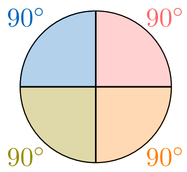
# RADIANS AND THE UNIT CIRCLE

## A RADIAN MEASURE

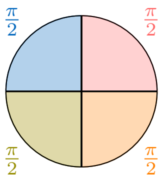
### A.1 CONVERTING DEGREES TO RADIANS IN TERMS OF $\pi$

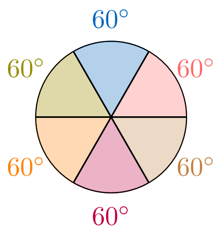



**Ex 1:** Convert to radians in terms of $\pi$:

$$90° = \boxed{\frac{\pi}{2}}$$

*Answer:*

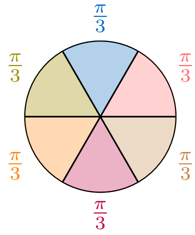
- **Method 1**: $90° = \left(90° \times \frac{\pi}{180°}\right)$
$$= \frac{\cancel{90°} \times \pi}{2 \times \cancel{90°}}$$
$$= \frac{\pi}{2}$$
- **Method 2**: $180° = \pi$
$$\frac{180°}{2} = \frac{\pi}{2} \quad \text{(dividing both sides by 2)}$$
$$90° = \frac{\pi}{2}$$
- 

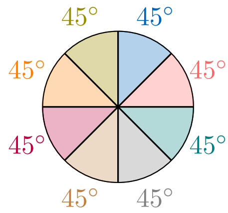




**Ex 2:** Convert to radians in terms of $\pi$:

$$60° = \boxed{\frac{\pi}{3}}$$

*Answer:*

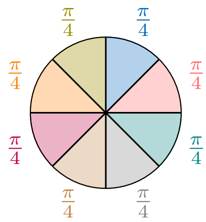
- **Method 1**: $60° = \left(60° \times \frac{\pi}{180°}\right)$
$$= \frac{\cancel{60°} \times \pi}{3 \times \cancel{60°}}$$
$$= \frac{\pi}{3}$$
- **Method 2**: $180° = \pi$
$$\frac{180°}{3} = \frac{\pi}{3} \quad \text{(dividing both sides by 3)}$$
$$60° = \frac{\pi}{3}$$
- 

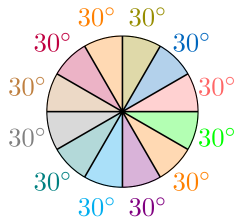




**Ex 3:** Convert to radians in terms of $\pi$:

$$45° = \boxed{\frac{\pi}{4}}$$

*Answer:*

- **Method 1**: $45° = \left(45° \times \frac{\pi}{180°}\right)$
$$= \frac{\cancel{45°} \times \pi}{4 \times \cancel{45°}}$$
$$= \frac{\pi}{4}$$
- **Method 2**: $180° = \pi$
$$\frac{180°}{4} = \frac{\pi}{4} \quad \text{(dividing both sides by 4)}$$
$$45° = \frac{\pi}{4}$$
- 

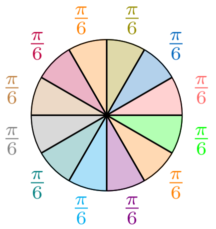




**Ex 4:** Convert to radians in terms of $\pi$:

$$30° = \boxed{\frac{\pi}{6}}$$

*Answer:*

- **Method 1**: $30° = \left(30° \times \frac{\pi}{180°}\right)$
$$= \frac{\cancel{30°} \times \pi}{6 \times \cancel{30°}}$$
$$= \frac{\pi}{6}$$
- **Method 2**: $180° = \pi$
$$\frac{180°}{6} = \frac{\pi}{6} \quad \text{(dividing both sides by 6)}$$
$$30° = \frac{\pi}{6}$$

- 


## A.2 CONVERTING DEGREES TO RADIANS

**Ex 5:** Convert to radians (rounded to 2 decimal places).

$$46.5^\circ \approx \boxed{0.81} \text{ rad}$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 46.5^\circ &= 46.5 \times \frac{\pi}{180} \text{ radians} \\ &\approx 0.81157... \\ &\approx 0.81 \text{ rad} \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in radians be $x$. We set up a proportion based on the fact that $180^\circ = \pi$ radians.

$$\begin{aligned} \frac{x}{\pi} &= \frac{46.5}{180} \\ x &= \pi \times \frac{46.5}{180} \\ x &\approx 0.81157... \\ x &\approx 0.81 \text{ rad} \end{aligned}$$

**Ex 6:** Convert to radians (rounded to 2 decimal places).

$$110^\circ \approx \boxed{1.92} \text{ rad}$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 110^\circ &= 110 \times \frac{\pi}{180} \text{ radians} \\ &\approx 1.91986... \\ &\approx 1.92 \text{ rad} \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in radians be $x$.

$$\begin{aligned} \frac{x}{\pi} &= \frac{110}{180} \\ x &= \pi \times \frac{110}{180} \\ x &\approx 1.91986... \\ x &\approx 1.92 \text{ rad} \end{aligned}$$

**Ex 7:** Convert to radians (rounded to 2 decimal places).

$$43^\circ \approx \boxed{0.75} \text{ rad}$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 43^\circ &= 43 \times \frac{\pi}{180} \text{ radians} \\ &\approx 0.75049... \\ &\approx 0.75 \text{ rad} \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in radians be $x$.

$$\begin{aligned} \frac{x}{\pi} &= \frac{43}{180} \\ x &= \pi \times \frac{43}{180} \\ x &\approx 0.75049... \\ x &\approx 0.75 \text{ rad} \end{aligned}$$

**Ex 8:** Convert to radians (rounded to 2 decimal places).

$$300^\circ \approx \boxed{5.24} \text{ rad}$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 300^\circ &= 300 \times \frac{\pi}{180} \text{ radians} \\ &\approx 5.23598... \\ &\approx 5.24 \text{ rad} \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in radians be $x$.

$$\begin{aligned} \frac{x}{\pi} &= \frac{300}{180} \\ x &= \pi \times \frac{300}{180} \\ x &\approx 5.23598... \\ x &\approx 5.24 \text{ rad} \end{aligned}$$

## A.3 CONVERTING RADIANS TO DEGREES

**Ex 9:** Convert to degrees (rounded to the nearest integer).

$$1.25 \text{ rad} \approx \boxed{72}^\circ$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 1.25 \text{ rad} &= 1.25 \times \frac{180^\circ}{\pi} \\ &\approx 71.6197...^\circ \\ &\approx 72^\circ \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in degrees be $x$.

$$\begin{aligned} \frac{x}{180^\circ} &= \frac{1.25}{\pi} \\ x &= 180^\circ \times \frac{1.25}{\pi} \\ x &\approx 71.6197...^\circ \\ x &\approx 72^\circ \end{aligned}$$

**Ex 10:** Convert to degrees (rounded to the nearest integer).

$$0.7 \text{ rad} \approx \boxed{40}^\circ$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 0.7 \text{ rad} &= 0.7 \times \frac{180^\circ}{\pi} \\ &\approx 40.1070...^\circ \\ &\approx 40^\circ \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in degrees be $x$.

$$\begin{aligned} \frac{x}{180^\circ} &= \frac{0.7}{\pi} \\ x &= 180^\circ \times \frac{0.7}{\pi} \\ x &\approx 40.1070...^\circ \\ x &\approx 40^\circ \end{aligned}$$

**Ex 11:** Convert to degrees (rounded to the nearest integer).

$$4.5 \text{ rad} \approx \boxed{258}^\circ$$

*Answer:*

- **Method 1 (Using the conversion formula):**

$$\begin{aligned} 4.5 \text{ rad} &= 4.5 \times \frac{180^\circ}{\pi} \\ &\approx 257.831...^\circ \\ &\approx 258^\circ \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in degrees be $x$.

