the highest power ($-4x^2$), which is -4.
3. The **constant term** is the term without a variable, which is 1.

**Ex 3:** For the polynomial $Q(x) = -x^4 + 9x^2 - x$, state:

1. the degree is $\boxed{4}$
2. the leading coefficient is $\boxed{-1}$
3. the coefficient of $x^2$ is $\boxed{9}$
4. the constant term is $\boxed{0}$

*Answer:* The polynomial can be written with all terms as $Q(x) = -x^4 + 0x^3 + 9x^2 - x + 0$.

1. The **degree** is the highest power of $x$, which is 4.
2. The **leading coefficient** is the coefficient of the term with the highest power ($-x^4$), which is -1.
3. The **coefficient of** $x^2$ is the number multiplying the $x^2$ term, which is 9.
4. The **constant term** is the term without a variable. Since there is no such term, the constant term is 0.

### A.2 CLASSIFYING POLYNOMIALS BY DEGREE

**MCQ 4:** What is the correct classification for the polynomial $P(x) = 4 - 2x^3 + x$?

- ☐ Linear
- ☐ Quadratic
- ☒ Cubic
- ☐ Quartic

*Answer:* The degree of the polynomial is determined by the highest power of the variable $x$. In $P(x) = -2x^3+x+4$, the highest power is 3. A polynomial of degree 3 is called a **cubic** polynomial.

**MCQ 5:** What is the correct classification for the polynomial $P(x) = 5x + x^2 - 1$?

- ☐ Linear
- ☒ Quadratic
- ☐ Cubic
- ☐ Quartic

*Answer:* The degree of the polynomial is determined by the highest power of the variable $x$. In $P(x) = x^2+5x-1$, the highest power is 2. A polynomial of degree 2 is called a **quadratic** polynomial.

**MCQ 6:** What is the correct classification for the polynomial $P(x) = 6x^2 - x^3$?

- ☐ Linear
- ☐ Quadratic
- ☒ Cubic
- ☐ Quartic

*Answer:* The degree of a polynomial is its highest power of the variable $x$. By rewriting the polynomial in standard form, $P(x) = -x^3 + 6x^2$, we can see that the highest power is 3. A polynomial of degree 3 is called a **cubic** polynomial.

**MCQ 7:** What is the correct classification for the polynomial $P(x) = 3x - 5x^4 + 2$?

- ☐ Linear
- ☐ Quadratic
- ☐ Cubic
- ☒ Quartic

*Answer:* The degree of a polynomial is its highest power of the variable $x$. By rewriting the polynomial in standard form, $P(x) = -5x^4 + 3x + 2$, we can see that the highest power is 4. A polynomial of degree 4 is called a **quartic** polynomial.

**MCQ 8:** What is the correct classification for the polynomial $P(x) = 5 - 2x$?

- ☒ Linear

☐ Quadratic

☐ Cubic

☐ Quartic

*Answer:* The degree of the polynomial is determined by the highest power of the variable $x$. In $P(x) = -2x+5$, the highest power of $x$ is 1. A polynomial of degree 1 is called a **linear** polynomial.

### A.3 IDENTIFYING COEFFICIENTS

**Ex 9:** Find the values of $a$, $b$, and $c$ given the polynomial identity:

$$ax^2 + bx + c = 5x^2 - 7$$

$$a = \boxed{5},\ b = \boxed{0},\ c = \boxed{-7}$$

*Answer:* We equate the coefficients of the corresponding powers of $x$ on both sides. It is helpful to write the right-hand side with all terms present: $5x^2 + 0x - 7$.

- Coefficient of $x^2$: $a = 5$.
- Coefficient of $x$: $b = 0$.
- Constant term: $c = -7$.

**Ex 10:** Find the values of $a$, $b$, $c$, and $d$ given the polynomial identity:

$$ax^3 + bx^2 + cx + d = 4 - 9x + 2x^3$$

$$a = \boxed{2},\ b = \boxed{0},\ c = \boxed{-9},\ d = \boxed{4}$$

*Answer:* We first write the right-hand side in standard form (descending powers of $x$): $2x^3 + 0x^2 - 9x + 4$.Now we equate the coefficients of the corresponding powers of $x$.

- Coefficient of $x^3$: $a = 2$.
- Coefficient of $x^2$: $b = 0$.
- Coefficient of $x$: $c = -9$.
- Constant term: $d = 4$.

**Ex 11:** Find the values of $a$, $b$, and $c$ given the polynomial identity:

$$ax^4 + (b-2)x^3 + 5x = -3x^4 + 7x^3 + cx$$

$$a = \boxed{-3},\ b = \boxed{9},\ c = \boxed{5}$$

*Answer:* We equate the coefficients of the corresponding powers of $x$ on both sides of the identity.

- Coefficient of $x^4$: $a = -3$.
- Coefficient of $x^3$: $b - 2 = 7 \implies b = 9$.
- Coefficient of $x$: $c = 5$.

The solution is $a = -3, b = 9, c = 5$.

## B OPERATIONS WITH POLYNOMIALS

### B.1 PERFORMING LINEAR OPERATIONS

**Ex 12:** For $P(x) = 4x^3 + 2x^2 - 5x + 1$ and $Q(x) = x^3 - 3x^2 + 7$, find:

$$P(x) + Q(x) = \boxed{5x^3 - x^2 - 5x + 8}$$

*Answer:* We group like terms.

$$\begin{aligned} P(x) + Q(x) &= (4x^3 + 2x^2 - 5x + 1) + (x^3 - 3x^2 + 7) \\ &= (4x^3 + x^3) + (2x^2 - 3x^2) - 5x + (1 + 7) \\ &= 5x^3 - x^2 - 5x + 8 \end{aligned}$$

**Ex 13:** For $P(x) = 4x^3 + 2x^2 - 5x + 1$ and $Q(x) = x^3 - 3x^2 + 7$, find:

$$P(x) - Q(x) = \boxed{3x^3 + 5x^2 - 5x - 6}$$

*Answer:*

$$\begin{aligned} P(x) - Q(x) &= (4x^3 + 2x^2 - 5x + 1) - (x^3 - 3x^2 + 7) \\ &= 4x^3 + 2x^2 - 5x + 1 - x^3 + 3x^2 - 7 \\ &= (4x^3 - x^3) + (2x^2 + 3x^2) - 5x + (1 - 7) \\ &= 3x^3 + 5x^2 - 5x - 6 \end{aligned}$$

**Ex 14:** For $P(x) = 2x^2 - x + 5$ and $Q(x) = x^3 - 3x^2 + 4$, find:

$$2P(x) - 3Q(x) = \boxed{-3x^3 + 13x^2 - 2x - 2}$$

*Answer:*

$$\begin{aligned} 2P(x) - 3Q(x) &= 2(2x^2 - x + 5) - 3(x^3 - 3x^2 + 4) \\ &= (4x^2 - 2x + 10) - (3x^3 - 9x^2 + 12) \\ &= 4x^2 - 2x + 10 - 3x^3 + 9x^2 - 12 \\ &= -3x^3 + (4x^2 + 9x^2) - 2x + (10 - 12) \\ &= -3x^3 + 13x^2 - 2x - 2 \end{aligned}$$

**Ex 15:** For $P(x) = 2x^2 - 3x$ and $Q(x) = x^3 + x^2 - 1$, find:

$$2(P(x) + Q(x)) = \boxed{2x^3 + 6x^2 - 6x - 2}$$

*Answer:*

$$\begin{aligned} 2(P(x) + Q(x)) &= 2((2x^2 - 3x) + (x^3 + x^2 - 1)) \\ &= 2(x^3 + (2x^2 + x^2) - 3x - 1) \\ &= 2(x^3 + 3x^2 - 3x - 1) \\ &= 2x^3 + 6x^2 - 6x - 2 \end{aligned}$$

