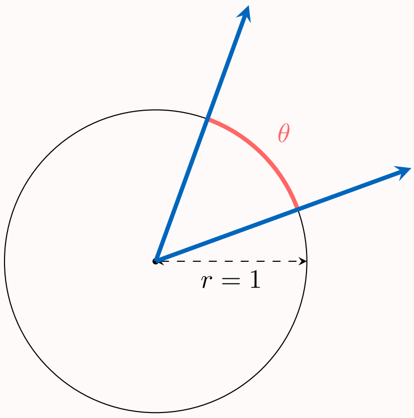
RADIANS AND THE UNIT CIRCLE

A RADIAN MEASURE

The measure of an angle describes what fraction of a full revolution it represents. While degrees ($360°$ in a circle) are a common unit, they are an arbitrary human invention. A more mathematically natural unit is the **radian**.

Definition Radian Measure

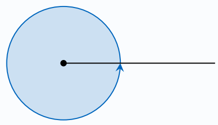
The **radian measure** of an angle θ is defined as the length of the arc it subtends on a unit circle (a circle with radius 1).



Proposition Angle of a Full Circle

The circumference of a unit circle is $C = 2\pi(1) = 2\pi$. Therefore, a full circle contains 2π radians.
This establishes the fundamental conversion: $360° = 2\pi$ **radians**, which simplifies to $180° = \pi$ **radians**.

2π rad $= 360°$



Method Converting Between Degrees and Radians

Based on the relationship $180° = \pi$ radians:

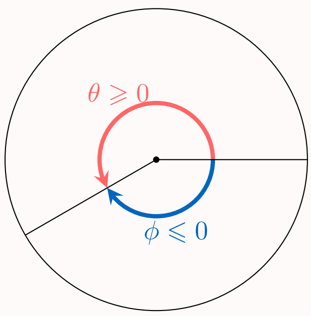
- To convert from degrees to radians, multiply by $\dfrac{\pi}{180}$.
- To convert from radians to degrees, multiply by $\dfrac{180}{\pi}$.

Ex: Convert $60°$ to radians.

Answer:
$$60° = 60° \times \frac{\pi}{180°} = \frac{\pi}{3}$$

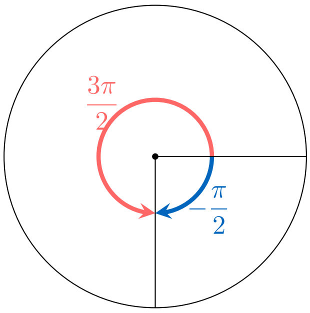
Definition Positive and Negative Angles

- A **positive angle measure** represents a **counterclockwise rotation**.
- A **negative angle measure** represents a **clockwise rotation**.

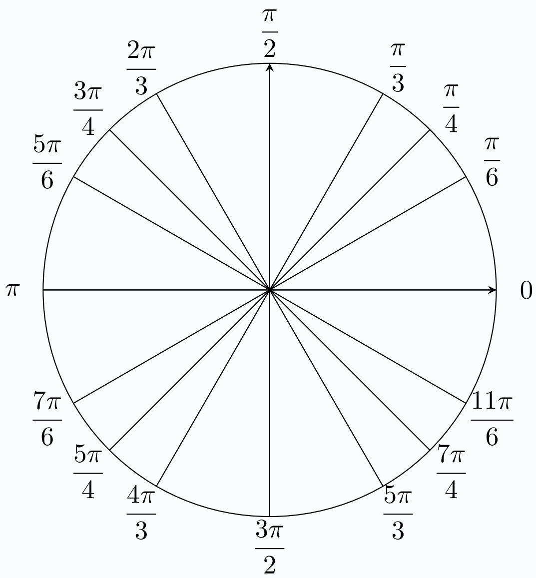


Ex: Draw the angles $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$.

Answer:



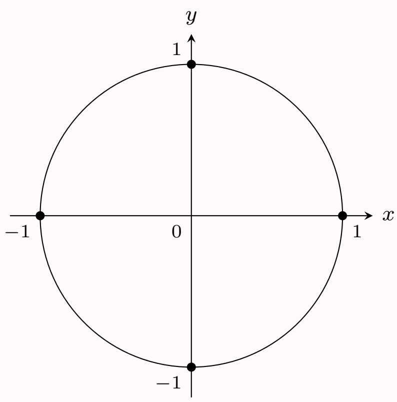
Proposition Reference Angles on the Unit Circle



