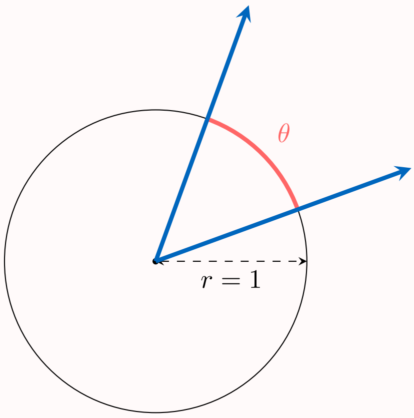
RADIANS AND THE UNIT CIRCLE

A RADIAN MEASURE

The measure of an angle describes what fraction of a full revolution it represents. While degrees ($360°$ in a circle) are a common unit, they are an arbitrary human invention. A more mathematically natural unit is the **radian**.

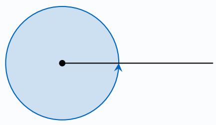
Definition Radian Measure

The **radian measure** of an angle θ is defined as the length of the arc it subtends on a unit circle (a circle with radius 1).



Proposition Angle of a Full Circle

The circumference of a unit circle is $C = 2\pi(1) = 2\pi$. Therefore, a full circle contains 2π radians.
This establishes the fundamental conversion: $360° = 2\pi$ **radians**, which simplifies to $180° = \pi$ **radians**.



Method Converting Between Degrees and Radians

Based on the relationship $180° = \pi$ radians:

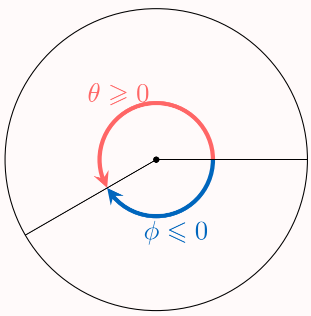
- To convert from degrees to radians, multiply by $\dfrac{\pi}{180}$.
- To convert from radians to degrees, multiply by $\dfrac{180}{\pi}$.

Ex: Convert $60°$ to radians.

Answer: $$60° = 60° \times \frac{\pi}{180°} = \frac{\pi}{3}$$

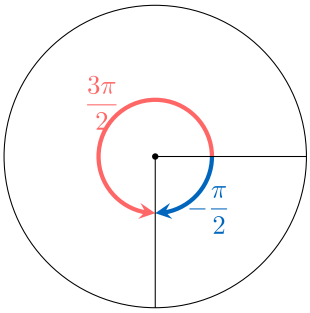
Definition Positive and Negative Angles

- A **positive angle measure** represents a **counterclockwise rotation**.
- A **negative angle measure** represents a **clockwise rotation**.

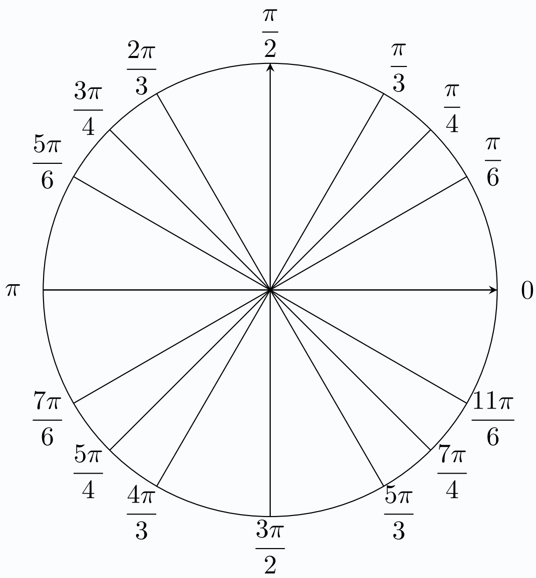


Ex: Draw the angles $\frac{3\pi}{2}$ and $-\frac{\pi}{2}$.