### B.2 EXPANDING POLYNOMIALS

**Ex 16:** For $P(x) = x - 1$ and $Q(x) = x^2 + 2x + 1$, find:

$$P(x)Q(x) = \boxed{x^3 + x^2 - x - 1}$$

*Answer:*

$$\begin{aligned}P(x)Q(x) &= (x-1)(x^2+2x+1)\\ &= x(x^2+2x+1) - 1(x^2+2x+1)\\ &= (x^3+2x^2+x) - (x^2+2x+1)\\ &= x^3+2x^2+x-x^2-2x-1\\ &= x^3+(2x^2-x^2)+(x-2x)-1\\ &= x^3+x^2-x-1\end{aligned}$$

**Ex 17:** For $P(x) = x-1$ and $Q(x) = x^2+x+1$, find:

$$P(x)Q(x) = \boxed{x^3-1}$$

*Answer:*

$$\begin{aligned}P(x)Q(x) &= (x-1)(x^2+x+1)\\ &= x(x^2+x+1) - 1(x^2+x+1)\\ &= (x^3+x^2+x) - (x^2+x+1)\\ &= x^3+x^2+x-x^2-x-1\\ &= x^3+(x^2-x^2)+(x-x)-1\\ &= x^3-1\end{aligned}$$

*Note: This is the well-known algebraic identity for the difference of two cubes:* $(a-b)(a^2+ab+b^2) = a^3-b^3$.

**Ex 18:** For $P(x) = x^2-3x+1$, find:

$$(P(x))^2 = \boxed{x^4-6x^3+11x^2-6x+1}$$

*Answer:*

$$\begin{aligned}(P(x))^2 &= (x^2-3x+1)^2\\ &= (x^2-3x+1)(x^2-3x+1)\\ &= x^2(x^2-3x+1) - 3x(x^2-3x+1) + 1(x^2-3x+1)\\ &= (x^4-3x^3+x^2) - (3x^3-9x^2+3x) + (x^2-3x+1)\\ &= x^4-3x^3+x^2-3x^3+9x^2-3x+x^2-3x+1\\ &= x^4-6x^3+11x^2-6x+1\end{aligned}$$

**Ex 19:** For $P(x) = x-1$, $Q(x) = x+1$, and $R(x) = x+2$, find:

$$P(x)Q(x)R(x) = \boxed{x^3+2x^2-x-2}$$

*Answer:* It is often easiest to multiply two of the polynomials first, especially if they form a recognizable pattern.

$$\begin{aligned}P(x)Q(x)R(x) &= [(x-1)(x+1)](x+2)\\ &= (x^2-1)(x+2)\\ &= x^2(x+2) - 1(x+2)\\ &= (x^3+2x^2) - (x+2)\\ &= x^3+2x^2-x-2\end{aligned}$$

*Note: Recognizing that* $(x-1)(x+1)$ *is the difference of two squares,* $x^2-1$, *simplifies the calculation significantly.*

## B.3 IDENTIFYING COEFFICIENTS

**Ex 20:** Find the values of $a$, $b$, and $c$ given the polynomial identity:

$$(x-1)(ax^2+bx+c) = 2x^3-5x^2+8x-5, \forall x \in \mathbb{R}$$

$$a = \boxed{2},\ b = \boxed{-3},\ c = \boxed{5}$$

*Answer:* First, we expand the left-hand side of the identity.

$$\begin{aligned}(x-1)(ax^2+bx+c) &= x(ax^2+bx+c) - 1(ax^2+bx+c)\\ &= (ax^3+bx^2+cx) - (ax^2+bx+c)\\ &= ax^3+(b-a)x^2+(c-b)x-c\end{aligned}$$

Now we equate the coefficients of this expanded form with the right-hand side, $2x^3-5x^2+8x-5$.

- Coefficient of $x^3$: $a = 2$.
- Constant term: $-c = -5 \implies c = 5$.
- Coefficient of $x^2$: $b-a = -5$. Substituting $a = 2$, we get $b-2 = -5 \implies b = -3$.
- Check with coefficient of $x$: $c-b = 8$. Substituting our values, $5-(-3) = 8$, which is correct.

The solution is $a = 2, b = -3, c = 5$.

**Ex 21:** Find the values of $a$, $b$, and $c$ given the polynomial identity:

$$(x^2+2x-1)(ax^2+bx+c) = 2x^4+3x^3-x^2+7x-3, \forall x \in \mathbb{R}$$

$$a = \boxed{2},\ b = \boxed{-1},\ c = \boxed{3}$$

*Answer:* First, we expand the left-hand side of the identity.

$$\begin{aligned}(x^2+2x-1)(ax^2+bx+c) &= x^2(ax^2+bx+c) + 2x(ax^2+bx+c)\\ &= (ax^4+bx^3+cx^2) + (2ax^3+2bx^2+2\\ &= ax^4+(b+2a)x^3+(c+2b-a)x^2+(\end{aligned}$$

Now we equate the coefficients with the right-hand side, $2x^4+3x^3-x^2+7x-3$.

- Coefficient of $x^4$: $a = 2$.
- Constant term: $-c = -3 \implies c = 3$.
- Coefficient of $x^3$: $b+2a = 3$. Substituting $a = 2$, we get $b+2(2) = 3 \implies b = -1$.
- Check with other coefficients:
  - $x^2$: $c+2b-a = 3+2(-1)-2 = 3-2-2 = -1$. (Correct)
  - $x$: $2c-b = 2(3)-(-1) = 6+1 = 7$. (Correct)

The solution is $a = 2, b = -1, c = 3$.

**Ex 22:** Find the values of $a$ and $b$ given the polynomial identity:

$$(2x+a)(x^2+bx-1) = 2x^3+5x^2+x-3, \forall x \in \mathbb{R}$$

$$a = \boxed{3},\ b = \boxed{1}$$

*Answer:* First, we expand the left-hand side of the identity.

$$\begin{aligned}(2x+a)(x^2+bx-1) &= 2x(x^2+bx-1)+a(x^2+bx-1)\\ &= (2x^3+2bx^2-2x)+(ax^2+abx-a)\\ &= 2x^3+(2b+a)x^2+(ab-2)x-a\end{aligned}$$

Now we equate the coefficients with the right-hand side, $2x^3+5x^2+x-3$.

- Constant term: $-a=-3 \implies a=3$.
- Coefficient of $x^2$: $2b+a=5$. Substituting $a=3$, we get $2b+3=5 \implies 2b=2 \implies b=1$.
- Check with coefficient of $x$: $ab-2=1$. Substituting our values, $(3)(1)-2=3-2=1$. (Correct)

The solution is $a=3, b=1$.

**Ex 23:** Find the values of $a$, $b$, and $c$ given the polynomial identity:

$$x^2-2x+3=a(x-b)^2+c, \forall x \in \mathbb{R}$$

$$a=\boxed{1},\ b=\boxed{1},\ c=\boxed{2}$$

*Answer:* First, we expand the right-hand side of the identity.

$$\begin{aligned}a(x-b)^2+c &= a(x^2-2bx+b^2)+c\\ &= ax^2-2abx+(ab^2+c)\end{aligned}$$

Now we equate the coefficients of this expanded form with the left-hand side, $x^2-2x+3$.

