PYTHAGOREAN THEOREM

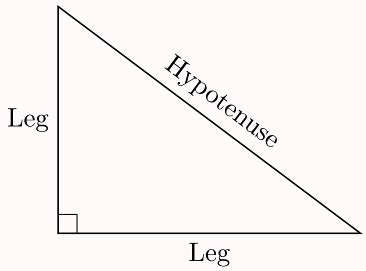
One of the oldest and most famous mathematical concepts is the Pythagorean theorem. Named after the ancient Greek mathematician Pythagoras, this theorem establishes a fundamental relationship between the sides of a right-angled triangle.

A RIGHT-ANGLED TRIANGLE

Definition Right-Angled Triangle

A **right-angled triangle** is a triangle that has one right angle (90°).

- The two sides that form the right angle are called the **legs** of the triangle.
- The side opposite the right angle, which is also the longest side, is called the **hypotenuse**.



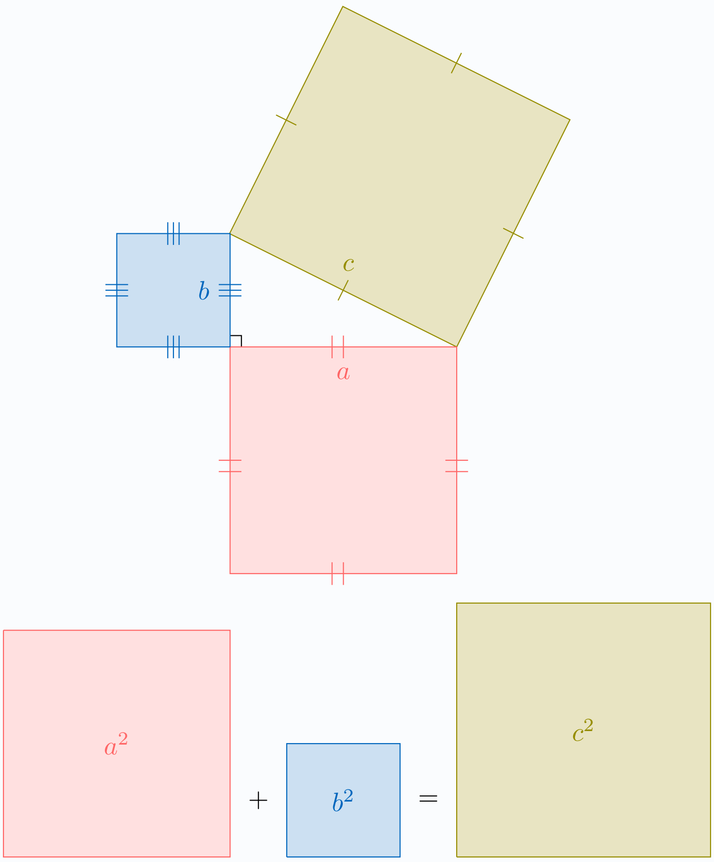
B PYTHAGOREAN THEOREM

Theorem Pythagorean Theorem

For any right-angled triangle with legs of length a and b and a hypotenuse of length c, the square of the hypotenuse is equal to the sum of the squares of the legs:

$$a^2 + b^2 = c^2.$$

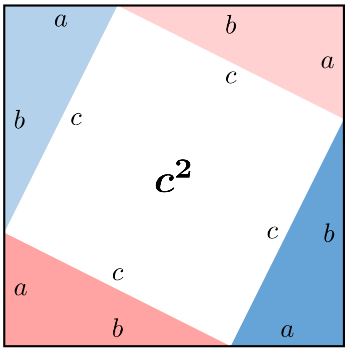
This means the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.



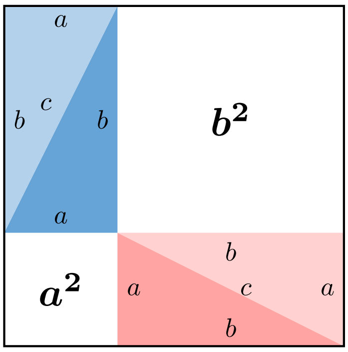
Proof

- The geometric proof involves rearranging four identical right-angled triangles within a square of side length $a+b$.