Answer:



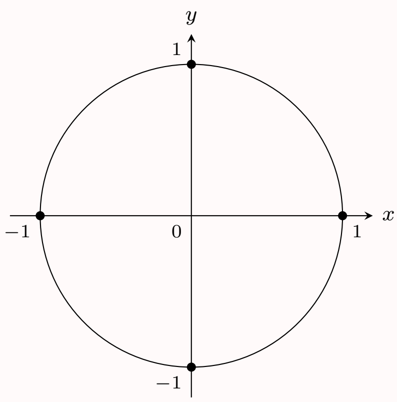
Proposition **Reference Angles on the Unit Circle**



B TRIGONOMETRY ON THE UNIT CIRCLE

Definition **Unit circle**

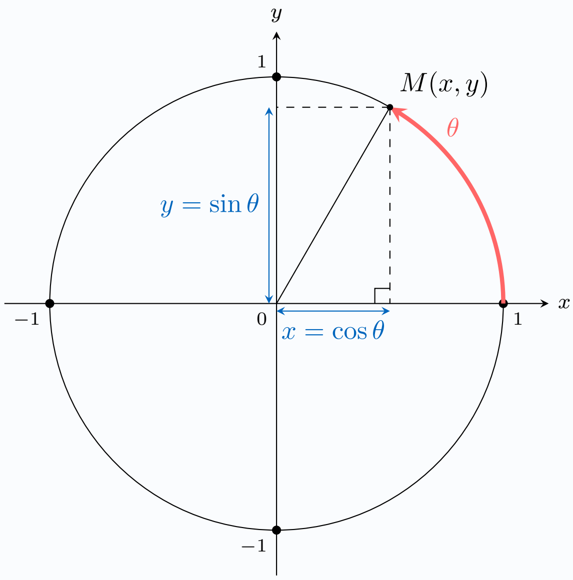
The **unit circle** is a circle with a radius of 1 centered at the origin.



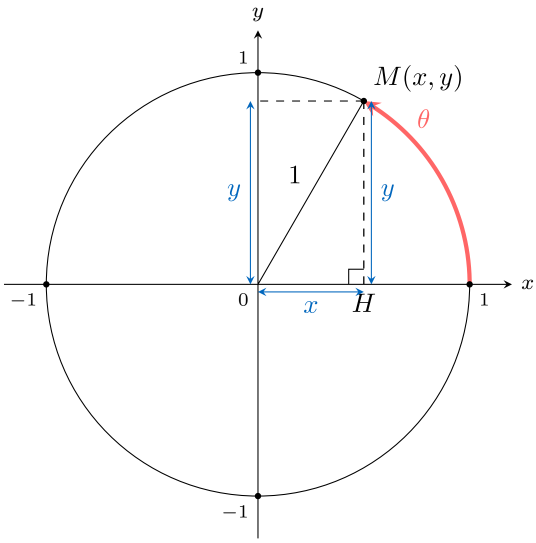
Proposition **Relationship between Angle and Coordinates**

For any angle θ, measured counterclockwise from the positive x-axis, the corresponding point $M(x, y)$ on the circle defines the values of cosine and sine.

- The x-coordinate is the cosine of the angle: $\cos\theta = x$
- The y-coordinate is the sine of the angle: $\sin\theta = y$



Proof



Using right-angled triangle trigonometry in the right triangle OHM:

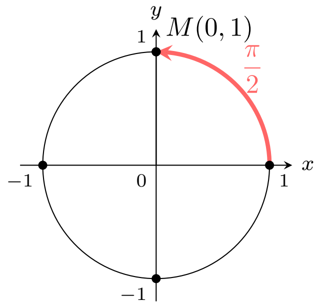
$$\begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ &= \frac{OH}{OM}\\ &= \frac{x}{1}\\ &= x\end{aligned}$$

and

$$\begin{aligned}\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\ &= \frac{HM}{OM}\\ &= \frac{y}{1}\\ &= y\end{aligned}$$

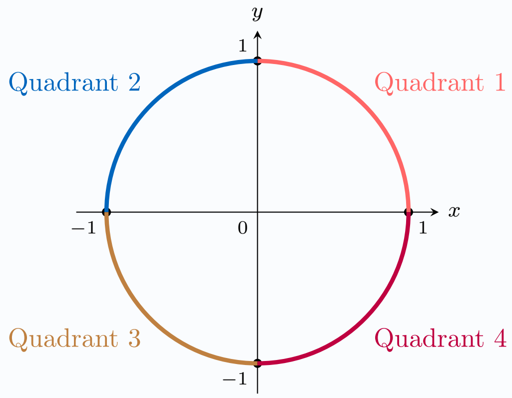
Ex: Find the values $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.