- Coefficient of $x^2$: $a=1$.
- Coefficient of $x$: $-2ab=-2$. Substituting $a=1$, we get $-2(1)b=-2 \implies b=1$.
- Constant term: $ab^2+c=3$. Substituting $a=1$ and $b=1$, we get $(1)(1)^2+c=3 \implies 1+c=3 \implies c=2$.

The solution is $a=1, b=1, c=2$.
*Alternatively, one can use the method of completing the square on the left side:*

$$\begin{aligned}x^2-2x+3 &= (x^2-2x+1)-1+3\\ &= (x-1)^2+2\end{aligned}$$

*Comparing this to* $a(x-b)^2+c$ *directly gives* $a=1, b=1, c=2$.

# C THE DIVISION ALGORITHM

## C.1 PERFORMING POLYNOMIAL DIVISION

**Ex 24:** Write the division with remainder of $x^2+3x+5$ by $x+1$:

$$x^2+3x+5=(x+1)\times\boxed{(x+2)}+\boxed{3}$$

*Answer:* We perform the long division of $x^2+3x+5$ by $x+1$:

$$\begin{array}{r|l} & x+2 \\ \hline x+1 & x^2+3x+5 \\ & -x^2-x \\ \hline & 2x+5 \\ & -2x-2 \\ \hline & 3 \end{array}$$

So $x^2+3x+5=(x+1)\times(x+2)+3$

**Ex 25:** Write the division with remainder of $2x^2+5x-4$ by $x+3$:

$$2x^2+5x-4=(x+3)\times\boxed{(2x-1)}+\boxed{-1}$$

*Answer:* We perform the long division of $2x^2+5x-4$ by $x+3$:

$$\begin{array}{r|l} & 2x-1 \\ \hline x+3 & 2x^2+5x-4 \\ & -2x^2-6x \\ \hline & -x-4 \\ & x+3 \\ \hline & -1 \end{array}$$

So $2x^2+5x-4=(x+3)\times(2x-1)+(-1)$

**Ex 26:** Write the division with remainder of $x^3$ by $x^2-1$:

$$x^3=(x^2-1)\times\boxed{(x)}+\boxed{x}$$

*Answer:* We perform the long division of $x^3$ by $x^2-1$.

$$\begin{array}{r|l} & x \\ \hline x^2-1 & x^3 \\ & -x^3+x \\ \hline & x \end{array}$$

The quotient is $x$ and the remainder is $x$.
So $x^3=(x^2-1)\times(x)+x$.

**Ex 27:** Write the division with remainder of $2x^3-2x-1$ by $x^2+2x+1$:

$$2x^3-2x-1=(x^2+2x+1)\times\boxed{(2x-4)}+\boxed{4x+3}$$

*Answer:* We perform the long division of $2x^3-2x-1$ by $x^2+2x+1$.

$$\begin{array}{r|l} & 2x-4 \\ \hline x^2+2x+1 & 2x^3 \qquad -2x-1 \\ & -2x^3-4x^2-2x \\ \hline & -4x^2-4x-1 \\ & 4x^2+8x+4 \\ \hline & 4x+3 \end{array}$$

The quotient is $2x-4$ and the remainder is $4x+3$.
So $2x^3-2x-1=(x^2+2x+1)\times(2x-4)+(4x+3)$.

## C.2 VERIFYING DIVISIBILITY

**MCQ 28:** Is the polynomial $D(x)=x-2$ a divisor of $P(x)=x^3-4x^2+x+6$?

☒ Yes

☐ No

*Answer:* According to the Condition for Divisibility, $D(x)$ is a divisor of $P(x)$ if and only if the remainder of the division is zero. We perform the long division:

$$
\begin{array}{r}
x^2 - 2x - 3 \\
x - 2 \overline{\big) \, x^3 - 4x^2 \quad + x + 6} \\
\underline{- x^3 + 2x^2 \qquad\qquad} \\
- 2x^2 \quad + x \quad \\
\underline{2x^2 - 4x \quad} \\
- 3x + 6 \\
\underline{3x - 6} \\
0
\end{array}
$$

The remainder is 0. Therefore, $x-2$ is a divisor of $x^3-4x^2+x+6$.

**MCQ 29:** Is the polynomial $D(x) = x + 1$ a divisor of $P(x) = x^3 + 2x^2 - x - 5$?

☐ Yes

☒ No

*Answer:* According to the Condition for Divisibility, $D(x)$ is a divisor of $P(x)$ if and only if the remainder of the division is zero. We perform the long division:

$$
\begin{array}{r}
x^2 \quad + x - 2 \\
x + 1 \overline{\big) \, x^3 + 2x^2 \quad - x - 5} \\
\underline{- x^3 \quad - x^2 \qquad\qquad} \\
x^2 \quad - x \quad \\
\underline{- x^2 \quad - x \quad} \\
- 2x - 5 \\
\underline{2x + 2} \\
- 3
\end{array}
$$

The remainder is -3, which is not zero. Therefore, $x + 1$ is not a divisor of $x^3 + 2x^2 - x - 5$.

**MCQ 30:** Is the polynomial $D(x) = x - 1$ a divisor of $P(x) = 2x^3 - x^2 - 7x + 6$?

☒ Yes

☐ No

*Answer:* According to the Condition for Divisibility, $D(x)$ is a divisor of $P(x)$ if and only if the remainder of the division is zero. We perform the long division:

$$
\begin{array}{r}
2x^2 \quad + x - 6 \\
x - 1 \overline{\big) \, 2x^3 \quad - x^2 - 7x + 6} \\
\underline{- 2x^3 + 2x^2 \qquad\qquad} \\
x^2 - 7x \quad \\
\underline{- x^2 \quad + x \quad} \\
- 6x + 6 \\
\underline{6x - 6} \\
0
\end{array}
$$

The remainder is 0. Therefore, $x-1$ is a divisor of $2x^3-x^2-7x+6$.

## C.3 FINDING COEFFICIENTS OF FACTORS

**Ex 31:** The polynomial $P(x) = 2x^3 + x^2 - 4x + 1$ can be written in the form $(x-1)(ax^2+bx+c)$, where $a, b$, and $c$ are constants. Determine the values of $a, b$, and $c$.

$$a = \boxed{2}, \, b = \boxed{3}, \, c = \boxed{-1}$$

*Answer:* There are two common methods to solve this problem.

- **Method 1: Expansion and Identification**
  First, we expand the factored form of the polynomial.

  $$\begin{aligned}(x-1)(ax^2+bx+c) &= x(ax^2+bx+c) - 1(ax^2+bx+c) \\ &= (ax^3+bx^2+cx) - (ax^2+bx+c) \\ &= ax^3 + (b-a)x^2 + (c-b)x - c\end{aligned}$$

  Now we equate the coefficients of this expanded form with the given polynomial, $2x^3 + x^2 - 4x + 1$.

  - Coefficient of $x^3$: $a = 2$.
  - Constant term: $-c = 1 \implies c = -1$.
  - Coefficient of $x^2$: $b - a = 1$. Substituting $a = 2$, we get $b - 2 = 1 \implies b = 3$.

  We can check our result with the coefficient of $x$: $c - b = -1 - 3 = -4$, which is correct.
  The solution is $a = 2, b = 3, c = -1$.

- **Method 2: Polynomial Long Division**
  If $P(x) = (x-1)(ax^2+bx+c)$, then $(ax^2+bx+c)$ is the quotient when $P(x)$ is divided by $(x-1)$.

  $$
  \begin{array}{r}
  2x^2 + 3x - 1 \\
  x - 1 \overline{\big) \, 2x^3 \quad + x^2 - 4x + 1} \\
  \underline{- 2x^3 + 2x^2 \qquad\qquad} \\
  3x^2 - 4x \quad \\
  \underline{- 3x^2 + 3x \quad} \\
  - x + 1 \\
  \underline{x - 1} \\
  0
  \end{array}
  $$

  The division gives a quotient of $2x^2+3x-1$ and a remainder of 0.
  By comparing the quotient with the form $ax^2 + bx + c$, we can identify the coefficients:$a = 2, b = 3, c = -1$.