$$\begin{aligned} \frac{x}{180^\circ} &= \frac{4.5}{\pi} \\ x &= 180^\circ \times \frac{4.5}{\pi} \\ x &\approx 257.831...^\circ \\ x &\approx 258^\circ \end{aligned}$$

**Ex 12:** Convert to degrees (rounded to the nearest integer).

$$2 \text{ rad} \approx \boxed{115}^\circ$$

*Answer:*

- **Method 1 (Using the conversion formula):**

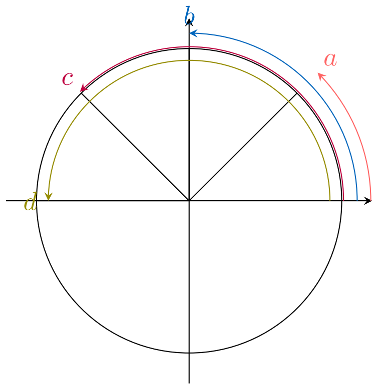
$$\begin{aligned} 2 \text{ rad} &= 2 \times \frac{180^\circ}{\pi} \\ &\approx 114.591...^\circ \\ &\approx 115^\circ \end{aligned}$$

- **Method 2 (Using proportion):** Let the angle in degrees be $x$.

$$\begin{aligned} \frac{x}{180^\circ} &= \frac{2}{\pi} \\ x &= 180^\circ \times \frac{2}{\pi} \\ x &\approx 114.591...^\circ \\ x &\approx 115^\circ \end{aligned}$$

## A.4 CONVERTING REFERENCE ANGLES TO RADIANS

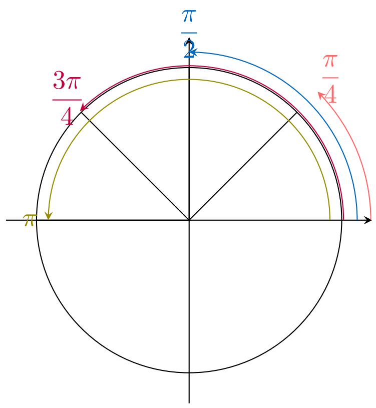
**Ex 13:** Convert to radians in terms of $\pi$:



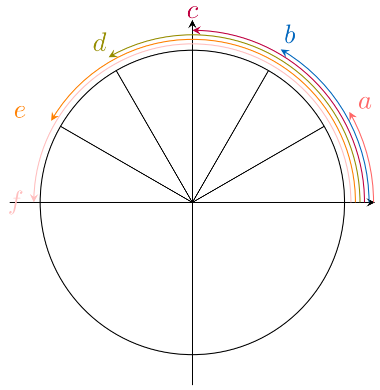

- $a = \boxed{\frac{\pi}{4}}$
- $b = \boxed{\frac{\pi}{2}}$
- $c = \boxed{\frac{3\pi}{4}}$
- $d = \boxed{\pi}$

*Answer:* All angles are multiples of $\frac{\pi}{4}$:

- $1 \times \frac{\pi}{4} = \frac{\pi}{4}$
- $2 \times \frac{\pi}{4} = \frac{\pi}{2}$
- $3 \times \frac{\pi}{4} = \frac{3\pi}{4}$
- $4 \times \frac{\pi}{4} = \pi$



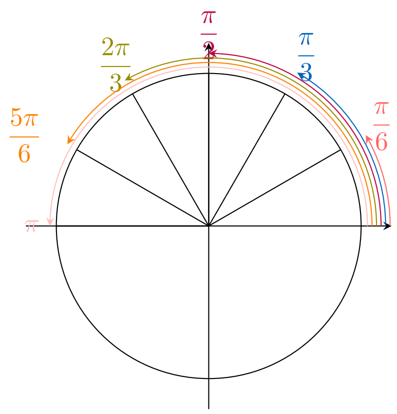

**Ex 14:** Convert to radians in terms of $\pi$:




- $a = \boxed{\frac{\pi}{6}}$
- $b = \boxed{\frac{\pi}{3}}$
- $c = \boxed{\frac{\pi}{2}}$
- $d = \boxed{\frac{2\pi}{3}}$
- $e = \boxed{\frac{5\pi}{6}}$
- $f = \boxed{\pi}$

*Answer:* All angles are multiples of $\frac{\pi}{6}$:

- 1 time: $\frac{\pi}{6}$
- 2 times: $2 \times \frac{\pi}{6} = \frac{\pi}{3}$
- 3 times: $3 \times \frac{\pi}{6} = \frac{\pi}{2}$
- 4 times: $4 \times \frac{\pi}{6} = \frac{2\pi}{3}$
- 5 times: $5 \times \frac{\pi}{6} = \frac{5\pi}{6}$
- 6 times: $6 \times \frac{\pi}{6} = \pi$




## A.5 CALCULATING TRIGONOMETRIC VALUES

**Ex 15:** Calculate (round to 2 decimal places)

$$\cos\left(\frac{\pi}{4}\right) \approx \boxed{0.71}$$

*Answer:* To calculate this on your calculator, you can either:

- Set the calculator to radian mode and compute $\cos\left(\frac{\pi}{4}\right) \approx 0.71$
- Or, if in degree mode, first convert the angle to degrees: $\frac{\pi}{4} \times \frac{180°}{\pi} = 45°$, then $\cos(45°) \approx 0.71$

**Ex 16:** Calculate (round to 2 decimal places)

$$\cos\left(\frac{\pi}{6}\right) \approx \boxed{0.87}$$

*Answer:* To calculate this on your calculator, you can either:

- Set the calculator to radian mode and compute $\cos\left(\frac{\pi}{6}\right) \approx 0.87$
- Or, if in degree mode, first convert the angle to degrees: $\frac{\pi}{6} \times \frac{180°}{\pi} = 30°$, then $\cos(30°) \approx 0.87$

**Ex 17:** Calculate (round to 1 decimal place)

$$\sin\left(\frac{5\pi}{6}\right) \approx \boxed{0.5}$$

*Answer:* To calculate this on your calculator, you can either:

- Set the calculator to radian mode and compute $\sin\left(\frac{5\pi}{6}\right) \approx 0.5$
- Or, if in degree mode, first convert the angle to degrees: $\frac{5\pi}{6} \times \frac{180°}{\pi} = 150°$, then $\sin(150°) = 0.5$

**Ex 18:** Calculate (round to 2 decimal places)

$$\sin\left(-\frac{\pi}{5}\right) \approx \boxed{-0.59}$$

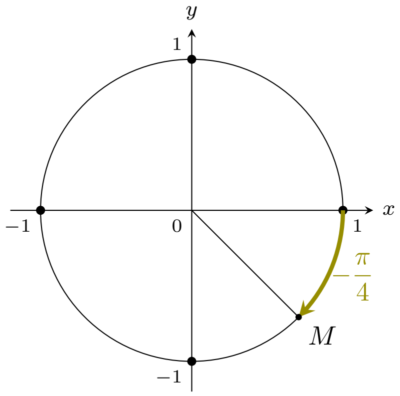
*Answer:* To calculate this on your calculator, you can either:

- Set the calculator to radian mode and compute $\sin\left(-\frac{\pi}{5}\right) \approx -0.59$
- Or, if in degree mode, first convert the angle to degrees: $-\frac{\pi}{5} \times \frac{180°}{\pi} = -36°$, then $\sin(-36°) \approx -0.59$