Answer: On the unit circle, the point corresponding to the angle $\frac{\pi}{2}$ has coordinates $(0, 1)$:



$\cos\left(\frac{\pi}{2}\right) = 0$ x-coordinate
$\sin\left(\frac{\pi}{2}\right) = 1$ y-coordinate

Proposition Sign of Sine and Cosine



Quadrant	$\cos\theta$	$\sin\theta$
1	$+$	$+$
2	$-$	$+$
3	$-$	$-$
4	$+$	$-$

C TRIGONOMETRIC IDENTITIES

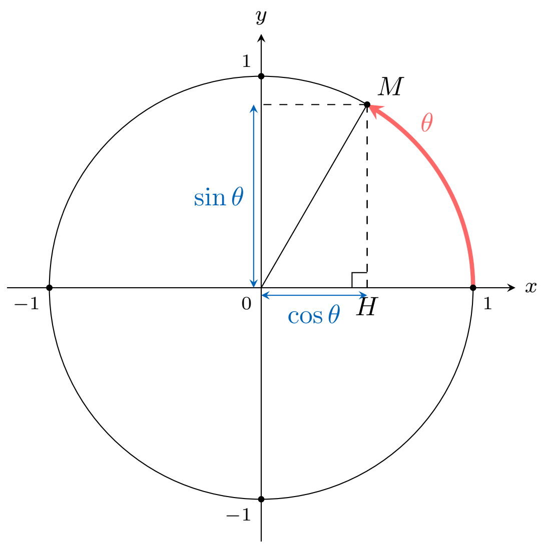
Proposition Pythagorean Identity

For any angle θ:

$$\cos^2\theta + \sin^2\theta = 1$$

Proof

Let $M(\cos\theta, \sin\theta)$ be the point on the unit circle at the angle θ.



By the Pythagorean theorem in the right triangle OHM:

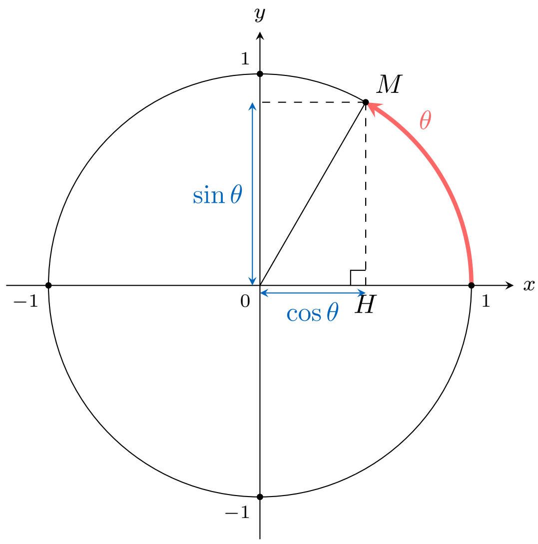
$$OH^2 + HM^2 = OM^2$$
$$(\cos\theta)^2 + (\sin\theta)^2 = 1^2$$
$$\cos^2\theta + \sin^2\theta = 1$$

Proposition Maximum and Minimum of Trigonometric Ratios

$$-1 \leqslant \cos\theta \leqslant 1 \quad \text{and} \quad -1 \leqslant \sin\theta \leqslant 1$$

Proof

- Geometric proof:



The length OH lies between -1 and 1 on the x-axis. As $OH = \cos\theta$, we have $-1 \leqslant \cos\theta \leqslant 1$. Similarly, HM lies between -1 and 1 on the y-axis, so $-1 \leqslant \sin\theta \leqslant 1$.