**Ex 32:** The polynomial $P(x) = x^3 - x^2 - 5x + 2$ can be written in the form $(x+2)(ax^2+bx+c)$, where $a, b$, and $c$ are constants. Determine the values of $a, b$, and $c$.

$$a = \boxed{1}, \, b = \boxed{-3}, \, c = \boxed{1}$$

*Answer:* There are two common methods to solve this problem.

- **Method 1: Expansion and Identification**
  First, we expand the factored form of the polynomial.

  $$\begin{aligned}(x+2)(ax^2+bx+c) &= x(ax^2+bx+c) + 2(ax^2+bx+c) \\ &= (ax^3+bx^2+cx) + (2ax^2+2bx+2c) \\ &= ax^3 + (b+2a)x^2 + (c+2b)x + 2c\end{aligned}$$

  Now we equate the coefficients of this expanded form with the given polynomial, $x^3 - x^2 - 5x + 2$.

  - Coefficient of $x^3$: $a = 1$.
  - Constant term: $2c = 2 \implies c = 1$.
  - Coefficient of $x^2$: $b + 2a = -1$. Substituting $a = 1$, we get $b + 2(1) = -1 \implies b = -3$.

We can check our result with the coefficient of $x$: $c+2b = 1+2(-3) = 1-6 = -5$, which is correct.
The solution is $a=1, b=-3, c=1$.

- **Method 2: Polynomial Long Division**
  If $P(x) = (x+2)(ax^2+bx+c)$, then $(ax^2+bx+c)$ is the quotient when $P(x)$ is divided by $(x+2)$.

$$\begin{array}{r|l} & x^2 - 3x + 1 \\ \hline x+2 & x^3 - x^2 - 5x + 2 \\ & -x^3 - 2x^2 \\ \hline & -3x^2 - 5x \\ & 3x^2 + 6x \\ \hline & x + 2 \\ & -x - 2 \\ \hline & 0 \end{array}$$

  The division gives a quotient of $x^2-3x+1$ and a remainder of 0.
  By comparing the quotient with the form $ax^2+bx+c$, we can identify the coefficients:$a=1, b=-3, c=1$.

**Ex 33:** The polynomial $P(x) = 3x^3 - 11x^2 + 7x - 3$ can be written in the form $(x-3)(ax^2+bx+c)$, where $a, b$, and $c$ are constants. Determine the values of $a, b$, and $c$.

$$a = \boxed{3},\ b = \boxed{-2},\ c = \boxed{1}$$

*Answer:* There are two common methods to solve this problem.

- **Method 1: Expansion and Identification**
  First, we expand the factored form of the polynomial.

$$\begin{aligned}(x-3)(ax^2+bx+c) &= x(ax^2+bx+c) - 3(ax^2+bx+c) \\ &= (ax^3+bx^2+cx) - (3ax^2+3bx+3c) \\ &= ax^3 + (b-3a)x^2 + (c-3b)x - 3c\end{aligned}$$

  Now we equate the coefficients of this expanded form with the given polynomial, $3x^3 - 11x^2 + 7x - 3$.
  - Coefficient of $x^3$: $a = 3$.
  - Constant term: $-3c = -3 \implies c = 1$.
  - Coefficient of $x^2$: $b - 3a = -11$. Substituting $a=3$, we get $b - 3(3) = -11 \implies b - 9 = -11 \implies b = -2$.

  We can check our result with the coefficient of $x$: $c - 3b = 1 - 3(-2) = 1 + 6 = 7$, which is correct.
  The solution is $a=3, b=-2, c=1$.

- **Method 2: Polynomial Long Division**
  If $P(x) = (x-3)(ax^2+bx+c)$, then $(ax^2+bx+c)$ is the quotient when $P(x)$ is divided by $(x-3)$.

$$\begin{array}{r|l} & 3x^2 - 2x + 1 \\ \hline x-3 & 3x^3 - 11x^2 + 7x - 3 \\ & -3x^3 + 9x^2 \\ \hline & -2x^2 + 7x \\ & 2x^2 - 6x \\ \hline & x - 3 \\ & -x + 3 \\ \hline & 0 \end{array}$$

  The division gives a quotient of $3x^2-2x+1$ and a remainder of 0.
  By comparing the quotient with the form $ax^2+bx+c$, we can identify the coefficients:$a=3, b=-2, c=1$.

# D THE REMAINDER AND FACTOR THEOREMS

## D.1 APPLYING THE REMAINDER THEOREM

**Ex 34:** Use the Remainder Theorem to find the remainder when $2x^3 - 5x^2 + 3x + 7$ is divided by $x - 2$.

$$R(x) = \boxed{9}$$

*Answer:* Let $P(x) = 2x^3 - 5x^2 + 3x + 7$.
According to the Remainder Theorem, the remainder when dividing by $x-2$ is $P(2)$.

$$\begin{aligned}P(2) &= 2(2)^3 - 5(2)^2 + 3(2) + 7 \\ &= 2(8) - 5(4) + 6 + 7 \\ &= 16 - 20 + 6 + 7 \\ &= 9\end{aligned}$$

When $2x^3 - 5x^2 + 3x + 7$ is divided by $x-2$, the remainder is 9.We can check the result by applying the long division:

$$\begin{array}{r|l} & 2x^2 - x + 1 \\ \hline x-2 & 2x^3 - 5x^2 + 3x + 7 \\ & -2x^3 + 4x^2 \\ \hline & -x^2 + 3x \\ & x^2 - 2x \\ \hline & x + 7 \\ & -x + 2 \\ \hline & 9 \end{array}$$

**Ex 35:** Use the Remainder theorem to find the remainder when $x^4 - 3x^3 + x - 4$ is divided by $x+2$.

$$R(x) = \boxed{34}$$

*Answer:* Let $P(x) = x^4 - 3x^3 + x - 4$.
According to the Remainder Theorem, the remainder when dividing by $x+2$ (which is $x-(-2)$) is $P(-2)$.

$$\begin{aligned}P(-2) &= (-2)^4 - 3(-2)^3 + (-2) - 4 \\ &= 16 - 3(-8) - 2 - 4 \\ &= 16 + 24 - 2 - 4 \\ &= 34\end{aligned}$$

When $x^4 - 3x^3 + x - 4$ is divided by $x+2$, the remainder is 34.
We can check the result by applying the long division:

$$\begin{array}{r|l} & x^3 - 5x^2 + 10x - 19 \\ \hline x+2 & x^4 - 3x^3 + x - 4 \\ & -x^4 - 2x^3 \\ \hline & -5x^3 \\ & 5x^3 + 10x^2 \\ \hline & 10x^2 + x \\ & -10x^2 - 20x \\ \hline & -19x - 4 \\ & 19x + 38 \\ \hline & 34 \end{array}$$

**Ex 36:** Use the Remainder Theorem to find the remainder when $x^3 - 2x^2 - 5x + 8$ is divided by $x + 1$.

$$R(x) = \boxed{10}$$

*Answer:* Let $P(x) = x^3 - 2x^2 - 5x + 8$.
According to the Remainder Theorem, the remainder when dividing by $x + 1$ (which is $x - (-1)$) is $P(-1)$.

$$\begin{aligned} P(-1) &= (-1)^3 - 2(-1)^2 - 5(-1) + 8 \\ &= -1 - 2(1) + 5 + 8 \\ &= -1 - 2 + 5 + 8 \\ &= 10 \end{aligned}$$

When $x^3 - 2x^2 - 5x + 8$ is divided by $x + 1$, the remainder is 10.We can check the result by applying the long division:

$$\begin{array}{r|l} & x^2 - 3x - 2 \\ \hline x+1 & x^3 - 2x^2 - 5x + 8 \\ & -x^3 - x^2 \\ \hline & -3x^2 - 5x \\ & 3x^2 + 3x \\ \hline & -2x + 8 \\ & 2x + 2 \\ \hline & 10 \end{array}$$

## D.2 VERIFYING DIVISIBILITY

**MCQ 37:** Is $(x - 1)$ a factor of the polynomial $P(x) = x^3 - 2x^2 - 5x + 6$?

☒ Yes

☐ No

*Answer:* There are two ways to check for divisibility.