# B TRIGONOMETRY ON THE UNIT CIRCLE

## B.1 EXPRESSING THE COORDINATES OF A POINT ON THE UNIT CIRCLE

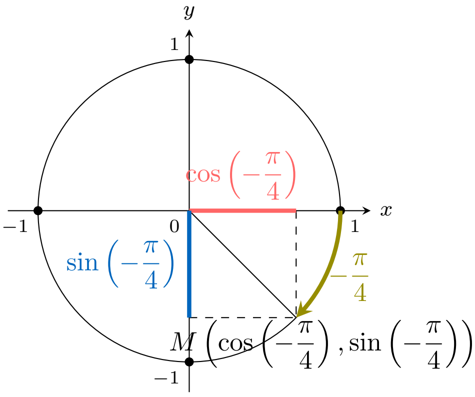
**Ex 19:**



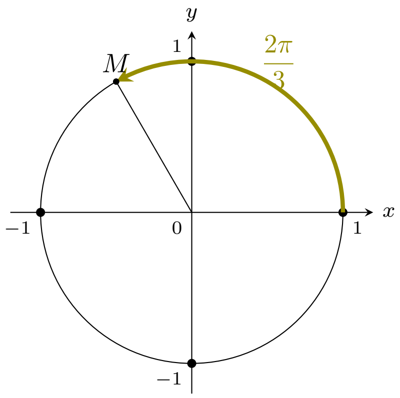

Determine the coordinates of point $M$ in terms of sine and cosine:

$$M(\boxed{\cos(-\frac{\pi}{4})}, \boxed{\sin(-\frac{\pi}{4})}).$$

*Answer:*



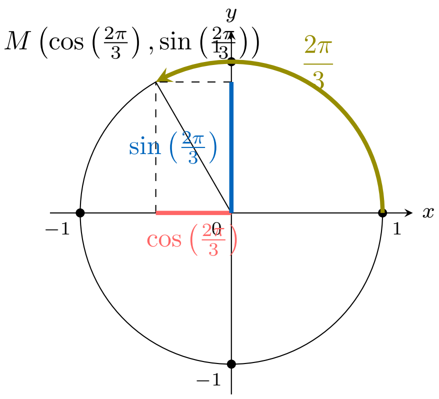

**Ex 20:**



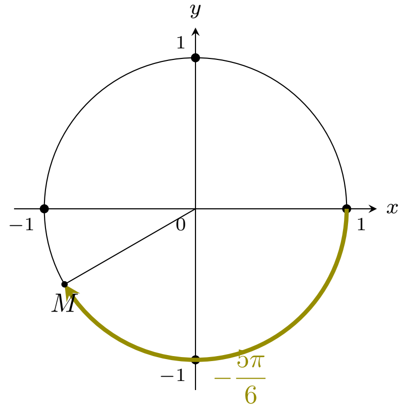

Determine the coordinates of point $M$ in terms of sine and cosine:

$$M(\boxed{\cos(\frac{2\pi}{3})}, \boxed{\sin(\frac{2\pi}{3})}).$$

*Answer:*



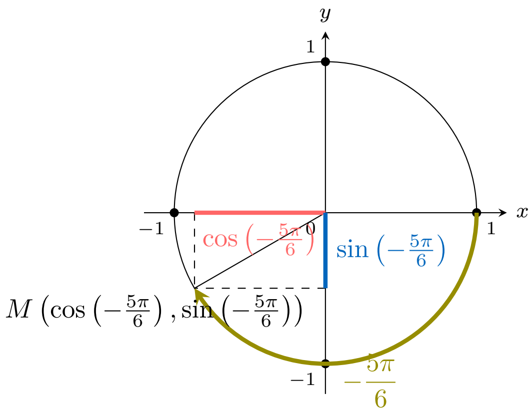

**Ex 21:**




Determine the coordinates of point $M$ in terms of sine and cosine:

$$M(\boxed{\cos(-\frac{5\pi}{6})}, \boxed{\sin(-\frac{5\pi}{6})}).$$

*Answer:*

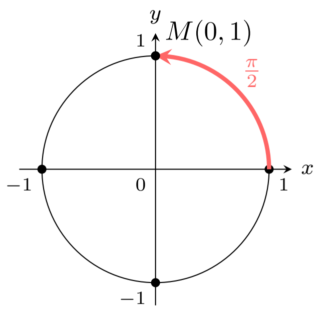



## B.2 FINDING SINE AND COSINE VALUES ON THE UNIT CIRCLE

**Ex 22:** Find the values:

- $\cos\left(\frac{\pi}{2}\right) = \boxed{0}$
- $\sin\left(\frac{\pi}{2}\right) = \boxed{1}$

*Answer:* On the unit circle, the point corresponding to the angle $\frac{\pi}{2}$ has coordinates (0,1):



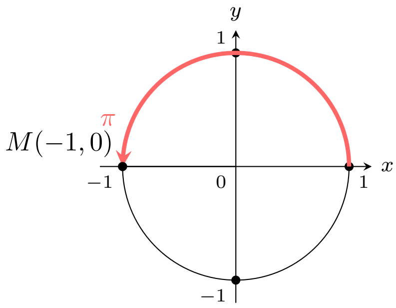

$\cos\left(\frac{\pi}{2}\right) = 0$ $x$-coordinate
$\sin\left(\frac{\pi}{2}\right) = 1$ $y$-coordinate

**Ex 23:** Find the values:

- $\cos(\pi) = \boxed{-1}$

- $\sin(\pi) = \boxed{0}$

*Answer:* On the unit circle, the point corresponding to the angle $\pi$ has coordinates $(-1, 0)$:

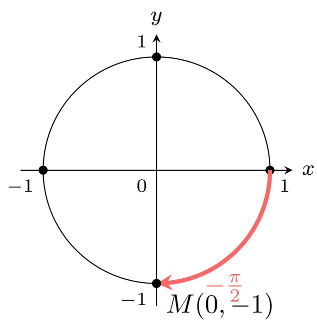



$\cos(\pi) = -1$ $x$-coordinate
$\sin(\pi) = 0$ $y$-coordinate

**Ex 24:** Find the values:

- $\cos\left(-\frac{\pi}{2}\right) = \boxed{0}$
- $\sin\left(-\frac{\pi}{2}\right) = \boxed{-1}$

*Answer:* On the unit circle, the point corresponding to the angle $-\frac{\pi}{2}$ has coordinates $(0, -1)$:

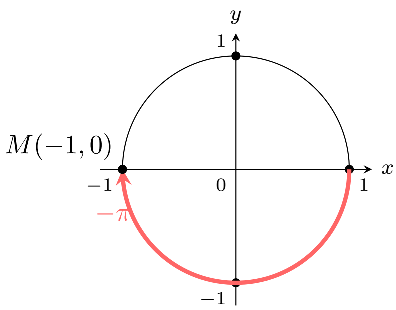



$\cos\left(-\frac{\pi}{2}\right) = 0$ $x$-coordinate
$\sin\left(-\frac{\pi}{2}\right) = -1$ $y$-coordinate

**Ex 25:** Find the values:

- $\cos(-\pi) = \boxed{-1}$
- $\sin(-\pi) = \boxed{0}$

*Answer:* On the unit circle, the point corresponding to the angle $-\pi$ has coordinates $(-1, 0)$:

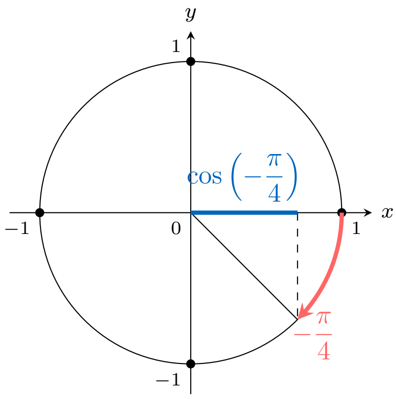



$\cos(-\pi) = -1$ $x$-coordinate
$\sin(-\pi) = 0$ $y$-coordinate

## B.3 DETERMINING THE SIGN OF SINE AND COSINE

**Ex 26:** Determine the sign of $\cos\left(-\frac{\pi}{4}\right)$: $\boxed{+}$.

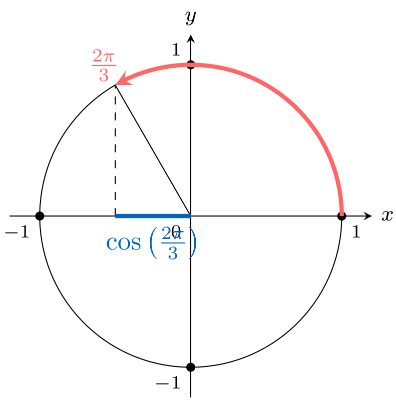
*Answer:* On the unit circle, the angle $-\frac{\pi}{4}$ (or $-45°$) lies in Quadrant 4, where the $x$-coordinate (cosine) is positive:




Thus, $\cos\left(-\frac{\pi}{4}\right)$ is positive.