- Analytical proof (for cosine; the proof for sine is analogous):

$$\begin{aligned} 0 &\leqslant \sin^2\theta && \text{(a square is always non-negative)} \\ \cos^2\theta &\leqslant \cos^2\theta + \sin^2\theta && \text{(add } \cos^2\theta \text{ to both sides)} \\ \cos^2\theta &\leqslant 1 && (\cos^2\theta + \sin^2\theta = 1) \\ |\cos\theta| &\leqslant 1 && \text{(taking square roots)} \\ -1 \leqslant \cos\theta &\leqslant 1 \end{aligned}$$

The same reasoning applied to $\sin\theta$ gives $-1 \leqslant \sin\theta \leqslant 1$.

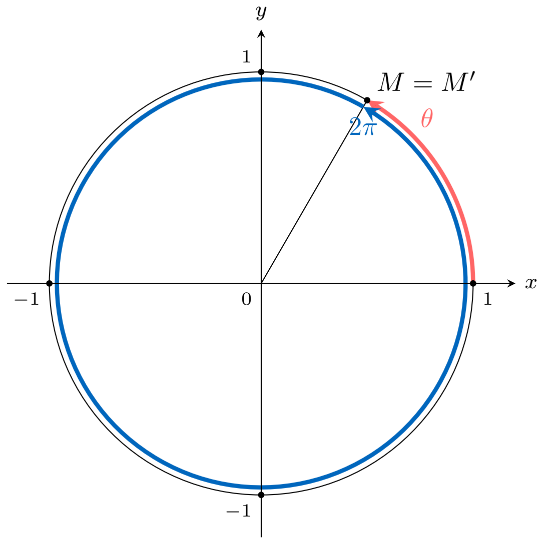
Proposition Periodicity Identity

For any angle θ and any integer k:

$$\cos(\theta + 2k\pi) = \cos\theta \quad \text{and} \quad \sin(\theta + 2k\pi) = \sin\theta$$

Proof

Let $M(\cos\theta, \sin\theta)$ be the point on the unit circle at the angle θ.
Let $M'(\cos(\theta + 2\pi), \sin(\theta + 2\pi))$ be the point on the unit circle at the angle $\theta + 2\pi$.

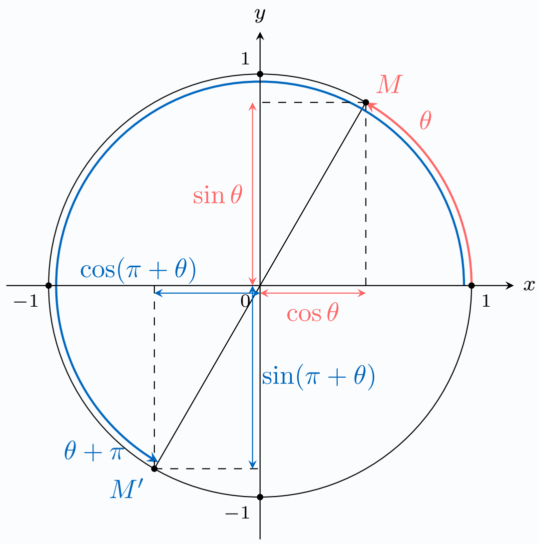


Since 2π is a full revolution, the points M and M' coincide on the unit circle: $M' = M$.
Thus, $\cos(\theta + 2\pi) = \cos\theta$ and $\sin(\theta + 2\pi) = \sin\theta$.
The same argument holds for any multiple of 2π, i.e. for any integer k.

Proposition **Add π to Trigonometric Ratios**

Reflection through the origin:

$$\sin(\pi + \theta) = -\sin\theta$$
$$\cos(\pi + \theta) = -\cos\theta$$



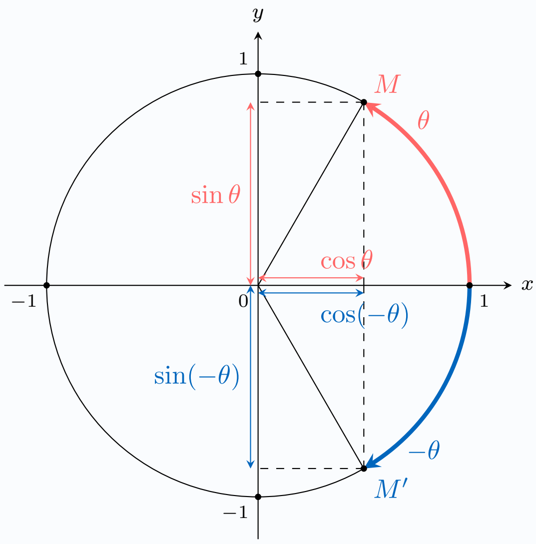
Proof

Let θ be an angle.
Let $M(\cos\theta, \sin\theta)$ be the point on the unit circle at the angle θ.
Let $M'(\cos(\pi + \theta), \sin(\pi + \theta))$ be the point on the unit circle at the angle $\pi + \theta$.
A rotation by an angle π is a half-turn about the origin O, which is the same as a point reflection through O. Therefore, the coordinates of M' are the opposites of the coordinates of M.
So $\sin(\pi + \theta) = -\sin\theta$
$\cos(\pi + \theta) = -\cos\theta$

Proposition **Opposite of Trigonometric Ratios**

Reflection in the x-axis:

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$



Proof

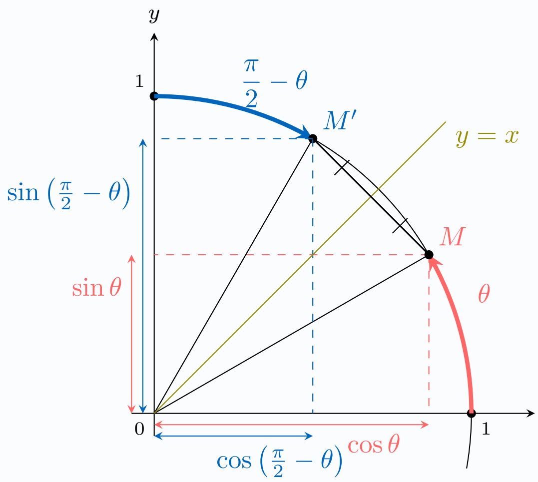
Let $M(\cos\theta, \sin\theta)$ be the point on the unit circle at the angle θ.
Let $M'(\cos(-\theta), \sin(-\theta))$ be the point on the unit circle at the angle $-\theta$.
The point M' is the reflection of M across the x-axis. Therefore, the x-coordinate of M' is the same as that of M, and the y-coordinate of M' is the opposite of the y-coordinate of M.
So $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

Proposition **Identities with $\frac{\pi}{2} - \theta$**

Reflection across the line $y = x$:

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$



Proof

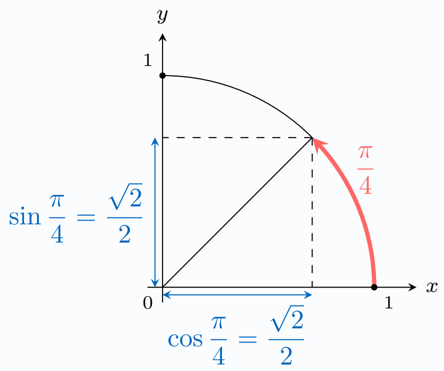
Let θ be an angle.
Let $M(\cos\theta, \sin\theta)$ be the point on the unit circle at the angle θ.
Let $M'(\cos(\frac{\pi}{2} - \theta), \sin(\frac{\pi}{2} - \theta))$ be the point on the unit circle at the angle $\frac{\pi}{2} - \theta$.
The points M and M' are symmetric with respect to the line $y = x$, so the x-coordinate of M is the y-coordinate of M', and the y-coordinate of M is the x-coordinate of M'.
So $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