- **Method 1: Using the Factor Theorem**
  By the Factor Theorem, $(x - 1)$ is a factor of $P(x)$ if and only if $P(1) = 0$.

$$\begin{aligned} P(1) &= (1)^3 - 2(1)^2 - 5(1) + 6 \\ &= 1 - 2 - 5 + 6 \\ &= 0 \end{aligned}$$

  Since $P(1) = 0$, $(x - 1)$ is a factor of $P(x)$.

- **Method 2: Using Long Division**
  By the Condition for Divisibility, $(x - 1)$ is a factor if the remainder of the division is zero.

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x-1 & x^3 - 2x^2 - 5x + 6 \\ & -x^3 + x^2 \\ \hline & -x^2 - 5x \\ & x^2 - x \\ \hline & -6x + 6 \\ & 6x - 6 \\ \hline & 0 \end{array}$$

  The remainder is 0. Therefore, $(x - 1)$ is a factor of $P(x)$.

**MCQ 38:** Is $(x + 2)$ a factor of the polynomial $P(x) = x^3 + x^2 - 4x - 4$?

☒ Yes

☐ No

*Answer:* There are two ways to check for divisibility.

- **Method 1: Using the Factor Theorem**
  By the Factor Theorem, $(x + 2)$ is a factor of $P(x)$ if and only if $P(-2) = 0$.

$$\begin{aligned} P(-2) &= (-2)^3 + (-2)^2 - 4(-2) - 4 \\ &= -8 + 4 + 8 - 4 \\ &= 0 \end{aligned}$$

  Since $P(-2) = 0$, $(x + 2)$ is a factor of $P(x)$.

- **Method 2: Using Long Division**
  By the Condition for Divisibility, $(x + 2)$ is a factor if the remainder of the division is zero.

$$\begin{array}{r|l} & x^2 - x - 2 \\ \hline x+2 & x^3 + x^2 - 4x - 4 \\ & -x^3 - 2x^2 \\ \hline & -x^2 - 4x \\ & x^2 + 2x \\ \hline & -2x - 4 \\ & 2x + 4 \\ \hline & 0 \end{array}$$

  The remainder is 0. Therefore, $(x + 2)$ is a factor of $P(x)$.

**MCQ 39:** Is $(x - 3)$ a factor of the polynomial $P(x) = 2x^3 - 5x^2 - 4x + 5$?

☐ Yes

☒ No

*Answer:* There are two ways to check for divisibility.

- **Method 1: Using the Factor Theorem**
  By the Factor Theorem, $(x - 3)$ is a factor of $P(x)$ if and only if $P(3) = 0$.

$$\begin{aligned} P(3) &= 2(3)^3 - 5(3)^2 - 4(3) + 5 \\ &= 2(27) - 5(9) - 12 + 5 \\ &= 54 - 45 - 12 + 5 \\ &= 2 \end{aligned}$$

  Since $P(3) \neq 0$, $(x - 3)$ is not a factor of $P(x)$.

- **Method 2: Using Long Division**
  By the Condition for Divisibility, $(x - 3)$ is a factor if the remainder of the division is zero.

$$\begin{array}{r|l} & 2x^2 + x - 1 \\ \hline x-3 & 2x^3 - 5x^2 - 4x + 5 \\ & -2x^3 + 6x^2 \\ \hline & x^2 - 4x \\ & -x^2 + 3x \\ \hline & -x + 5 \\ & x - 3 \\ \hline & 2 \end{array}$$

  The remainder is 2, which is not zero. Therefore, $(x - 3)$ is not a factor of $P(x)$.

## D.3 FINDING UNKNOWN COEFFICIENTS

**Ex 40:** When the polynomial $P(x) = 2x^3 + ax^2 - 5x + 1$ is divided by $x + 1$, the remainder is 7. Find the value of $a$.

$$a = \boxed{3}$$

*Answer:* By the Remainder Theorem, the remainder when dividing by $(x + 1)$ is $P(-1)$. We are given that this remainder is 7.

$$P(-1) = 7$$

$$\begin{aligned} 2(-1)^3 + a(-1)^2 - 5(-1) + 1 &= 7 \\ 2(-1) + a(1) + 5 + 1 &= 7 \\ -2 + a + 6 &= 7 \\ a + 4 &= 7 \\ a &= 3 \end{aligned}$$

**Ex 41:** When the polynomial $P(x) = x^3+2x^2+ax-8$ is divided by $x - 3$, the remainder is 10. Find the value of $a$.

$$a = \boxed{-9}$$

*Answer:* By the Remainder Theorem, the remainder when dividing by $(x - 3)$ is $P(3)$. We are given that this remainder is 10.

$$P(3) = 10$$

$$\begin{aligned} (3)^3 + 2(3)^2 + a(3) - 8 &= 10 \\ 27 + 2(9) + 3a - 8 &= 10 \\ 27 + 18 + 3a - 8 &= 10 \\ 37 + 3a &= 10 \\ 3a &= -27 \\ a &= -9 \end{aligned}$$

**Ex 42:** Given that $(x+5)$ is a factor of $P(x) = x^3+ax^2-11x+30$, find the value of $a$.

$$a = \boxed{8/5}$$

*Answer:* Since $(x+5)$ is a factor, we know from the Factor Theorem that $P(-5) = 0$.

$$\begin{aligned} P(-5) = (-5)^3 + a(-5)^2 - 11(-5) + 30 &= 0 \\ -125 + a(25) + 55 + 30 &= 0 \\ 25a - 40 &= 0 \\ 25a &= 40 \\ a &= \frac{40}{25} = \frac{8}{5} \end{aligned}$$

## D.4 FACTORISING POLYNOMIALS GIVEN A FACTOR

**Ex 43:** Consider the polynomial $P(x) = x^3 + kx^2 - 3x + 6$.

1. Find the value of $k$ given that $(x - 2)$ is a factor of $P(x)$.

$$k = \boxed{-2}$$

2. Hence, fully factorise $P(x)$.

$$P(x) = \boxed{(x-2)(x-\sqrt{3})(x+\sqrt{3})}$$

*Answer:*

1. **Finding the value of k**
   By the Factor Theorem, if $(x - 2)$ is a factor of $P(x)$, then $P(2)$ must be equal to 0.

$$\begin{aligned} P(2) = (2)^3 + k(2)^2 - 3(2) + 6 &= 0 \\ 8 + 4k - 6 + 6 &= 0 \\ 8 + 4k &= 0 \\ 4k &= -8 \\ k &= -2 \end{aligned}$$

2. **Factorising the polynomial**
   Now that we know $k = -2$, the polynomial is $P(x) = x^3 - 2x^2 - 3x + 6$.
   Since we know $(x-2)$ is a factor, we can find the other factor by performing polynomial long division.

