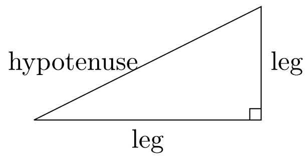
# PYTHAGOREAN THEOREM

One of the oldest and most famous mathematical concepts is the Pythagorean theorem. Named after the ancient Greek mathematician Pythagoras, this theorem establishes a fundamental relationship between the sides of a right-angled triangle.

## A RIGHT-ANGLED TRIANGLE

**Definition Right-Angled Triangle**

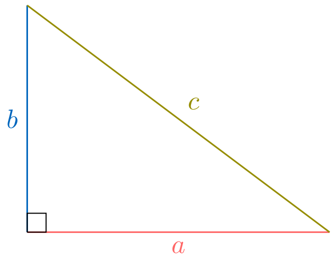
A **right-angled triangle** is a triangle with one right angle. The two sides forming this right angle are called the **legs**, and the longest side, opposite the right angle, is called the **hypotenuse**.




## B PYTHAGOREAN THEOREM

**Discover:**

1. Draw two right-angled triangles, similar to the one shown below:



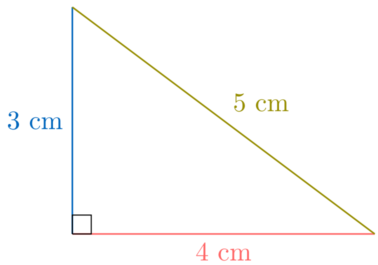

2. Measure the sides of these right-angled triangles to the nearest millimeter and complete the table:

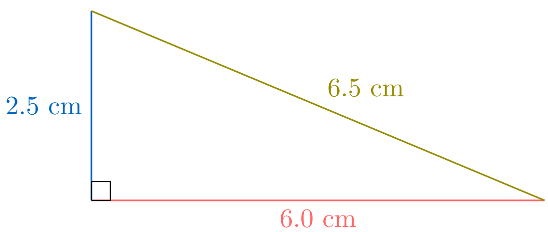
| | $a$ | $b$ | $c$ | $a^2$ | $b^2$ | $c^2$ | $a^2+b^2$ |
|---|---|---|---|---|---|---|---|
| Triangle 1 | | | | | | | |
| Triangle 2 | | | | | | | |

3. Can you identify any relationship between the values in the columns for $a^2$, $b^2$, and $c^2$?
4. Can you express this relationship as an equation?

*Answer:*

1. You have drawn two right-angled triangles. For example:




2. Here are example measurements of the sides and their squares:

| | $a$ | $b$ | $c$ | $a^2$ | $b^2$ | $c^2$ | $a^2+b^2$ |
|---|---|---|---|---|---|---|---|
| Triangle 1 | 4 | 3 | 5 | 16 | 9 | 25 | 25 |
| Triangle 2 | 6 | 2.5 | 6.5 | 36 | 6.25 | 42.25 | 42.25 |

3. From the table, the sum of the squares of the legs ($a^2+b^2$) equals the square of the hypotenuse ($c^2$) for both triangles:

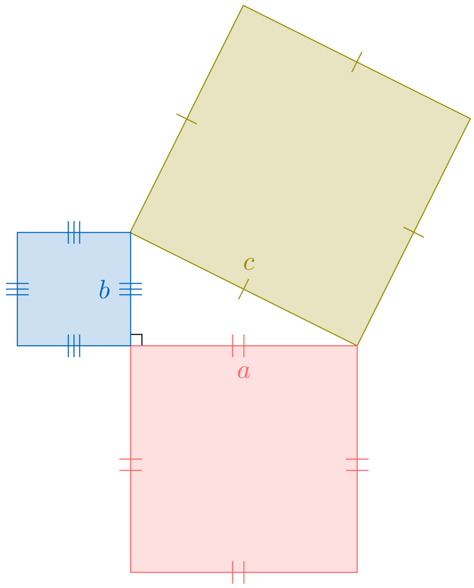
$$16+9=25$$
$$36+6.25=42.25$$

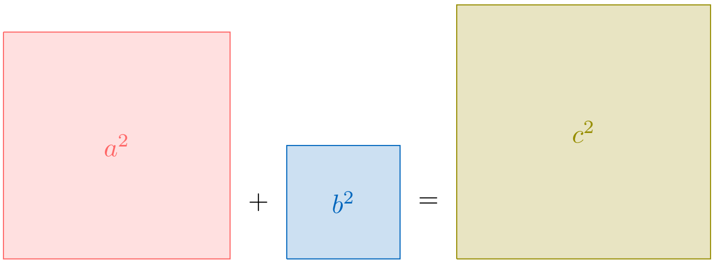
4. This relationship is expressed as the equation $a^2 + b^2 = c^2$, known as the Pythagorean theorem, which states that the sum of the squares of the legs equals the square of the hypotenuse in a right-angled triangle.

