PROBABILITY

Ever wondered if it'll rain tomorrow or if you'll win a game? That's probability! It's a math way to guess how likely things are to happen. Like when the weather app says there's a 90% chance of rain, that's probability telling us it's very likely. What else do you think we could use probability for?

A ALGEBRA OF EVENTS

A.1 SAMPLE SPACE

Have you ever flipped a coin and wondered if it'd land on heads or tails? Or rolled a die and guessed what number you'd get? These are random experiments—things we do where the result isn't certain until it happens.

Definition Outcome

An **outcome** is one possible result of a random experiment.

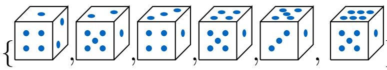
Definition Sample Space

The **sample space** is the list of all possible outcomes of a random experiment.

Ex: What's the sample space when you flip a coin?

Answer: It's {Heads, Tails} = {}, or just {H, T} for short.

Ex: What's the sample space when you roll a six-sided die?

Answer: It's {1, 2, 3, 4, 5, 6} = {}.

A.2 EVENTS

Definition Event

An **event** is a set of outcomes from all possible outcomes.

In math, we use capital letters like E to name events. So, we might say E is the event "it's sunny tomorrow."

Ex: You roll a die. Let E be the event of rolling an even number. What's E?

Answer: The sample space is $\{1, 2, 3, 4, 5, 6\}$, and $E = \{2, 4, 6\}$ because those are the even numbers.

A.3 COMPLEMENTARY EVENT

Definition Complementary Event

The **complementary event** of an event E is everything in the sample space that isn't in E. We call it E' (E-prime).

Ex: You roll a die, and E is rolling an even number. What's E'?

Answer: $E = \{2, 4, 6\}$, so E' is all the other numbers: $E' = \{1, 3, 5\}$. These are the odd numbers!

A.4 MULTI-STEP RANDOM EXPERIMENTS

A multi-step random experiment is one that involves a sequence of actions, where each action (or step) has its own set of possible outcomes. In our example of tossing two coins, the experiment is multi-step because it involves two separate coin tosses:

- The first coin toss (step 1) can result in Heads (H) or Tails (T).
- The second coin toss (step 2) can also result in Heads (H) or Tails (T).

The overall outcome of the experiment is given by combining the outcomes of each step. For instance, the outcome HT means that the first coin landed on Heads and the second coin landed on Tails.Using different representations—such as grids, tables, tree diagrams, or simply listing the outcomes—helps us organize and visualize all the possible combinations that result from the multiple steps.

Method Representations of Multi-Step Random Experiment

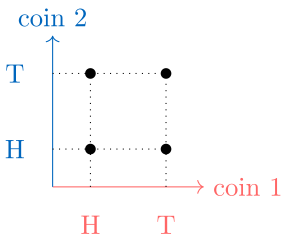
When an experiment involves more than one step, we can represent the sample space (the set of all possible outcomes) in several ways:

- using a **grid** (to visually map combinations along two axes),
- using a **table** (to organize outcomes in rows and columns),
- using a **tree diagram** (to follow each sequential step), or
- by **listing** all possible outcomes.

Ex: For the random experiment of tossing two coins, display the sample space by:

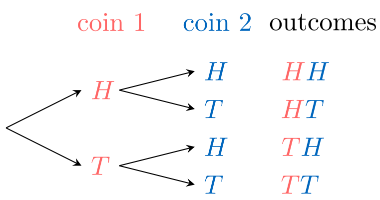
1. using a grid
2. using a table
3. using a tree diagram
4. listing all possible outcomes

Answer:

1. 

2.

coin 2 / coin 1	H	T
H	HH	HT
T	TH	TT

3. 

4. $\{HH, HT, TH, TT\}$

A.5 E OR F

Definition E or F

The **union of two events** E and F, denoted as E **or** F or $E \cup F$, is the event that occurs if either event E happens, event F happens, or both events happen simultaneously.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an even number. So, $E = \{2, 4, 6\}$.
Let event F be the event of rolling a number less than 4. So, $F = \{1, 2, 3\}$.
Find the event E or F.