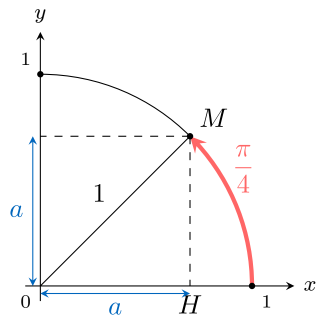
D MULTIPLES OF $\frac{\pi}{4}$

Proposition Coordinates for Angle $\frac{\pi}{4}$

$$\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$



Proof



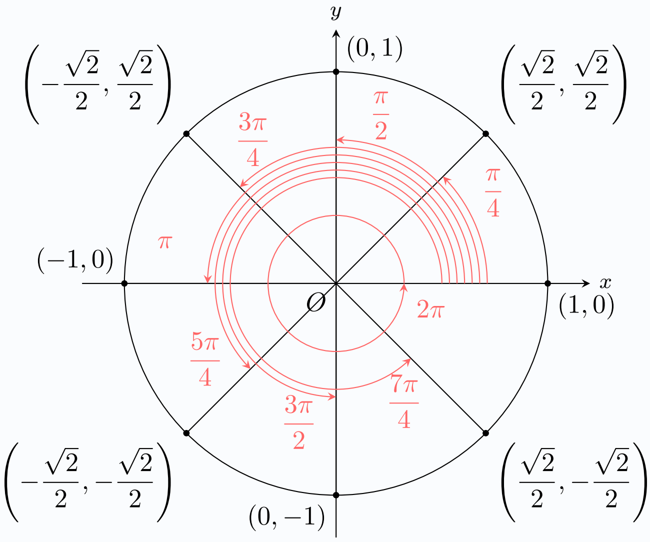
The sum of the angles in a triangle is π, so $\angle OMH=\pi-\frac{\pi}{4}-\frac{\pi}{2}=\frac{\pi}{4}$.
Since $\angle OMH=\angle MOH$, the triangle OHM is isosceles.
Let $a=OH=HM$.

$$\begin{aligned} a^2+a^2&=1^2 \quad \text{(Pythagorean theorem in the right triangle } OHM\text{)}\\ 2a^2&=1\\ a^2&=\frac{1}{2}\\ a&=\frac{\sqrt{2}}{2} \quad \text{since } a\geqslant 0 \end{aligned}$$

So $M\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.
As $\cos\theta$ is the x-coordinate of M and $\sin\theta$ is the y-coordinate of M,

$$\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

Proposition **Multiples of $\frac{\pi}{4}$**



Proof

The coordinates of each point are found by using reflection symmetries over the axes or the origin.

The signs of the coordinates are determined by the quadrant in which the angle lies.

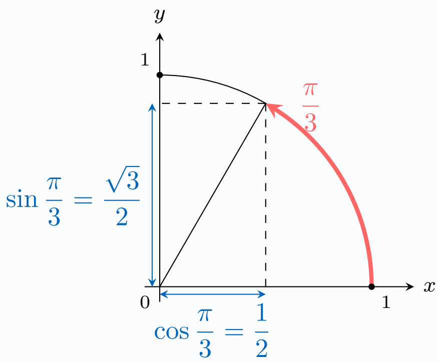
Ex: Find $\cos\frac{3\pi}{4}$.

Answer: $\cos\frac{3\pi}{4}=-\frac{\sqrt{2}}{2}$

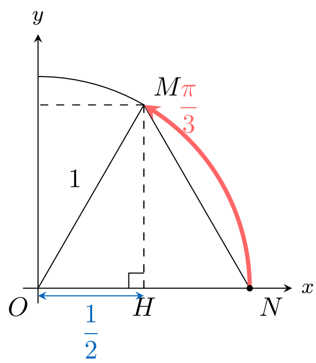
E MULTIPLES OF $\frac{\pi}{6}$

Proposition **Coordinates of Angle $\frac{\pi}{3}$**

$$\cos\frac{\pi}{3}=\frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$



Proof



Let $\angle MON=\frac{\pi}{3}$.

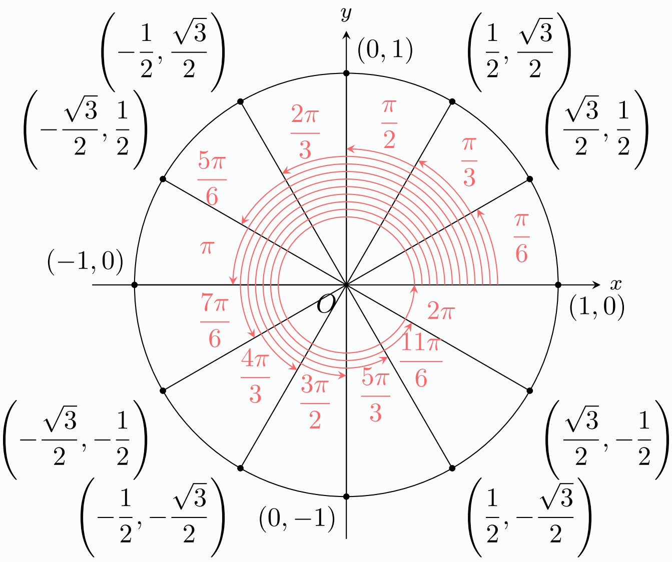
As $ON = OM = 1$, the triangle OMN is isosceles. So $\angle MON = \angle MNO = \frac{\pi}{3}$.
Since the sum of the angles in a triangle is π, we have $\angle OMN = \frac{\pi}{3}$.
So the triangle OMN is equilateral.
The altitude MH bisects the base ON.
Thus $OH = \frac{1}{2}$.