**Ex 27:** Determine the sign of $\cos\left(\frac{2\pi}{3}\right)$: $\boxed{-}$.

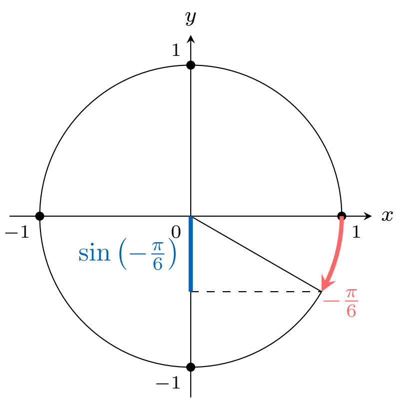
*Answer:* On the unit circle, the angle $\frac{2\pi}{3}$ (or $120°$) lies in Quadrant 2, where the $x$-coordinate (cosine) is negative:




Thus, $\cos\left(\frac{2\pi}{3}\right)$ is negative.

**Ex 28:** Determine the sign of $\sin\left(-\frac{\pi}{6}\right)$: $\boxed{-}$.

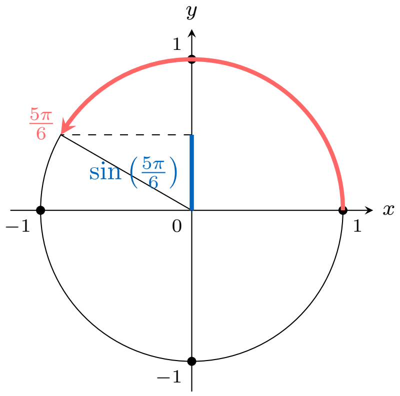
*Answer:* On the unit circle, the angle $-\frac{\pi}{6}$ (or $-30°$) lies in Quadrant 4, where the $y$-coordinate (sine) is negative:




Thus, $\sin\left(-\frac{\pi}{6}\right)$ is negative.

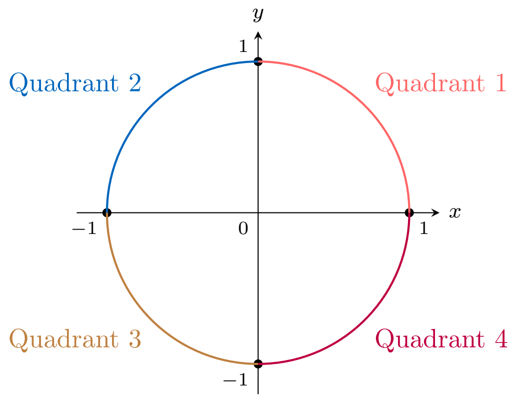
**Ex 29:** Determine the sign of $\sin\left(\frac{5\pi}{6}\right)$: $\boxed{+}$.

*Answer:* On the unit circle, the angle $\frac{5\pi}{6}$ (or $150°$) lies in Quadrant 2, where the $y$-coordinate (sine) is positive:




Thus, $\sin\left(\frac{5\pi}{6}\right)$ is positive.

**Ex 30:**




- For Quadrant 1, $\cos\theta$ is $\boxed{+}$ and $\sin\theta$ is $\boxed{+}$.
- For Quadrant 2, $\cos\theta$ is $\boxed{-}$ and $\sin\theta$ is $\boxed{+}$.
- For Quadrant 3, $\cos\theta$ is $\boxed{-}$ and $\sin\theta$ is $\boxed{-}$.
- For Quadrant 4, $\cos\theta$ is $\boxed{+}$ and $\sin\theta$ is $\boxed{-}$.

*Answer:*

| Quadrant | $\cos\theta$ | $\sin\theta$ |
|---|---|---|
| 1 | + | + |
| 2 | − | + |
| 3 | − | − |
| 4 | + | − |

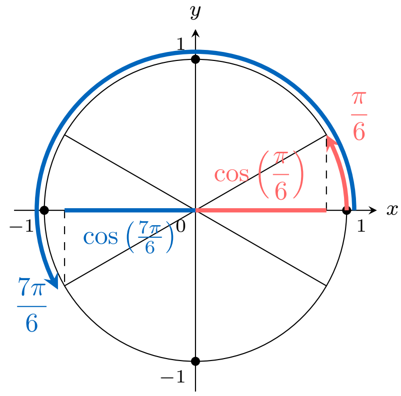
# C TRIGONOMETRIC IDENTITIES

## C.1 EXPRESSING TRIGONOMETRIC VALUES IN TERMS OF REFERENCE ANGLES

**Ex 31:** Express $\cos\left(\frac{7\pi}{6}\right)$ in terms of sine or cosine of $\frac{\pi}{6}$ (use a unit circle):

$$\cos\left(\frac{7\pi}{6}\right) = \boxed{-\cos(\frac{\pi}{6})}$$

*Answer:*

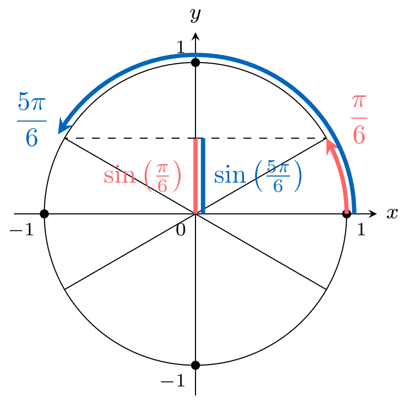



$$\cos\left(\frac{7\pi}{6}\right) = \cos\left(\pi + \frac{\pi}{6}\right)$$
$$= -\cos\left(\frac{\pi}{6}\right)$$

**Ex 32:** Express $\sin\left(\frac{5\pi}{6}\right)$ in terms of sine or cosine of $\frac{\pi}{6}$ (use a unit circle):

$$\sin\left(\tfrac{5\pi}{6}\right) = \boxed{\sin(\frac{\pi}{6})}$$

*Answer:*

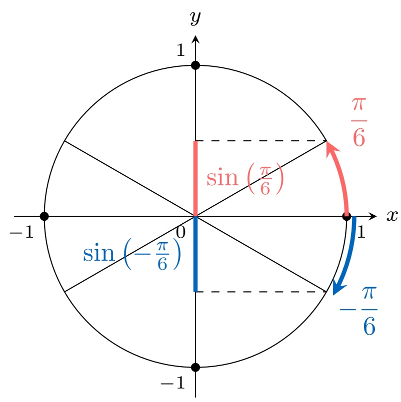



$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right)$$
$$= \sin\left(\frac{\pi}{6}\right)$$

**Ex 33:** Express $\sin\left(-\frac{\pi}{6}\right)$ in terms of sine or cosine of $\frac{\pi}{6}$ (use a unit circle):

$$\sin\left(-\tfrac{\pi}{6}\right) = \boxed{-\sin(\frac{\pi}{6})}$$

*Answer:*

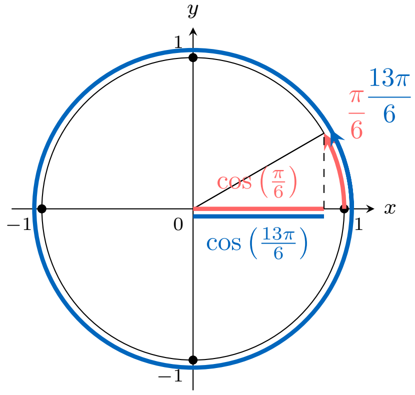



$$\sin\left(-\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$$

**Ex 34:** Express $\cos\left(\frac{13\pi}{6}\right)$ in terms of cosine or sine of $\frac{\pi}{6}$ (use a unit circle):

$$\cos\left(\frac{13\pi}{6}\right) = \boxed{\cos(\frac{\pi}{6})}$$

*Answer:*

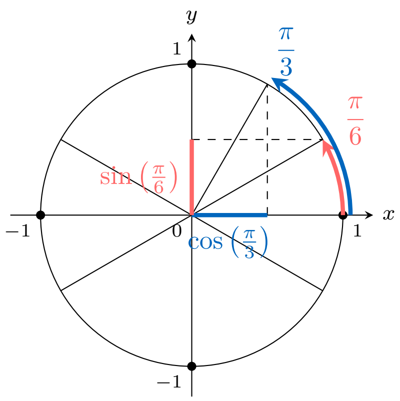



$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(2\pi + \frac{\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{6}\right)\end{aligned}$$