Answer: E or F includes all outcomes from both event sets.
So, E or $F = E \cup F = \{1, 2, 3, 4, 6\}$.

A.6 E AND F

Definition E and F

The **intersection of two events** E and F, denoted as E **and** F or $E \cap F$, is the set of all outcomes that are common to both E and F. The event E and F occurs if and only if both E and F happen simultaneously.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an odd number. So, $E = \{1, 3, 5\}$.
Let event F be the event of rolling a number less than 4. So, $F = \{1, 2, 3\}$.
Find the event E and F.

Answer: E and F includes all outcomes that are common to both E and F.
So, E and $F = E \cap F = \{1, 3\}$.

A.7 MUTUALLY EXCLUSIVE

Definition Mutually Exclusive

Two events E and F are said to be **mutually exclusive** if they cannot occur simultaneously. In other words, the occurrence of event E excludes the possibility of event F occurring, and vice versa. This is mathematically represented as:

$$E \text{ and } F = E \cap F = \emptyset,$$

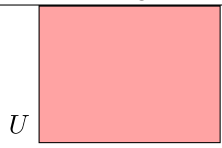
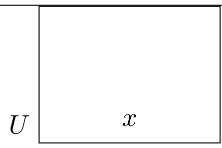
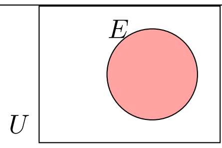
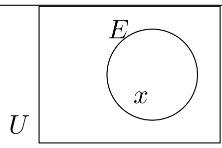
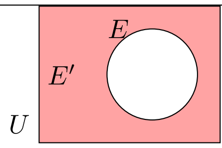
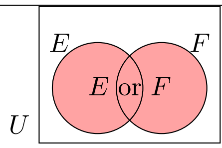
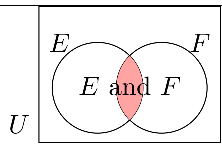
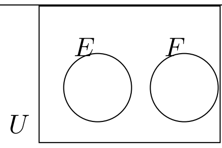
where $\emptyset$ denotes the empty set, indicating that there are no common outcomes between E and F.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an odd number. So, $E = \{1, 3, 5\}$.
Let event F be the event of rolling an even number. So, $F = \{2, 4, 6\}$.
Show that E and F are mutually exclusive.

Answer: Since there are no common outcomes between E and F, we have E and $F = E \cap F = \emptyset$. Thus, the events E and F are mutually exclusive.

A.8 VENN DIAGRAM

Definition Correspondence Between Set Theory and Probabilistic Vocabulary

Notation	Set Vocabulary	Probabilistic Vocabulary	Venn Diagram
U	Universal set	Sample space	
x	Element of U	Outcome	
$\emptyset$	Empty set	Impossible event	
E	Subset of U	Event	
$x \in E$	x is an element of E	x is an outcome of E	
E'	Complement of E in U	Complement of E in U	
E or F	Union of E and F: $E \cup F$	E or F	
E and F	Intersection of E and F: $E \cap F$	E and F	
$E \cap F = \emptyset$	E and F are disjoint	E and F are mutually exclusive	

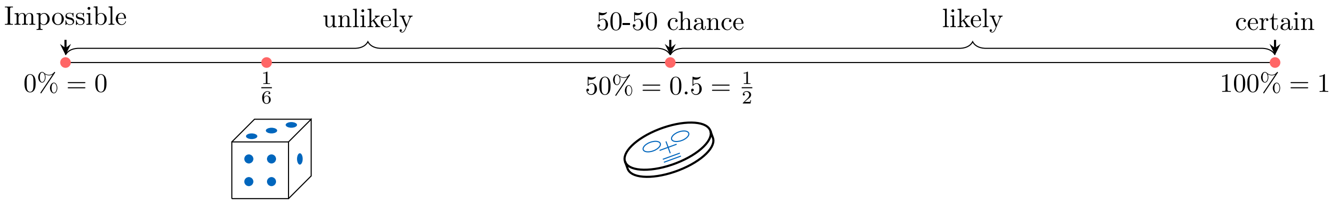
B PROBABILITY

B.1 PROBABILITY AXIOMS

The probability of an event E, denoted by $P(E)$, is a value between 0 and 1 that represents the chance or likelihood of that event occurring.