$$\begin{array}{r|l} & x^2 \qquad\quad - 3 \\ \hline x-2 & x^3 - 2x^2 - 3x + 6 \\ & -x^3 + 2x^2 \\ \hline & \qquad\quad -3x + 6 \\ & \qquad\quad 3x - 6 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

   The division shows that $P(x) = (x - 2)(x^2 - 3)$.
   The quadratic factor $x^2 - 3$ can be factorised further as a difference of two squares: $x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$.
   Therefore, the fully factorised form is:

$$P(x) = (x-2)(x-\sqrt{3})(x+\sqrt{3})$$

**Ex 44:** Consider the polynomial $P(x) = x^3 - 2x^2 + kx + 6$.

1. Find the value of $k$ given that $(x + 2)$ is a factor of $P(x)$.

$$k = \boxed{-5}$$

2. Hence, fully factorise $P(x)$.

$$P(x) = \boxed{(x+2)(x-1)(x-3)}$$

*Answer:*

1. **Finding the value of k**
   By the Factor Theorem, if $(x + 2)$ is a factor of $P(x)$, then $P(-2)$ must be equal to 0.

$$\begin{aligned} P(-2) = (-2)^3 - 2(-2)^2 + k(-2) + 6 &= 0 \\ -8 - 2(4) - 2k + 6 &= 0 \\ -8 - 8 - 2k + 6 &= 0 \\ -10 - 2k &= 0 \\ -2k &= 10 \\ k &= -5 \end{aligned}$$

2. **Factorising the polynomial**
   Now that we know $k = -5$, the polynomial is $P(x) = x^3 - 2x^2 - 5x + 6$.
   Since we know $(x + 2)$ is a factor, we can find the other quadratic factor by performing polynomial long division.

$$
\begin{array}{r|l}
 & x^2 - 4x + 3 \\
\hline
x+2 & x^3 - 2x^2 - 5x + 6 \\
 & -\,x^3 - 2x^2 \\
\hline
 & \quad -\,4x^2 - 5x \\
 & \quad\; 4x^2 + 8x \\
\hline
 & \qquad 3x + 6 \\
 & \qquad -\,3x - 6 \\
\hline
 & \qquad\quad 0
\end{array}
$$

The division shows that $P(x) = (x+2)(x^2 - 4x + 3)$.
The quadratic factor $x^2 - 4x + 3$ can be factorised further into $(x-1)(x-3)$.
Therefore, the fully factorised form is:

$$P(x) = (x+2)(x-1)(x-3)$$

## E QUADRATIC EQUATIONS WITH COMPLEX ROOTS

### E.1 SOLVING QUADRATIC EQUATIONS

**Ex 45:** Solve the equation $z^2 + 1 = 0$ for real numbers and for complex numbers.

*Answer:* The equation can be written as $z^2 = -1$.

- For real numbers ($\mathbb{R}$): There are no real solutions because the square of any real number is non-negative.
- For complex numbers ($\mathbb{C}$):

$$
\begin{aligned}
&z^2 = -1 \\
\Leftrightarrow &z = \pm i
\end{aligned}
$$

**Ex 46:** Solve the equation $z^2 + 2 = 0$ for real numbers and for complex numbers.

*Answer:* The problem is to find all numbers $z$ such that $z^2 + 2 = 0$. This equation can be rewritten as $z^2 = -2$.

- For real numbers ($\mathbb{R}$): There are no real solutions because the square of any real number is non-negative.
- For complex numbers ($\mathbb{C}$):

$$
\begin{aligned}
&z^2 = -2 \\
\Leftrightarrow &z = \pm i\sqrt{2}
\end{aligned}
$$

The solutions are $z_1 = -i\sqrt{2}$ and $z_2 = i\sqrt{2}$.

**Ex 47:** Solve the equation $z^2 - 4z + 5 = 0$ for complex numbers.

*Answer:* We solve the quadratic equation using the quadratic formula. First, we compute the discriminant $\Delta$.
For $az^2 + bz + c = 0$, we have $a = 1$, $b = -4$, and $c = 5$.

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (-4)^2 - 4(1)(5) \\
&= 16 - 20 \\
&= -4
\end{aligned}
$$

Since the discriminant is negative, the solutions are two complex conjugate numbers.

$$
\begin{aligned}
z &= \frac{-b \pm i\sqrt{-\Delta}}{2a} \\
&= \frac{4 \pm i\sqrt{4}}{2} \\
&= \frac{4 \pm 2i}{2} \\
&= 2 \pm i
\end{aligned}
$$

The solutions are $z_1 = 2 - i$ and $z_2 = 2 + i$.

**Ex 48:** Solve the equation $z^2 + 2z + 2 = 0$ for complex numbers.

*Answer:* We solve the quadratic equation using the quadratic formula. First, we compute the discriminant $\Delta$.
For $az^2 + bz + c = 0$, we have $a = 1$, $b = 2$, and $c = 2$.

$$
\begin{aligned}
\Delta &= b^2 - 4ac \\
&= (2)^2 - 4(1)(2) \\
&= 4 - 8 \\
&= -4
\end{aligned}
$$

Since the discriminant is negative, the solutions are two complex conjugate numbers.

$$
\begin{aligned}
z &= \frac{-b \pm i\sqrt{-\Delta}}{2a} \\
&= \frac{-2 \pm i\sqrt{4}}{2} \\
&= \frac{-2 \pm 2i}{2} \\
&= -1 \pm i
\end{aligned}
$$

The solutions are $z_1 = -1 - i$ and $z_2 = -1 + i$.

### E.2 FACTORING POLYNOMIALS

**Ex 49:** Let $P(x) = x^3 - 4x^2 - 7x + 10$.

1. Show that $(x-1)$ is a factor of $P(x)$.
2. Hence, fully factorise $P(x)$ into a product of three linear factors.

*Answer:*

1. By the Factor Theorem, if $(x-1)$ is a factor, then $P(1) = 0$.

   $$P(1) = (1)^3 - 4(1)^2 - 7(1) + 10 = 1 - 4 - 7 + 10 = 0$$

   Since $P(1) = 0$, $(x-1)$ is a factor of $P(x)$.
2. Since $(x-1)$ is a factor, we divide $P(x)$ by $(x-1)$ using long division to find the other factors.

$$
\begin{array}{r|l}
 & x^2 \;\; - 3x - 10 \\
\hline
x-1 & x^3 - 4x^2 \;\; - 7x + 10 \\
 & -\,x^3 \;\; + x^2 \\
\hline
 & \quad -\,3x^2 \;\; - 7x \\
 & \quad\; 3x^2 \;\; - 3x \\
\hline
 & \qquad -\,10x + 10 \\
 & \qquad\; 10x - 10 \\
\hline
 & \qquad\quad 0
\end{array}
$$

The quotient is $x^2-3x-10$. We now factorise this quadratic:

$$x^2 - 3x - 10 = (x-5)(x+2)$$

Therefore, the full factorisation is $P(x) = (x-1)(x-5)(x+2)$.

**Ex 50:** Let $P(x) = x^3 - x^2 + x - 1$.

1. Show that $(x-1)$ is a factor of $P(x)$.
2. Hence, fully factorise $P(x)$ into a product of linear factors over the complex numbers.

*Answer:*

1. By the Factor Theorem, $(x-1)$ is a factor of $P(x)$ if and only if $P(1) = 0$.

$$P(1) = (1)^3 - (1)^2 + (1) - 1 = 1 - 1 + 1 - 1 = 0$$

Since $P(1) = 0$, $(x-1)$ is a factor of $P(x)$.