B TRIGONOMETRY ON THE UNIT CIRCLE

Definition Unit circle

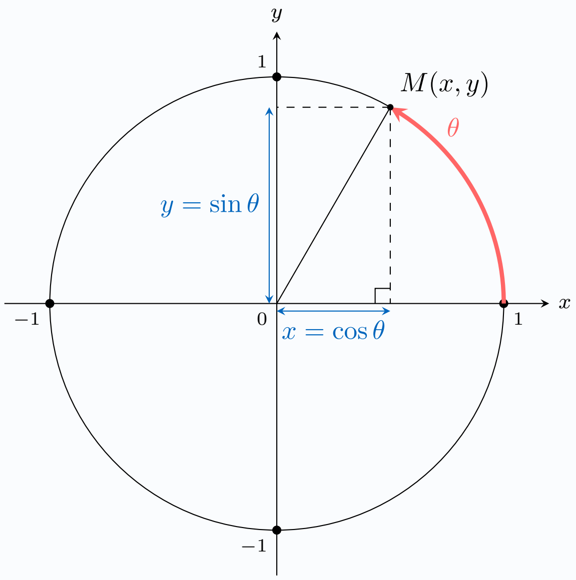
The **unit circle** is a circle with a radius of 1 centered at the origin.



Proposition Relationship between Angle and Coordinates

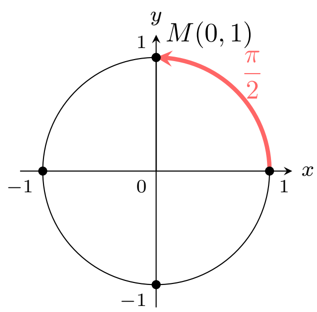
For any angle θ, measured counterclockwise from the positive x-axis, the corresponding point $M(x, y)$ on the circle defines the values of cosine and sine.

- The x-coordinate is the cosine of the angle: $\cos\theta = x$
- The y-coordinate is the sine of the angle: $\sin\theta = y$



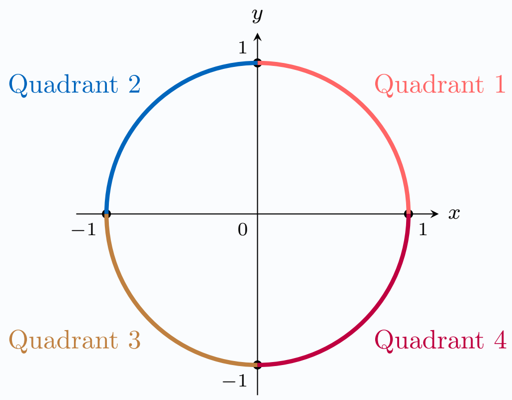
Ex: Find the values $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.

Answer: On the unit circle, the point corresponding to the angle $\frac{\pi}{2}$ has coordinates $(0, 1)$:



$\cos\left(\frac{\pi}{2}\right) = 0 \quad x\text{-coordinate}$
$\sin\left(\frac{\pi}{2}\right) = 1 \quad y\text{-coordinate}$

Proposition **Sign of Sine and Cosine**



Quadrant	$\cos\theta$	$\sin\theta$
1	$+$	$+$
2	$-$	$+$
3	$-$	$-$
4	$+$	$-$

C TRIGONOMETRIC IDENTITIES

Proposition Pythagorean Identity

For any angle θ:

$$\cos^2\theta + \sin^2\theta = 1$$

Proposition Maximum and Minimum of Trigonometric Ratios

$$-1 \leqslant \cos\theta \leqslant 1 \quad \text{and} \quad -1 \leqslant \sin\theta \leqslant 1$$

Proposition Periodicity Identity

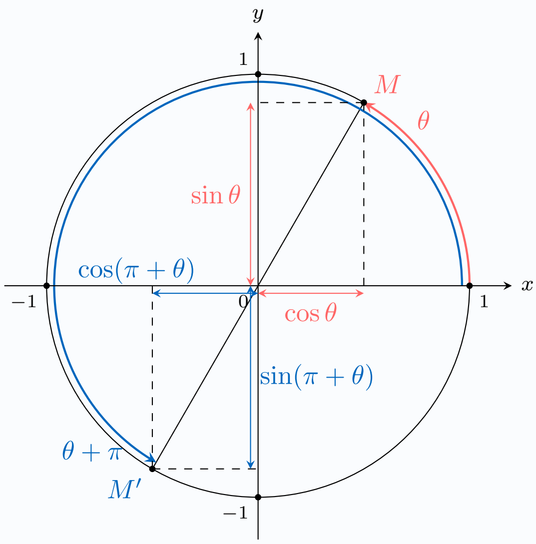
For any angle θ and any integer k:

$$\cos(\theta + 2k\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2k\pi) = \sin\theta$$

Proposition Add π to Trigonometric Ratios

Reflection through the origin:

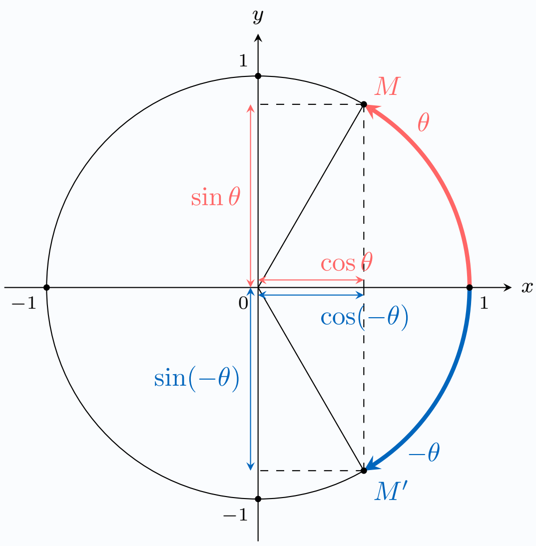
$$\sin(\pi + \theta) = -\sin\theta$$
$$\cos(\pi + \theta) = -\cos\theta$$



Proposition Opposite of Trigonometric Ratios

Reflection in the x-axis:

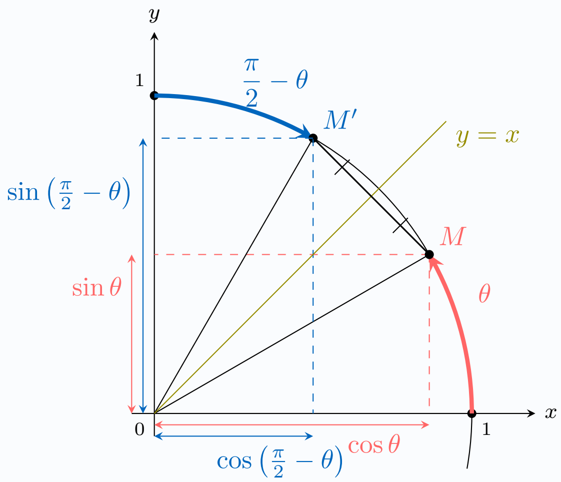
$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$



Proposition Identities with $\frac{\pi}{2}-\theta$

Reflection across the line $y = x$:

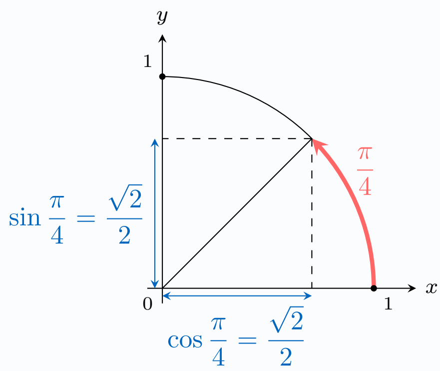
$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$
$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$



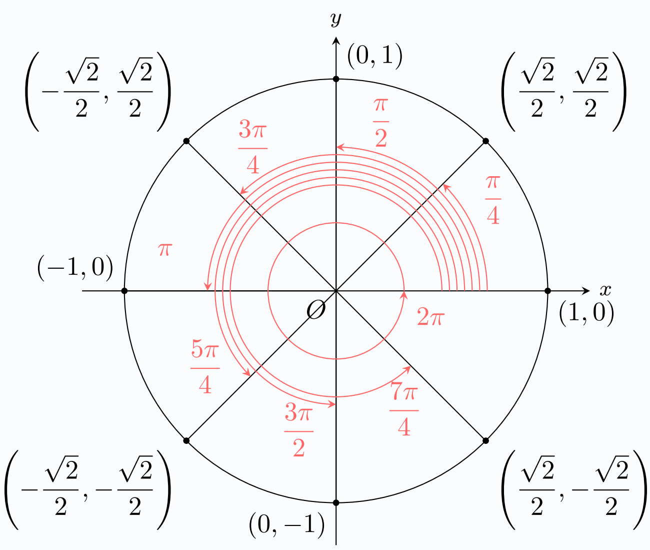
D MULTIPLES OF $\frac{\pi}{4}$

Proposition Coordinates for Angle $\frac{\pi}{4}$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$



Proposition Multiples of $\frac{\pi}{4}$



The signs of the coordinates are determined by the quadrant in which the angle lies.