$$\begin{aligned} OH^2 + HM^2 &= OM^2 \quad \text{(Pythagorean theorem for the right triangle } OHM\text{)} \\ \left(\frac{1}{2}\right)^2 + HM^2 &= 1 \\ HM^2 &= \frac{3}{4} \\ HM &= \frac{\sqrt{3}}{2} \quad \text{since } HM \geqslant 0 \end{aligned}$$

As $\cos\theta$ is the x-coordinate of M and $\sin\theta$ is the y-coordinate of M:

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Proposition Multiples of $\frac{\pi}{6}$

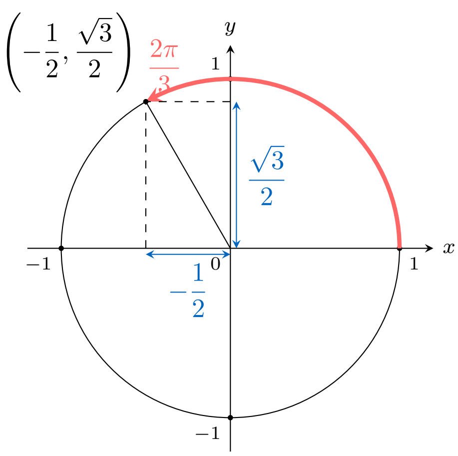


Proof

The coordinates of each point are found by applying reflection symmetries over the axes or the origin.

Ex: Find $\cos\frac{2\pi}{3}$ and $\sin\frac{2\pi}{3}$.

Answer:



$\cos\frac{2\pi}{3} = -\frac{1}{2}$ and $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$

F TANGENT FUNCTION

Definition Tangent Function

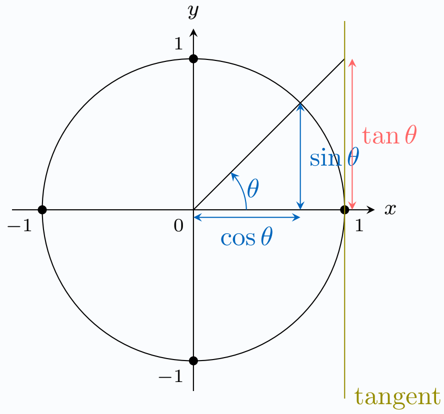
The **tangent** of an angle θ is defined, whenever $\cos\theta \neq 0$, as the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Equivalently, $\tan\theta$ is defined for all real θ such that $\theta \neq \frac{\pi}{2} + k\pi$ for any integer k.

Proposition Geometric Interpretation of Tangent

On the unit circle, for any angle θ with $\cos\theta \neq 0$, the ray from the origin forming an angle θ with the positive x-axis meets the vertical tangent line $x = 1$ at the point $(1, \tan\theta)$. In particular, $\tan\theta$ is the y-coordinate of this intersection point.



Proof

The ray from the origin at an angle θ passes through the point $P(\cos\theta, \sin\theta)$ on the unit circle.Provided $\cos\theta \neq 0$, the slope of the line through the origin and P is

$$\frac{\sin\theta}{\cos\theta} = \tan\theta,$$

so the equation of this line is $y = (\tan\theta)\,x$.The intersection of this line with the vertical line $x = 1$ occurs when $y = (\tan\theta)\cdot 1 = \tan\theta$. Thus, the intersection point is $(1, \tan\theta)$.