2. Since $(x-1)$ is a factor, we can find the remaining quadratic factor by dividing $P(x)$ by $(x-1)$.

$$\begin{array}{r|l} & x^2 + 1 \\ x-1 & x^3 - x^2 + x - 1 \\ & -x^3 + x^2 \\ \hline & x - 1 \\ & -x + 1 \\ \hline & 0 \end{array}$$

The quotient is $x^2+1$. So, $P(x) = (x-1)(x^2+1)$.
To fully factorise over the complex numbers, we must factor the quadratic term. The roots of $x^2+1=0$ are given by $x^2 = -1$, which are $x = i$ and $x = -i$.
Therefore, the quadratic factor is $(x-i)(x+i)$.
The full factorisation is $P(x) = (x-1)(x-i)(x+i)$.

**Ex 51:** Let $P(x) = x^3 - 2x + 4$.

1. Show that $(x+2)$ is a factor of $P(x)$.
2. Hence, fully factorise $P(x)$ into a product of linear factors over the complex numbers.

*Answer:*

1. By the Factor Theorem, $(x+2)$ is a factor of $P(x)$ if and only if $P(-2) = 0$.

$$\begin{aligned} P(-2) &= (-2)^3 - 2(-2) + 4 \\ &= -8 + 4 + 4 \\ &= 0 \end{aligned}$$

Since $P(-2) = 0$, $(x+2)$ is a factor of $P(x)$.

2. Since $(x+2)$ is a factor, we can find the remaining quadratic factor by dividing $P(x)$ by $(x+2)$. We use $x^3 + 0x^2 - 2x + 4$ for the division.

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ x+2 & x^3 \quad - 2x + 4 \\ & -x^3 - 2x^2 \\ \hline & -2x^2 - 2x \\ & 2x^2 + 4x \\ \hline & 2x + 4 \\ & -2x - 4 \\ \hline & 0 \end{array}$$

The quotient is $x^2-2x+2$. So, $P(x) = (x+2)(x^2-2x+2)$.
To fully factorise over the complex numbers, we find the roots of the quadratic factor $x^2 - 2x + 2 = 0$ using the quadratic formula.
The discriminant is $\Delta = b^2 - 4ac = (-2)^2 - 4(1)(2) = 4 - 8 = -4$.

$$x = \frac{-(-2) \pm i\sqrt{-\Delta}}{2(1)} = \frac{2 \pm i\sqrt{4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

The roots are $1+i$ and $1-i$. The corresponding factors are $(x-(1+i))$ and $(x-(1-i))$.
The full factorisation is $P(x) = (x+2)(x-1-i)(x-1+i)$.

# F THE FUNDAMENTAL THEOREM OF ALGEBRA

## F.1 APPLYING THE CONJUGATE ROOT THEOREM

**Ex 52:** A polynomial $P(x)$ has real coefficients.
Given that $r_1 = 1+i$ is a root of the equation $P(x) = 0$, find another root.

Another root is: $\boxed{1-i}$

*Answer:* According to the **Conjugate Root Theorem**, if a polynomial has real coefficients, then any complex roots must occur in conjugate pairs.
The conjugate of a complex number $a+bi$ is $a-bi$.
Therefore, if $r_1 = 1+i$ is a root, its conjugate, $r_2 = 1-i$, must also be a root.

**Ex 53:** A polynomial $P(x)$ has real coefficients.Given that $r_1 = -2+3i$ is a root of the equation $P(x) = 0$, find another root.

Another root is: $\boxed{-2-3i}$

*Answer:* According to the **Conjugate Root Theorem**, if a polynomial has real coefficients, then any complex roots must occur in conjugate pairs.
The conjugate of a complex number $a+bi$ is $a-bi$.
Therefore, if $r_1 = -2+3i$ is a root, its conjugate, $r_2 = -2-3i$, must also be a root.

**Ex 54:** A polynomial $P(x)$ has real coefficients.Given that $r_1 = 5i$ is a root of the equation $P(x) = 0$, find another root.

Another root is: $\boxed{-5i}$

*Answer:* According to the **Conjugate Root Theorem**, if a polynomial has real coefficients, then any complex roots must occur in conjugate pairs.
The complex number $5i$ can be written as $0+5i$. The conjugate of a complex number $a+bi$ is $a-bi$.
Therefore, if $r_1 = 0+5i$ is a root, its conjugate, $r_2 = 0-5i = -5i$, must also be a root.

# G SUM AND PRODUCT OF ROOTS THEOREM

## G.1 APPLYING VIETA'S FORMULAS

**Ex 55:** For the equation $3x^2 + 6x - 8 = 0$, find:

1. the sum of the roots. $\boxed{-2}$
2. the product of the roots. $\boxed{-8/3}$

*Answer:* The polynomial is quadratic ($n = 2$) with $a_2 = 3, a_1 = 6, a_0 = -8$.

1. **Sum of roots**: $-\frac{a_1}{a_2} = -\frac{6}{3} = -2$.
2. **Product of roots**: $(-1)^2 \frac{a_0}{a_2} = (1)\frac{-8}{3} = -\frac{8}{3}$.

**Ex 56:** For the equation $x^4 - 5x^3 + 2x - 1 = 0$, find:

1. the sum of the roots. $\boxed{5}$
2. the product of the roots. $\boxed{-1}$

*Answer:* The polynomial is quartic ($n = 4$) with $a_4 = 1, a_3 = -5, a_0 = -1$. Note that $a_2 = 0$.

1. **Sum of roots**: $-\frac{a_3}{a_4} = -\frac{-5}{1} = 5$.
2. **Product of roots**: $(-1)^4 \frac{a_0}{a_4} = (1)\frac{-1}{1} = -1$.

**Ex 57:** For the equation $2x^3 - 5x^2 + 7 = 0$, find:

1. the sum of the roots. $\boxed{5/2}$
2. the product of the roots. $\boxed{-7/2}$

*Answer:* The polynomial is cubic ($n = 3$) with $a_3 = 2, a_2 = -5, a_1 = 0, a_0 = 7$.

1. **Sum of roots**: $-\frac{a_2}{a_3} = -\frac{-5}{2} = \frac{5}{2}$.
2. **Product of roots**: $(-1)^3 \frac{a_0}{a_3} = (-1)\frac{7}{2} = -\frac{7}{2}$.

**Ex 58:** For the equation $-x^5 + 2x^4 - 3x^3 + x - 10 = 0$, find:

1. the sum of the roots. $\boxed{2}$
2. the product of the roots. $\boxed{-10}$

*Answer:* The polynomial is of degree 5 ($n = 5$) with $a_5 = -1, a_4 = 2, a_0 = -10$.

1. **Sum of roots**: $-\frac{a_4}{a_5} = -\frac{2}{-1} = 2$.
2. **Product of roots**: $(-1)^5 \frac{a_0}{a_5} = (-1)\frac{-10}{-1} = (-1)(10) = -10$.

## G.2 FINDING ALL ROOTS OF A POLYNOMIAL

**Ex 59:** Given that $r_1 = 1 + i$ is a root of the polynomial $P(x) = x^3 - 4x^2 + 6x - 4$, find the remaining roots.

*Answer:* Let the roots be $r_1, r_2, r_3$. We are given $r_1 = 1 + i$.