  1. In the first arrangement, the four right-angled triangles form a smaller square in the center of the larger square:



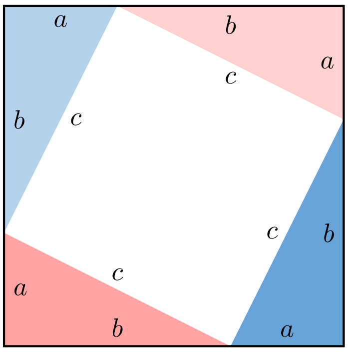
  2. In the second arrangement, the four right-angled triangles form two smaller squares within the larger square:



  3. Since the area not covered by the triangles is equal in both arrangements, we have:

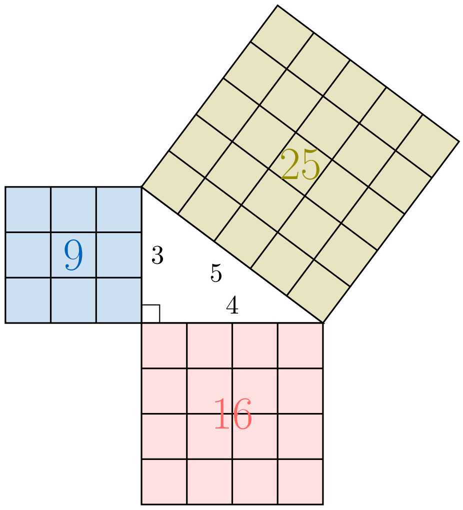
$$a^2 + b^2 = c^2.$$

- Algebraic proof: Using four copies of the same triangle arranged symmetrically around a central square with side c, inside a larger square with side $a + b$:



The larger square has side $a + b$, with area $(a + b)^2$. The four triangles and the central square of side c have the same total area as the larger square:

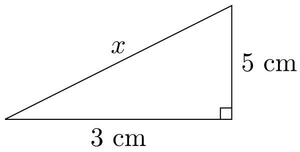
$$\begin{aligned}
(a+b)^2 &= \overbrace{c^2}^{\text{central square area}} + 4 \times \overbrace{\frac{ab}{2}}^{\text{triangle area}} \\
a^2 + 2ab + b^2 &= c^2 + 2ab \\
a^2 + b^2 &= c^2.
\end{aligned}$$



$$4^2 + 3^2 = 5^2$$
$$16 + 9 = 25$$

This diagram provides a visual representation of the Pythagorean theorem.

Ex: Find the length of the hypotenuse, x.

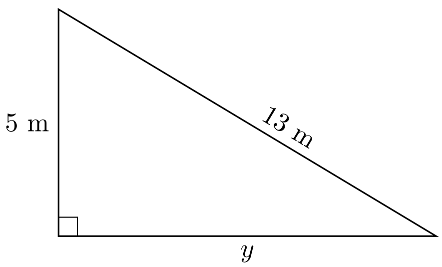


Answer: By the Pythagorean theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 5^2 &= x^2 \\ 9 + 25 &= x^2 \\ 34 &= x^2 \\ x &= \sqrt{34}\,\text{cm} \end{aligned}$$

The hypotenuse has a length of $\sqrt{34}$ cm (approximately 5.83 cm).

Ex: Find the length of the unknown leg, y.



Answer: By the Pythagorean theorem, with $c = 13$ being the hypotenuse:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ y^2 + 5^2 &= 13^2 \\ y^2 + 25 &= 169 \\ y^2 &= 169 - 25 \\ y^2 &= 144 \\ y &= \sqrt{144} = 12\,\text{m} \end{aligned}$$

The unknown leg has a length of 12 m.

C VERIFYING RIGHT-ANGLED TRIANGLES

Theorem Converse of the Pythagorean Theorem

If a triangle has side lengths a, b, and c, where c is the longest side, and they satisfy $a^2 + b^2 = c^2$, then the triangle is a right-angled triangle, and the right angle is opposite the longest side, c.

Ex: Is a triangle with side lengths 5 cm, 12 cm, and 13 cm a right-angled triangle?

Answer: We check if the sum of the squares of the two shorter sides equals the square of the longest side. The longest side is 13 cm.

- $5^2 + 12^2 = 25 + 144 = 169$
- $13^2 = 169$

Since $5^2 + 12^2 = 13^2$, the triangle **is** right-angled by the converse of the Pythagorean theorem.

Theorem Contrapositive of the Pythagorean Theorem

If a triangle has side lengths a, b, and c, where c is the longest side, and the sum of the squares of the two shorter sides is **not** equal to the square of the longest side ($a^2 + b^2 \neq c^2$), then the triangle **is not** a right-angled triangle.

Proof

This theorem is the **contrapositive** of the original Pythagorean theorem. Since a statement and its contrapositive are logically equivalent, the contrapositive is also true.

Ex: Is a triangle with sides of lengths 5, 8, and 9 right-angled?

Answer: The two shorter sides are 5 and 8, and the longest side is 9.

- $5^2 + 8^2 = 25 + 64 = 89$
- $9^2 = 81$

Since $89 \neq 81$, the triangle is **not** right-angled by the contrapositive of the Pythagorean theorem.