Proposition Tangent Values for Common Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

The values in other quadrants follow from the symmetries of the unit circle and the fact that sine and cosine are 2π-periodic.

G ANGLE SUM AND DIFFERENCE IDENTITIES

Proposition Cosine of Difference

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$

Proof

Consider two unit vectors on the unit circle:

$$\vec{u} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} \cos B \\ \sin B \end{pmatrix}.$$

The angle between $\vec{u}$ and $\vec{v}$ is $A - B$.
We can express the dot product $\vec{u} \cdot \vec{v}$ in two ways:

1. Using the geometric definition:

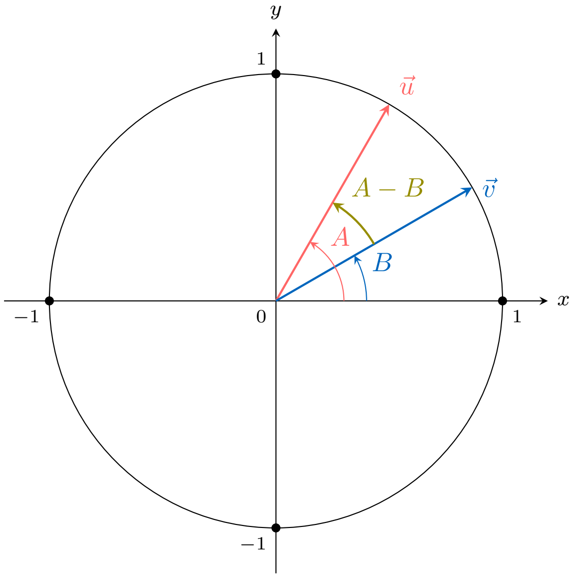
$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos(A - B) = 1 \cdot 1 \cdot \cos(A - B) = \cos(A - B).$$

2. Using the component form:

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} \cos A \\ \sin A \end{pmatrix} \cdot \begin{pmatrix} \cos B \\ \sin B \end{pmatrix} = (\cos A)(\cos B) + (\sin A)(\sin B).$$

Equating the two expressions gives the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B.$$



Proposition Cosine of Sum

$$\cos(A + B) = \cos A \cos B - \sin A \sin B.$$

Proof

Substitute B by $-B$ in the cosine difference formula:

$$\begin{aligned} \cos(A + B) &= \cos\big(A - (-B)\big) \\ &= \cos A \cos(-B) + \sin A \sin(-B) \\ &= \cos A \cos B + \sin A(-\sin B) \quad (\text{since } \cos(-\theta) = \cos\theta \text{ and } \sin(-\theta) = -\sin\theta) \\ &= \cos A \cos B - \sin A \sin B. \end{aligned}$$

Proposition Sine of Sum and Difference

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$
$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Proof

Using the complementary angle identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$:

$$\begin{aligned}\sin(A+B) &= \cos\left(\frac{\pi}{2} - (A+B)\right) \\ &= \cos\left(\left(\frac{\pi}{2} - A\right) - B\right) \\ &= \cos\left(\frac{\pi}{2} - A\right)\cos B + \sin\left(\frac{\pi}{2} - A\right)\sin B \\ &= \sin A\cos B + \cos A\sin B.\end{aligned}$$

For the difference, write

$$\begin{aligned}\sin(A-B) &= \sin\bigl(A + (-B)\bigr) \\ &= \sin A\cos(-B) + \cos A\sin(-B) \\ &= \sin A\cos B - \cos A\sin B.\end{aligned}$$

Proposition **Tangent of Sum and Difference**

For angles A and B such that all expressions below are defined,

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B},$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

Proof

Assume $\cos A\cos B \neq 0$ and $\cos(A+B) \neq 0$ so that all ratios below are defined:

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\ &= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} \\ &= \frac{\dfrac{\sin A\cos B + \cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B}} \quad \text{(divide numerator and denominator by } \cos A\cos B\text{)} \\ &= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}} \\ &= \frac{\tan A + \tan B}{1 - \tan A\tan B}.\end{aligned}$$

The formula for $\tan(A-B)$ is obtained by replacing B with $-B$ and using $\tan(-B) = -\tan B$.