- Since the polynomial $P(x)$ has real coefficients, the **Conjugate Root Theorem** states that the conjugate of $r_1$ must also be a root.
  Therefore, $r_2 = 1 - i$.
- The quadratic factor corresponding to these two complex roots is:

$$\begin{aligned}(x-(1+i))(x-(1-i)) &= ((x-1)-i)((x-1)+i)\\ &= (x-1)^2 - i^2\\ &= x^2 - 2x + 1 - (-1)\\ &= x^2 - 2x + 2\end{aligned}$$

  To find the third root, we can divide $P(x)$ by this factor.

$$\begin{array}{r|l} & x - 2 \\ x^2 - 2x + 2 & x^3 - 4x^2 + 6x - 4 \\ & -x^3 + 2x^2 - 2x \\ \hline & -2x^2 + 4x - 4 \\ & 2x^2 - 4x + 4 \\ \hline & 0 \end{array}$$

  The division gives a quotient of $(x - 2)$, which means the third root is $r_3 = 2$.
  Note : Alternatively, using Vieta's formulas, the sum of the roots must be $r_1 + r_2 + r_3 = -(-4)/1 = 4$.

$$(1+i) + (1-i) + r_3 = 4 \implies 2 + r_3 = 4 \implies r_3 = 2$$

The remaining roots are $\mathbf{1 - i}$ and **2**.

**Ex 60:** Given that $r_1 = 2 - i$ is a root of the polynomial $P(x) = x^3 - 3x^2 + x + 5$, find the remaining roots.

*Answer:* Let the roots be $r_1, r_2, r_3$. We are given $r_1 = 2 - i$.

- Since the polynomial $P(x)$ has real coefficients, the **Conjugate Root Theorem** states that the conjugate of $r_1$ must also be a root.
  Therefore, $r_2 = 2 + i$.
- The quadratic factor corresponding to these two complex roots is:

$$\begin{aligned}(x-(2-i))(x-(2+i)) &= ((x-2)+i)((x-2)-i)\\ &= (x-2)^2 - i^2\\ &= x^2 - 4x + 4 - (-1)\\ &= x^2 - 4x + 5\end{aligned}$$

  To find the third root, we can divide $P(x)$ by this factor.

$$\begin{array}{r|l} & x + 1 \\ x^2 - 4x + 5 & x^3 - 3x^2 + x + 5 \\ & -x^3 + 4x^2 - 5x \\ \hline & x^2 - 4x + 5 \\ & -x^2 + 4x - 5 \\ \hline & 0 \end{array}$$

The division gives a quotient of $(x+1)$, which means the third root is $r_3 = -1$.
Note : Alternatively, using Vieta's formulas, the product of the roots must be $r_1 r_2 r_3 = (-1)^3(5)/1 = -5$.

$$(2-i)(2+i)r_3 = -5 \implies (4-(-1))r_3 = -5 \implies 5r_3 = -5 \implies r_3 = -1$$

The remaining roots are $\mathbf{2+i}$ and $\mathbf{-1}$.

**Ex 61:** Consider the quartic polynomial $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$.

1. It is given that $z_1 = 1-2i$ is a root of the equation $P(x) = 0$.
   (a) Since $P(x)$ has real coefficients, write down another complex root, $z_2$.
   (b) Find a real quadratic factor of $P(x)$ corresponding to the roots $z_1$ and $z_2$.
2. Hence, find the other two roots of the equation $P(x) = 0$.
3. Using the four roots of $P(x) = 0$, verify the product of the roots using Vieta's formulas.

*Answer:*

1. (a) By the Complex Conjugate Root Theorem, since $P(x)$ has real coefficients and $1-2i$ is a root, its conjugate must also be a root. So, $z_2 = 1+2i$.
   (b) The quadratic factor is $(x-z_1)(x-z_2)$. Sum of these roots: $(1-2i)+(1+2i) = 2$. Product of these roots: $(1-2i)(1+2i) = 1^2 - (2i)^2 = 1-(-4) = 5$. The real quadratic factor is $x^2 - (\text{sum})x + (\text{product}) = x^2 - 2x + 5$.
2. We find the other quadratic factor by dividing $P(x)$ by $(x^2 - 2x + 5)$.

$$\begin{array}{r|l}
 & x^2 - 4x + 5 \\
\hline
x^2 - 2x + 5 & x^4 - 6x^3 + 18x^2 - 30x + 25 \\
 & -x^4 + 2x^3 - 5x^2 \\
\hline
 & -4x^3 + 13x^2 - 30x \\
 & 4x^3 - 8x^2 + 20x \\
\hline
 & 5x^2 - 10x + 25 \\
 & -5x^2 + 10x - 25 \\
\hline
 & 0
\end{array}$$

The other factor is $x^2 - 4x + 5$. We find the roots of this factor by solving $x^2 - 4x + 5 = 0$. Using the quadratic formula:

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} \\ &= \frac{4 \pm \sqrt{16-20}}{2} \\ &= \frac{4 \pm \sqrt{-4}}{2} \\ &= \frac{4 \pm 2i}{2} \end{aligned}$$

The other two roots are $2+i$ and $2-i$.

3. The four roots are $\{1-2i, 1+2i, 2+i, 2-i\}$.
The product of these roots is:

$$(1-2i)(1+2i)(2+i)(2-i) = (5)(2^2 - i^2) = 5(4-(-1)) = 5 \times 5 = 25$$

From Vieta's formulas for $P(x) = x^4 - 6x^3 + 18x^2 - 30x + 25$, the product is $(-1)^4 \frac{a_0}{a_4} = (1)\frac{25}{1} = 25$.
The results match, verifying the product.

**Ex 62:** Consider the quartic polynomial $P(x) = x^4 - 7x^3 + 14x^2 + 2x - 20$.

1. It is given that $z_1 = 3+i$ is a root of the equation $P(x) = 0$.
   (a) Since $P(x)$ has real coefficients, write down another complex root, $z_2$.
   (b) Find a real quadratic factor of $P(x)$ corresponding to the roots $z_1$ and $z_2$.
2. Hence, find the other two roots of the equation $P(x) = 0$.
3. Using the four roots of $P(x) = 0$, verify the sum of the roots using Vieta's formulas.

*Answer:*

1. (a) By the Complex Conjugate Root Theorem, since $P(x)$ has real coefficients and $3+i$ is a root, its conjugate must also be a root. So, $z_2 = 3-i$.
   (b) The quadratic factor is $(x-z_1)(x-z_2)$. Sum of these roots: $(3+i)+(3-i) = 6$. Product of these roots: $(3+i)(3-i) = 3^2 - i^2 = 9-(-1) = 10$. The real quadratic factor is $x^2 - (\text{sum})x + (\text{product}) = x^2 - 6x + 10$.
2. We find the other quadratic factor by dividing $P(x)$ by $(x^2 - 6x + 10)$.

$$\begin{array}{r|l}
 & x^2 - x - 2 \\
\hline
x^2 - 6x + 10 & x^4 - 7x^3 + 14x^2 + 2x - 20 \\
 & -x^4 + 6x^3 - 10x^2 \\
\hline
 & -x^3 + 4x^2 + 2x \\
 & x^3 - 6x^2 + 10x \\
\hline
 & -2x^2 + 12x - 20 \\
 & 2x^2 - 12x + 20 \\
\hline
 & 0
\end{array}$$

The other factor is $x^2 - x - 2$. We find the roots of this factor by solving $x^2 - x - 2 = 0$.

$$x^2 - x - 2 = (x-2)(x+1) = 0$$

The other two roots are 2 and $-1$.

3. The four roots are $\{3+i, 3-i, 2, -1\}$.
The sum of these roots is:

$$(3+i) + (3-i) + 2 + (-1) = 6 + 1 = 7$$

From Vieta's formulas for $P(x) = x^4 - 7x^3 + 14x^2 + 2x - 20$, the sum is $-\frac{a_{n-1}}{a_n} = -\frac{-7}{1} = 7$.
The results match, verifying the sum.