**Ex 35:** Express $\cos\left(\frac{\pi}{3}\right)$ in terms of sine or cosine of $\frac{\pi}{6}$ (use a unit circle):

$$\cos\left(\frac{\pi}{3}\right) = \boxed{\sin(\frac{\pi}{6})}$$

*Answer:*




$$\begin{aligned}\cos\left(\frac{\pi}{3}\right) &= \cos\left(\frac{\pi}{2} - \frac{\pi}{6}\right)\\ &= \sin\left(\frac{\pi}{6}\right)\end{aligned}$$

## C.2 EXPLAINING TRIGONOMETRIC EQUALITIES

**Ex 36:** Explain why $\cos\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right)$.

*Answer:*

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{12\pi}{6} + \frac{\pi}{6}\right)\\ &= \cos\left(2\pi + \frac{\pi}{6}\right)\\ &= \cos\left(\frac{\pi}{6}\right)\end{aligned}$$

**Ex 37:** Explain why $\cos\left(\frac{7\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right)$.

*Answer:*

$$\begin{aligned}\cos\left(\frac{7\pi}{6}\right) &= \cos\left(\frac{6\pi}{6} + \frac{\pi}{6}\right)\\ &= \cos\left(\pi + \frac{\pi}{6}\right)\\ &= -\cos\left(\frac{\pi}{6}\right)\end{aligned}$$

**Ex 38:** Explain why $\sin\left(\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right)$.

*Answer:*

$$\sin\left(\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right) \quad (\text{since } \sin(-x) = -\sin(x))$$

**Ex 39:** Explain why $\sin\left(\frac{5\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$.

*Answer:*

$$\begin{aligned}\sin\left(\frac{5\pi}{4}\right) &= \sin\left(\pi + \frac{\pi}{4}\right)\\ &= -\sin\left(\frac{\pi}{4}\right)\end{aligned}$$

because $\sin(\pi + x) = -\sin(x)$.

**Ex 40:** Explain why $\sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right)$.

*Answer:*

$$\begin{aligned}\sin\left(\frac{5\pi}{2}\right) &= \sin\left(2\pi + \frac{\pi}{2}\right)\\ &= \sin\left(\frac{\pi}{2}\right)\end{aligned}$$

## C.3 FINDING EXACT TRIGONOMETRIC VALUES USING THE PYTHAGOREAN IDENTITY

**Ex 41:** Find the exact value of $\sin\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $0 \leq \theta \leq \pi$.

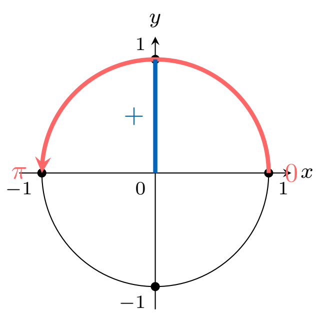
$$\sin\theta = \boxed{\frac{1}{2}}$$

*Answer:* We use the Pythagorean identity:

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= 1\\ \left(\frac{\sqrt{3}}{2}\right)^2 + \sin^2\theta &= 1\\ \frac{3}{4} + \sin^2\theta &= 1\\ \sin^2\theta &= 1 - \frac{3}{4}\\ \sin^2\theta &= \frac{1}{4}\\ \sin\theta &= \pm\frac{1}{2}\end{aligned}$$

To determine the sign, note that $0 \leq \theta \leq \pi$ corresponds to Quadrants I and II, where $\sin\theta \geq 0$.

So, $\boxed{\sin\theta = \frac{1}{2}}$.



**Ex 42:** Find the exact value of $\cos\theta$ if $\sin\theta = \frac{\sqrt{2}}{2}$ and $-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$.

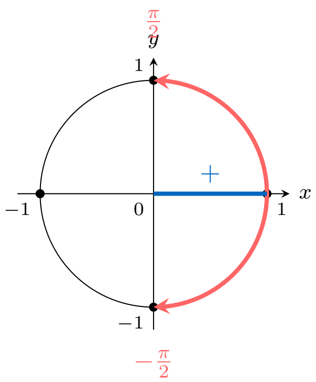
$$\cos\theta = \boxed{\frac{\sqrt{2}}{2}}$$

*Answer:* We use the Pythagorean identity:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\cos^2\theta + \left(\frac{\sqrt{2}}{2}\right)^2 &= 1\\
\cos^2\theta + \frac{2}{4} &= 1\\
\cos^2\theta + \frac{1}{2} &= 1\\
\cos^2\theta &= 1 - \frac{1}{2}\\
\cos^2\theta &= \frac{1}{2}\\
\cos\theta &= \pm\frac{\sqrt{2}}{2}
\end{aligned}$$

To determine the sign, note that $-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$ corresponds to Quadrants I and IV, where $\cos\theta$ is positive.

So, $\boxed{\cos\theta = \frac{\sqrt{2}}{2}}$.




**Ex 43:** Find the exact value of $\sin\theta$ if $\cos\theta = \frac{1}{2}$ and $-\pi \leq \theta < 0$.

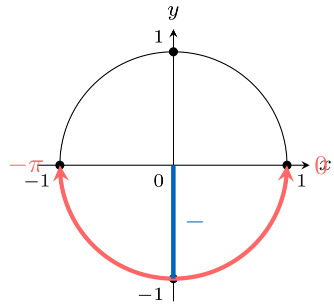
$$\sin\theta = \boxed{-\frac{\sqrt{3}}{2}}$$

*Answer:* We use the Pythagorean identity:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\left(\frac{1}{2}\right)^2 + \sin^2\theta &= 1\\
\frac{1}{4} + \sin^2\theta &= 1\\
\sin^2\theta &= 1 - \frac{1}{4}\\
\sin^2\theta &= \frac{3}{4}\\
\sin\theta &= \pm\frac{\sqrt{3}}{2}
\end{aligned}$$

To determine the sign, note that $-\pi \leq \theta < 0$ corresponds to Quadrants III and IV, where $\sin\theta < 0$.

So, $\boxed{\sin\theta = -\frac{\sqrt{3}}{2}}$.




**Ex 44:** Find the exact value of $\cos\theta$ if $\sin\theta = \frac{\sqrt{2}}{2}$ and $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

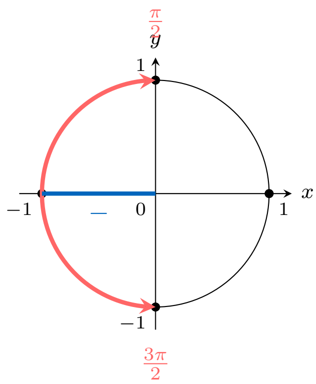
$$\cos\theta = \boxed{-\frac{\sqrt{2}}{2}}$$

*Answer:* We use the Pythagorean identity:

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1\\
\cos^2\theta + \left(\frac{\sqrt{2}}{2}\right)^2 &= 1\\
\cos^2\theta + \frac{1}{2} &= 1\\
\cos^2\theta &= 1 - \frac{1}{2}\\
\cos^2\theta &= \frac{1}{2}\\
\cos\theta &= \pm\frac{\sqrt{2}}{2}
\end{aligned}$$

To determine the sign, note that $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ corresponds to Quadrants II and III, where $\cos\theta < 0$.