- The probability of an impossible event is 0 or 0%.
- The probability of a certain event is 1 or 100%.
- The probability of any event is between 0 and 1, inclusive.

We can express probabilities using fractions, decimals, or percentages. For example, the probability of an event with a 50-50 chance can be written as $\frac{1}{2}$, 0.5, or 50%.
We can visualize probabilities on a number line:



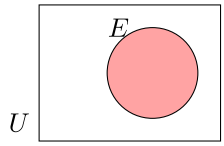
The main goal of probability theory is to develop tools and techniques to calculate the probabilities of different events. Probability theory is built upon a set of axioms that form its foundation. Let us state and explain these axioms.

Definition Probability Axioms

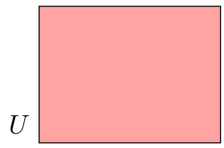
P is a **probability** if:

- $0 \leqslant P(E) \leqslant 1$, for any event E,
- $P(U) = 1$,
- If E and F are mutually exclusive events, then $P(E \text{ or } F) = P(E) + P(F)$.

- The probability of an event E, $P(E)$, is represented by the shaded area of the event in a Venn diagram:

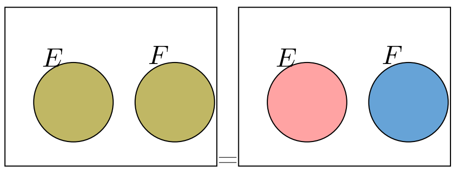


- The first axiom states that the probability of an event E is a value between 0 and 1, inclusive.
- The second axiom states that the probability of the sample space U is equal to 1, i.e., 100%. This is because the sample space U includes all possible outcomes of a random experiment, so the event U always occurs, and $P(U) = 1$. In a Venn diagram, this is represented as the entire shaded area of the sample space:



$P(U) = 1$

- The third axiom states that if two events are mutually exclusive (i.e., they cannot occur simultaneously), then the probability of their union is the sum of their individual probabilities. In a Venn diagram, for two mutually exclusive events (with no overlap), the total area of their union is the sum of the individual areas:



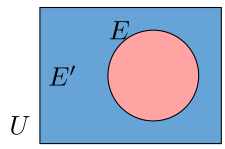
$P(E \text{ or } F) = P(E) + P(F)$

B.2 PROBABILITY RULES

Proposition Complement Rule

For any event E with complementary event E',

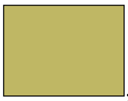
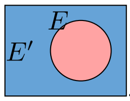
$$P(E) + P(E') = 1$$



Also,

$$P(E') = 1 - P(E)$$

Proof

The green area, $P(U)$, U ,is the sum of the red area, $P(E)$, and the blue area, $P(E')$, U 

.So,

$$P(E) + P(E') = P(U)$$

Since $P(U) = 1$, we have:

$$P(E) + P(E') = 1$$

Ex: Farid has a 0.8 (80%) chance of finishing his homework on time tonight (event E). What's the chance he doesn't finish on time ?

Answer: The complementary event E' represents the scenario where Farid does not complete his homework on time tonight. As $P(E) = 0.8$, by the complement rule, we get:

$$\begin{aligned} P(E') &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

So, there's a 20 % chance he doesn't finish on time!

Proposition Addition Law of Probability

For any events E and F,

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Ex: A local high school is holding a talent show where students can participate in singing, dancing, or both. The probability that a randomly selected student participates in singing is 0.4, the probability that a student participates in dancing is 0.3, and the probability that a student participates in both singing and dancing is 0.1. Find the probability that a randomly selected student participates in either singing or dancing.

Answer:

- Let S be the event that a student participates in singing, and D be the event that a student participates in dancing. We are given $P(S) = 0.4$, $P(D) = 0.3$, and $P(S \text{ and } D) = 0.1$.
- The probability of either singing or dancing is given by $P(S \text{ or } D)$.
- By the addition law of probability,

$$\begin{aligned} P(S \text{ or } D) &= P(S) + P(D) - P(S \text{ and } D) \\ &= 0.4 + 0.3 - 0.1 \\ &= 0.6 \end{aligned}$$

- Thus, the probability that a student participates in either singing or dancing is 0.6.