**Theorem Pythagorean Theorem**

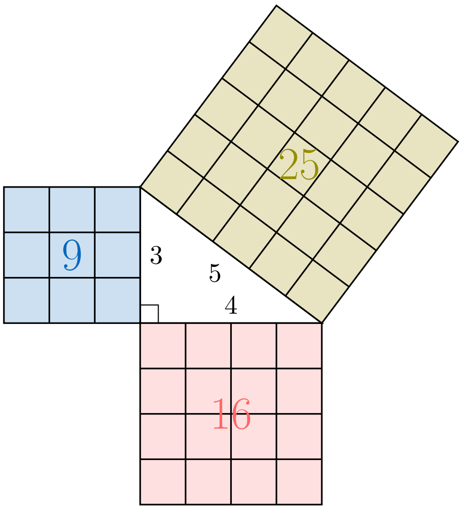
For any right-angled triangle with legs $a$ and $b$ and hypotenuse $c$, the following holds:

$$a^2 + b^2 = c^2$$



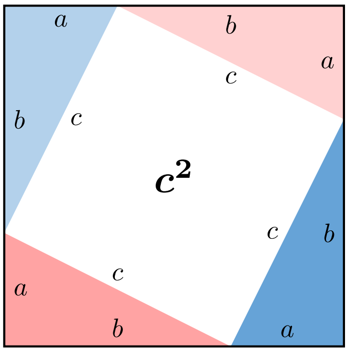

**Ex:**



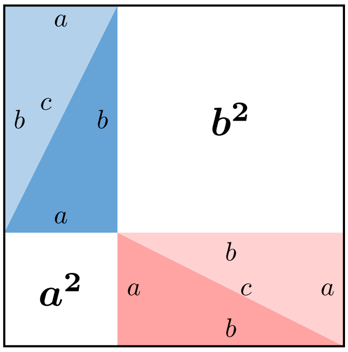

$$4^2 + 3^2 = 5^2$$
$$16 + 9 = 25$$

Proof

- The geometric proof involves rearranging four identical right-angled triangles within a square of side length $a+b$.
  1. In the first arrangement, the four right-angled triangles form a smaller square in the center of the larger square:



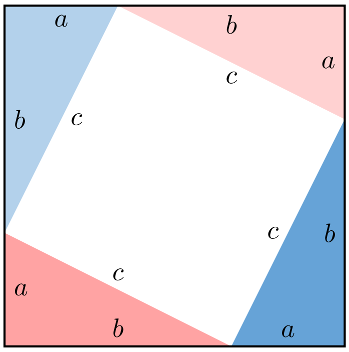

2. In the second arrangement, the four right-angled triangles form two smaller rectangles within the larger square:




3. Since the area not covered by the triangles is equal in both arrangements, we have:

$$a^2 + b^2 = c^2$$

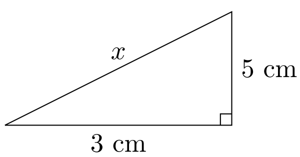
- Algebraic proof: Using four copies of the same triangle arranged symmetrically around a square with side $c$:




The larger square has side $a + b$, with area $(a + b)^2$. The four triangles and the central square of side $c$ have the same area as the larger square:

$$\begin{aligned} (a+b)^2 &= \overbrace{c^2}^{\text{central square area}} + 4 \times \overbrace{\frac{ab}{2}}^{\text{triangle area}} \\ a^2 + 2ab + b^2 &= c^2 + 2ab \\ a^2 + b^2 &= c^2 \end{aligned}$$

**Ex:** Find the length of the hypotenuse:



*Answer:*

$$\begin{aligned} x^2 &= 3^2 + 5^2 \quad \text{(Pythagorean theorem)} \\ x^2 &= 9 + 25 \\ x^2 &= 34 \\ x &= \sqrt{34} \quad \text{(since the length of a triangle side is positive)} \end{aligned}$$

Thus, the hypotenuse has length $\sqrt{34}$ cm.

## C VERIFYING RIGHT-ANGLED TRIANGLES

**Theorem Converse of the Pythagorean Theorem**

For any triangle with sides of lengths $a$, $b$, and $c$, if $a^2 + b^2 = c^2$, then the triangle is right-angled.

**Ex:** Is a triangle with sides of lengths 3, 4, and 5 right-angled?

*Answer:* The two shorter sides are 3 and 4:

$$3^2 + 4^2 = 9 + 16 = 25$$
$$5^2 = 25$$

Since $3^2 + 4^2 = 5^2$, the triangle is right-angled by the converse of the Pythagorean theorem.

**Theorem Contrapositive of the Pythagorean Theorem**

For any triangle with sides of lengths $a$, $b$, and $c$, where $c \geq a$ and $c \geq b$, if $a^2 + b^2 \neq c^2$, then the triangle is not right-angled.

Proof

This is the contrapositive of the Pythagorean theorem.

**Ex:** Is a triangle with sides of lengths 5, 8, and 9 right-angled?

*Answer:* The two shorter sides are 5 and 8, and the longest side is 9:

$$5^2 + 8^2 = 25 + 64 = 89$$
$$9^2 = 81$$

Since $5^2 + 8^2 \neq 9^2$, the triangle is not right-angled by the contrapositive of the Pythagorean theorem.