So, $\boxed{\cos\theta = -\frac{\sqrt{2}}{2}}$.



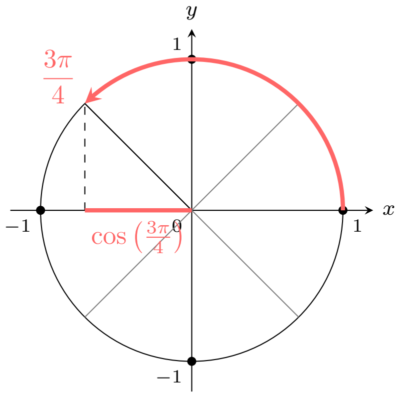

## D MULTIPLES OF $\frac{\pi}{4}$

### D.1 READING TRIGONOMETRIC VALUES FOR MULTIPLES OF $\pi/4$

**Ex 45:** Use a unit circle to find:

$$\cos\left(\frac{3\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

*Answer:*



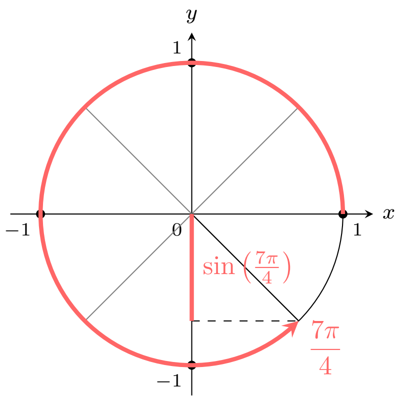

On the unit circle, the angle $3\pi/4$ is in the second quadrant, so the cosine (the $x$-coordinate) is negative:

$$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

**Ex 46:** Use a unit circle to find:

$$\sin\left(\frac{7\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

*Answer:*



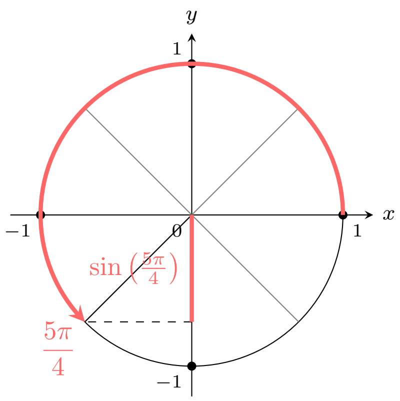

On the unit circle, the angle $7\pi/4$ is in the fourth quadrant, so the sine (the $y$-coordinate) is negative:

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

**Ex 47:** Use a unit circle to find:

$$\sin\left(\frac{5\pi}{4}\right) = \boxed{-\frac{\sqrt{2}}{2}}$$

*Answer:*



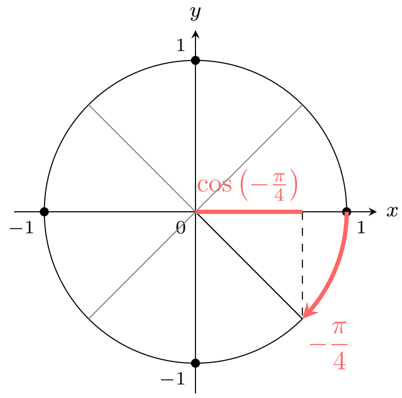

On the unit circle, the angle $5\pi/4$ is in the third quadrant, so the sine (the $y$-coordinate) is negative:

$$\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

**Ex 48:** Use a unit circle to find:

$$\cos\left(-\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$$

*Answer:*




On the unit circle, the angle $-\pi/4$ is in the fourth quadrant, so the cosine (the $x$-coordinate) is positive:

$$\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

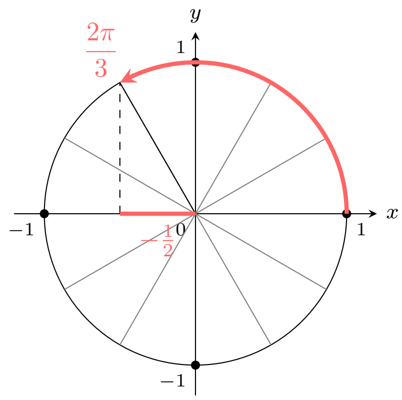
# E MULTIPLES OF $\frac{\pi}{6}$

## E.1 READING TRIGONOMETRIC VALUES FOR MULTIPLES OF $\pi/6$

**Ex 49:** Use a unit circle to find:

$$\cos\left(\frac{2\pi}{3}\right) = \boxed{-\frac{1}{2}}$$

*Answer:*



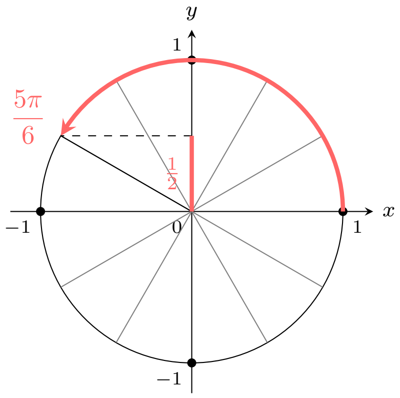

On the unit circle, the angle $2\pi/3$ is in the second quadrant, so the cosine (the $x$-coordinate) is negative:

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

**Ex 50:** Use a unit circle to find:

$$\sin\left(\frac{5\pi}{6}\right) = \boxed{\frac{1}{2}}$$

*Answer:*



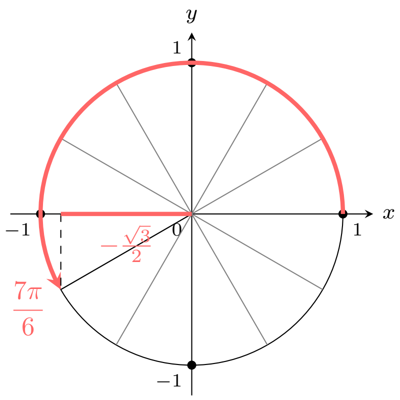

On the unit circle, the angle $5\pi/6$ is in the second quadrant, so the sine (the $y$-coordinate) is positive:

$$\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$$

**Ex 51:** Use a unit circle to find:

$$\cos\left(\frac{7\pi}{6}\right) = \boxed{-\frac{\sqrt{3}}{2}}$$

*Answer:*



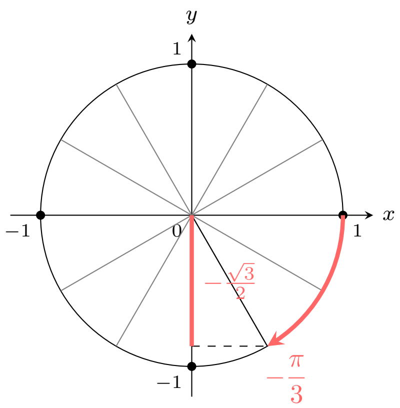

On the unit circle, the angle $7\pi/6$ is in the third quadrant, so the cosine (the $x$-coordinate) is negative:

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

**Ex 52:** Use a unit circle to find:

$$\sin\left(-\frac{\pi}{3}\right) = \boxed{-\frac{\sqrt{3}}{2}}$$

*Answer:*




On the unit circle, the angle $-\pi/3$ is in the fourth quadrant, so the sine (the $y$-coordinate) is negative:

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

# F TANGENT FUNCTION

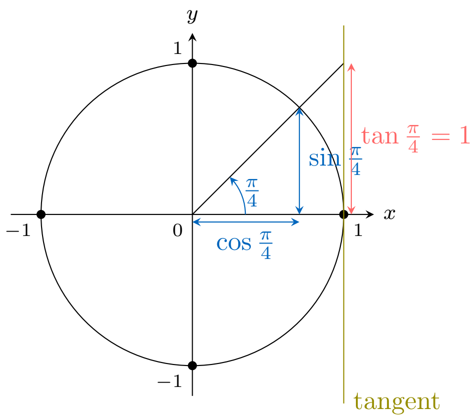
## F.1 EVALUATING THE TANGENT AT SPECIAL ANGLES