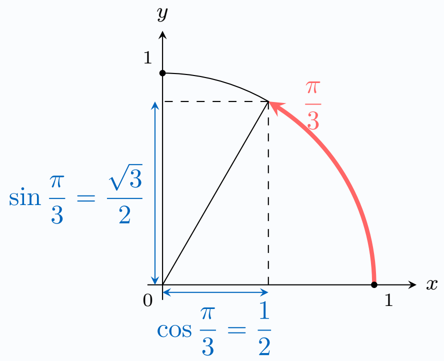
Ex: Find $\cos\dfrac{3\pi}{4}$.

Answer: $\cos\dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2}$

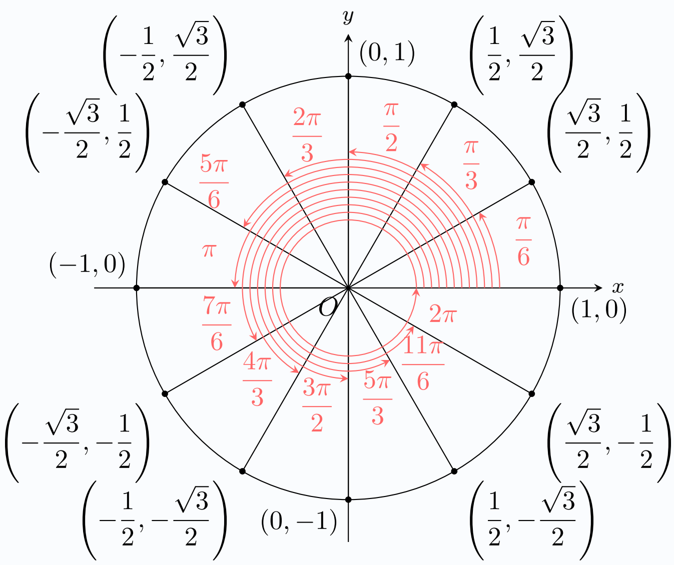
E MULTIPLES OF $\frac{\pi}{6}$

Proposition Coordinates of Angle $\frac{\pi}{3}$

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

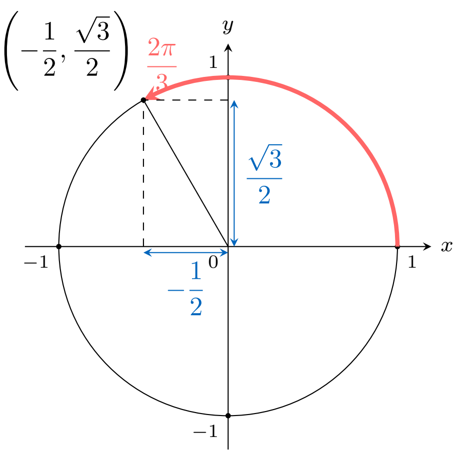


Proposition Multiples of $\frac{\pi}{6}$



Ex: Find $\cos\dfrac{2\pi}{3}$ and $\sin\dfrac{2\pi}{3}$.

Answer:



$\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

F TANGENT FUNCTION

Definition Tangent Function

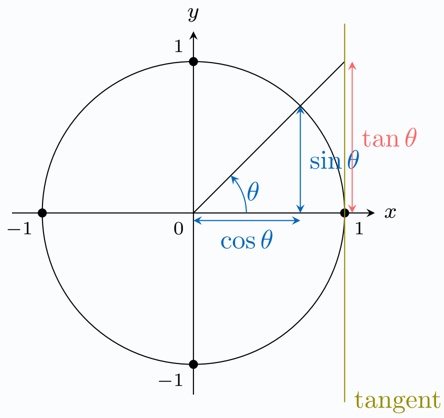
The **tangent** of an angle θ is defined, whenever $\cos\theta \neq 0$, as the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Equivalently, $\tan\theta$ is defined for all real θ such that $\theta \neq \frac{\pi}{2} + k\pi$ for any integer k.

Proposition Geometric Interpretation of Tangent

On the unit circle, for any angle θ with $\cos\theta \neq 0$, the ray from the origin forming an angle θ with the positive x-axis meets the vertical tangent line $x = 1$ at the point $(1, \tan\theta)$. In particular, $\tan\theta$ is the y-coordinate of this intersection point.



Proposition Tangent Values for Common Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

The values in other quadrants follow from the symmetries of the unit circle and the fact that sine and cosine are 2π-periodic.

G ANGLE SUM AND DIFFERENCE IDENTITIES

Proposition Cosine of Difference

$$\cos(A - B) = \cos A\cos B + \sin A\sin B.$$

Proposition Cosine of Sum

$$\cos(A + B) = \cos A\cos B - \sin A\sin B.$$

Proposition Sine of Sum and Difference

$$\sin(A + B) = \sin A\cos B + \cos A\sin B,$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B.$$

Proposition **Tangent of Sum and Difference**

For angles A and B such that all expressions below are defined,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$