**Ex 53:** Find:

$$\tan\left(\tfrac{\pi}{4}\right) = \boxed{1}$$

*Answer:*

$$\begin{aligned}\tan\left(\frac{\pi}{4}\right) &= \frac{\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}\\ &= \frac{\sqrt{2}/2}{\sqrt{2}/2}\\ &= 1\end{aligned}$$

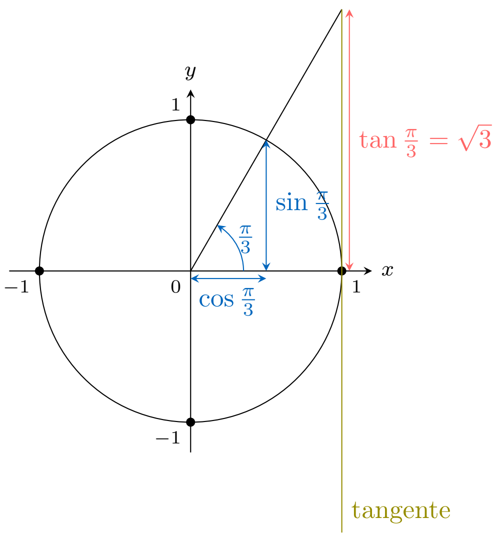



**Ex 54:** Find:

$$\tan\left(\tfrac{\pi}{3}\right) = \boxed{\sqrt{3}}$$

*Answer:*

$$\begin{aligned}\tan\left(\frac{\pi}{3}\right) &= \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}\\ &= \frac{\sqrt{3}/2}{1/2}\\ &= \sqrt{3}\end{aligned}$$

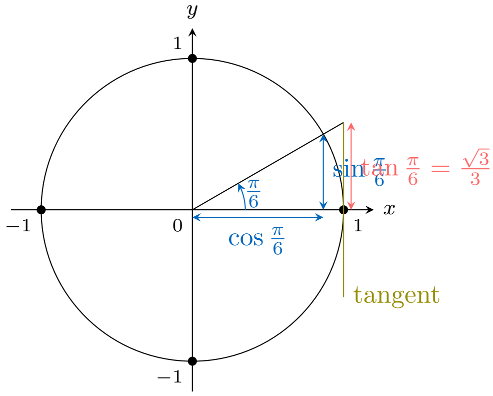


**Ex 55:** Find:

$$\tan\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{3}}$$

*Answer:*

$$\begin{aligned}\tan\left(\frac{\pi}{6}\right) &= \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}\\ &= \frac{1/2}{\sqrt{3}/2}\\ &= \frac{\sqrt{3}}{3}\end{aligned}$$

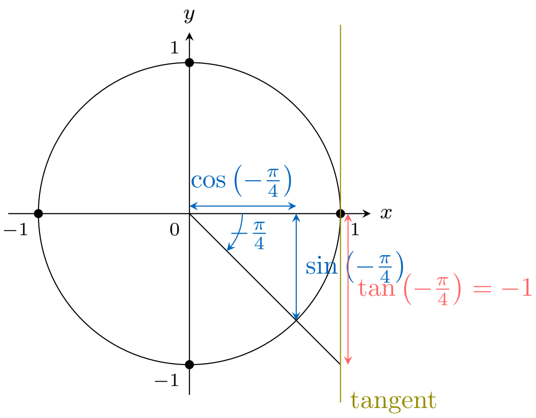



**Ex 56:** Find:

$$\tan\left(-\frac{\pi}{4}\right) = \boxed{-1}$$

*Answer:*

$$\begin{aligned}\tan\left(-\frac{\pi}{4}\right) &= \frac{\sin\left(-\frac{\pi}{4}\right)}{\cos\left(-\frac{\pi}{4}\right)}\\ &= \frac{-\sin\left(\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{4}\right)}\\ &= -\frac{\sqrt{2}/2}{\sqrt{2}/2}\\ &= -1\end{aligned}$$




**Ex 57:** Find:

$$\tan(0) = \boxed{0}$$

*Answer:*

$$\begin{aligned}\tan(0) &= \frac{\sin(0)}{\cos(0)}\\ &= \frac{0}{1}\\ &= 0\end{aligned}$$

## F.2 PROVING TANGENT PROPERTIES

**Ex 58:** Prove the identity: $\tan(-\theta) = -\tan\theta$.

*Answer:* Let $\theta$ be a real number such that $\cos(\theta) \neq 0$.

$$\begin{aligned}\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)}\\ &= \frac{-\sin\theta}{\cos\theta}\\ &= -\frac{\sin\theta}{\cos\theta}\\ &= -\tan\theta\end{aligned}$$

**Ex 59:** Prove the identity: $\tan(\theta+\pi) = \tan\theta$.

*Answer:* Let $\theta$ be a real number such that $\cos(\theta) \neq 0$.

$$\begin{aligned}\tan(\theta+\pi) &= \frac{\sin(\theta+\pi)}{\cos(\theta+\pi)}\\ &= \frac{-\sin\theta}{-\cos\theta}\\ &= \frac{\sin\theta}{\cos\theta}\\ &= \tan\theta\end{aligned}$$

**Ex 60:** Prove the identity: $\tan\left(\frac{\pi}{2}-\theta\right) = \frac{1}{\tan\theta}$.

*Answer:* Let $\theta$ be a real number such that $\cos(\theta) \neq 0$ and $\sin(\theta) \neq 0$.

$$\begin{aligned}\tan\left(\frac{\pi}{2}-\theta\right) &= \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}\\ &= \frac{\cos\theta}{\sin\theta}\\ &= \frac{1}{\frac{\sin\theta}{\cos\theta}}\\ &= \frac{1}{\tan\theta}\end{aligned}$$

**Ex 61:** Prove the identity: $\tan(\pi-\theta) = -\tan\theta$.

*Answer:* Let $\theta$ be a real number such that $\cos(\theta) \neq 0$.

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\sin(\pi-\theta)}{\cos(\pi-\theta)}\\ &= \frac{\sin\theta}{-\cos\theta}\\ &= -\frac{\sin\theta}{\cos\theta}\\ &= -\tan\theta\end{aligned}$$

# G ANGLE SUM AND DIFFERENCE IDENTITIES

## G.1 SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

**Ex 62:** Simplify:

$$\cos(2x)\cos(x) - \sin(2x)\sin(x) = \boxed{\cos(3x)}$$

*Answer:*

$$\begin{aligned}\cos(2x)\cos(x) - \sin(2x)\sin(x) &= \cos(2x+x) \quad \text{(cosine of sum identity)}\\ &= \cos 3x\end{aligned}$$

**Ex 63:** Simplify:

$$\sin(2x)\cos(x)+\cos(2x)\sin(x)=\boxed{\sin(3x)}$$

*Answer:*

$$\begin{aligned}\sin(2x)\cos(x)+\cos(2x)\sin(x)&=\sin(2x+x)\ \text{(sine of sum identity)}\\&=\sin 3x\end{aligned}$$

**Ex 64:** Simplify:

$$\cos(x)\cos(2x)+\sin(x)\sin(2x)=\boxed{\cos(x)}$$

*Answer:*

$$\begin{aligned}\cos(x)\cos(2x)+\sin(x)\sin(2x)&=\cos(x-2x)\ \text{(cosine of difference identity)}\\&=\cos(-x)\\&=\cos x\text{(since cosine is even)}\end{aligned}$$

**Ex 65:** Simplify:

$$\sin(x)\cos(2x)-\cos(x)\sin(2x)=\boxed{-\sin(x)}$$

*Answer:*

$$\begin{aligned}\sin(x)\cos(2x)-\cos(x)\sin(2x)&=\sin(x-2x)\ \text{(sine of difference identity)}\\&=\sin(-x)\\&=-\sin x\ \text{(since sine is odd)}\end{aligned}$$

## G.2 PROVING SUM-TO-PRODUCT AND HALF-ANGLE IDENTITIES

**Ex 66:** Prove the sum-to-product identity:

$$\cos(A+B)+\cos(A-B)=2\cos A\cos B$$

*Answer:* Let $A$ and $B$ be angles.

$$\begin{aligned}&\cos(A+B)+\cos(A-B)\\=&(\cos A\cos B-\sin A\sin B)+(\cos A\cos B+\sin A\sin B)\\=&2\cos A\cos B\end{aligned}$$

**Ex 67:** Prove the sum-to-product identity:

$$\sin(A+B)+\sin(A-B)=2\sin A\cos B$$

*Answer:* Let $A$ and $B$ be angles.

$$\begin{aligned}&\sin(A+B)+\sin(A-B)\\=&(\sin A\cos B+\cos A\sin B)+(\sin A\cos B-\cos A\sin B)\\=&2\sin A\cos B\end{aligned}$$

**Ex 68:** Prove the identity:

$$\sin x=\frac{2\tan(x/2)}{1+\tan^2(x/2)}$$

*Answer:* Let $x$ be a real number such that $\cos(x/2)\neq 0$.

$$\begin{aligned}\sin x&=\sin(2\cdot x/2)\\&=2\sin(x/2)\cos(x/2)\\&=\frac{2\sin(x/2)\cos(x/2)}{1}\\&=\frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)}&&\text{(Using the Pythagorean Identity)}\\&=\frac{\frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)}}{\frac{\cos^2(x/2)+\sin^2(x/2)}{\cos^2(x/2)}}&&\text{(Divide numerator and denominator by }\cos^2(x/2))\\&=\frac{2\frac{\sin(x/2)}{\cos(x/2)}}{1+\frac{\sin^2(x/2)}{\cos^2(x/2)}}&&\text{(Simplify each term)}\\&=\frac{2\tan(x/2)}{1+\tan^2(x/2)}&&\text{(By the definition of tangent)}\end{aligned}$$

**Ex 69:** Prove the identity:

$$\cos x=\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}$$

*Answer:* Soit $x$ un réel tel que $\cos(x/2)\neq 0$.
Let $x$ be a real number such that $\cos(x/2)\neq 0$.
Let $\theta=x/2$.

$$\begin{aligned}\cos x&=\cos(2\theta)\\&=\cos^2\theta-\sin^2\theta\\&=\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}\\&=\frac{(\cos^2\theta-\sin^2\theta)/\cos^2\theta}{(\cos^2\theta+\sin^2\theta)/\cos^2\theta}\\&=\frac{1-\tan^2\theta}{1+\tan^2\theta}\\&=\frac{1-\tan^2(x/2)}{1+\tan^2(x/2)}\end{aligned}$$

**Ex 70:**

1. Show that $\sin(A+B)+\sin(A-B)=2\sin A\cos B$.
2. Hence show that $\sin A\cos B=\frac{1}{2}\sin(A+B)+\frac{1}{2}\sin(A-B)$.
3. Hence write the following as sums:
   (a) $\sin 30°\cos\theta$
   (b) $\sin 6\alpha\cos\alpha$

*Answer:*

1. 
$$\begin{aligned}&\sin(A+B)+\sin(A-B)\\=&(\sin A\cos B+\cos A\sin B)+(\sin A\cos B-\cos A\sin B)\\=&2\sin A\cos B\end{aligned}$$

2. Divide both sides by 2:
$$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

3. (a) $\sin 30°\cos\theta=\frac{1}{2}[\sin(30°+\theta)+\sin(30°-\theta)]$

$$
\begin{aligned}
\text{(b)}\ \sin 6\alpha \cos\alpha &= \frac{1}{2}[\sin(6\alpha+\alpha)+\sin(6\alpha-\alpha)] \\
&= \frac{1}{2}[\sin 7\alpha + \sin 5\alpha]
\end{aligned}
$$

**Ex 71:**

1. Show that $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$.
2. Hence show that $\cos A\cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$.
3. Hence write the following as sums:
   (a) $\cos 30^\circ \cos\theta$
   (b) $\cos 6\alpha\cos\alpha$

*Answer:*

1.

$$
\begin{aligned}
&\cos(A+B)+\cos(A-B) \\
=&(\cos A\cos B - \sin A\sin B) + (\cos A\cos B + \sin A\sin B) \\
=&2\cos A\cos B
\end{aligned}
$$

2. Divide both sides by 2:

$$\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$$

3. (a) $\cos 30^\circ\cos\theta = \frac{1}{2}[\cos(30^\circ+\theta)+\cos(30^\circ-\theta)]$

$$
\begin{aligned}
\text{(b)}\ \cos 6\alpha\cos\alpha &= \frac{1}{2}[\cos(6\alpha+\alpha)+\cos(6\alpha-\alpha)] \\
&= \frac{1}{2}[\cos 7\alpha + \cos 5\alpha]
\end{aligned}
$$

## G.3 PROVING DOUBLE ANGLE IDENTITIES

**Ex 72:** Prove the double angle identity: $\sin 2\theta = 2\sin\theta\cos\theta$.

*Answer:* Let $\theta$ be an angle.

$$
\begin{aligned}
\sin 2\theta &= \sin(\theta+\theta) \\
&= \sin\theta\cos\theta + \cos\theta\sin\theta \\
&= 2\sin\theta\cos\theta
\end{aligned}
$$

**Ex 73:** Use the compound angle identities to prove the double angle identity: $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

*Answer:* Let $\theta$ be an angle.

$$
\begin{aligned}
\cos 2\theta &= \cos(\theta+\theta) \\
&= \cos\theta\cos\theta - \sin\theta\sin\theta \\
&= \cos^2\theta - \sin^2\theta
\end{aligned}
$$

**Ex 74:** Use the compound angle identities to prove the double angle identity: $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$.

*Answer:* Let $\theta$ be an angle.

$$
\begin{aligned}
\tan 2\theta &= \tan(\theta+\theta) \\
&= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} \\
&= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}
$$

## G.4 DERIVING TRIPLE ANGLE IDENTITIES

**Ex 75:** Consider the identity for $\cos(A+B)$.

1. By setting $A = B = \theta$, derive the double-angle formula for cosine: $\cos 2\theta = 2\cos^2\theta - 1$.
2. Using the result from part (1) or otherwise, express $\cos 3\theta$ in terms of $\cos\theta$.

*Answer:*

1.

$$
\begin{aligned}
\cos 2\theta &= \cos(\theta+\theta) \\
&= \cos\theta\cos\theta - \sin\theta\sin\theta \\
&= \cos^2\theta - \sin^2\theta \\
&= \cos^2\theta - (1-\cos^2\theta) \\
&= 2\cos^2\theta - 1
\end{aligned}
$$

2.

$$
\begin{aligned}
\cos 3\theta &= \cos(2\theta+\theta) \\
&= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta \\
&= (2\cos^2\theta - 1)\cos\theta - (2\cos\theta\sin\theta)\sin\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta \\
&= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta) \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}
$$

**Ex 76:** Consider the identity for $\sin(A+B)$.

1. By setting $A = B = \theta$, derive the double-angle formula for sine: $\sin 2\theta = 2\sin\theta\cos\theta$.
2. Using the result from part (1) or otherwise, express $\sin 3\theta$ in terms of $\sin\theta$.

*Answer:*

1.

$$
\begin{aligned}
\sin 2\theta &= \sin(\theta+\theta) \\
&= \sin\theta\cos\theta + \cos\theta\sin\theta \\
&= 2\sin\theta\cos\theta
\end{aligned}
$$

2.

$$
\begin{aligned}
\sin 3\theta &= \sin(2\theta+\theta) \\
&= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta \\
&= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \cos^2\theta\sin\theta - \sin^3\theta \\
&= 3\sin\theta\cos^2\theta - \sin^3\theta \\
&= 3\sin\theta(1-\sin^2\theta) - \sin^3\theta \\
&= 3\sin\theta - 3\sin^3\theta - \sin^3\theta \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}
$$