B.3 EQUALLY LIKELY

Sometimes, every outcome in an experiment has the same chance—like flipping a fair coin or rolling a fair die. We call these equally likely outcomes.

Definition Equally Likely

If all outcomes are equally likely, the probability of an event, E, is given by

$$P(E) = \frac{\text{number of outcomes in the event}}{\text{total number of all possible outcomes}}$$

Ex: What's the probability of rolling an even number with a fair six-sided die?

Answer:

- Sample space $= \{1, 2, 3, 4, 5, 6\}$ (6 outcomes).
- $E = \{2, 4, 6\}$ (3 outcomes).

- $$P(E) = \frac{3}{6} = \frac{1}{2}$$
  .

So, there's a $\frac{1}{2}$ chance (or 50%) of rolling an even number!

B.4 PROBABILITY OF INDEPENDENT EVENTS

Independent events are like you choosing an ice cream flavor and your friend picking a movie—your choice doesn't affect theirs, and theirs doesn't affect yours! For example, flipping a coin and rolling a die at the same time: whether you get heads or tails has no effect on what number you roll. Let's explore this idea together!

Definition Independent Events

Two events, A and B, are **independent** if the chance of both happening is just the product of their individual chances. Mathematically:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Ex: Imagine you do two totally separate actions:

1. Flipping a fair coin (heads or tails).
2. Rolling a fair six-sided die (1, 2, 3, 4, 5, or 6).

What's the probability of getting tails **and** rolling a number greater than 4 (like a 5 or 6)?

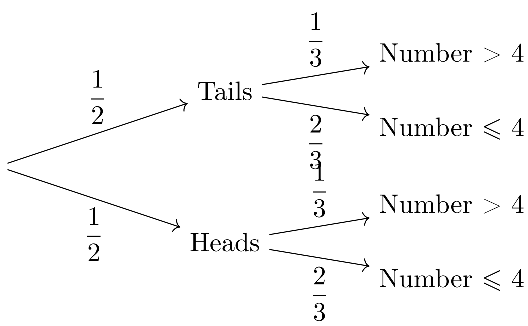
Answer: Let's break it down:

- These events are independent, so we multiply their probabilities.
- For the coin: You've got two options—heads or tails—and they're equally likely. So, $P(\text{"tails"}) = \frac{1}{2}$.
- For the die: There are six sides, and "greater than 4" means 5 or 6. That's 2 out of 6 possibilities, so $P(\text{"number} > 4\text{"}) = \frac{2}{6} = \frac{1}{3}$.
- Now, combine them:
$$\begin{aligned} P(\text{"tails" and "number} > 4\text{"}) &= P(\text{"tails"}) \times P(\text{"number} > 4\text{"}) \\ &= \frac{1}{2} \times \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$
- Result: There's a $\frac{1}{6}$ chance of landing tails and rolling a 5 or 6.

Method Finding the probability of two independent events using a probability tree diagram

Let's use a probability tree for our coin flip and die roll:

1. **Draw the Tree:** Start with two branches for the coin: "Heads" and "Tails." Then, from each, draw two more branches for the die: "Number $>$ 4" (5 or 6) and "Number $\leqslant$ 4" (1, 2, 3, or 4).
2. **Add Probabilities:** Label each branch. The coin gives $\frac{1}{2}$ for "Tails" and $\frac{1}{2}$ for "Heads." For the die, "Number $>$ 4" is $\frac{1}{3}$ (2 out of 6), and "Number $\leqslant$ 4" is $\frac{2}{3}$ (4 out of 6).



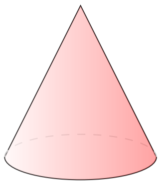
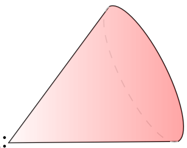
3. **Follow the Path:** To find $P(\text{tails and number} > 4)$, trace the “Tails” branch, then the “Number > 4” branch, and multiply:

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

4. **Wrap It Up:** The tree confirms our answer—$\frac{1}{6}$!

B.5 EXPERIMENTAL PROBABILITY

Isaac wants to know how a cone lands when he tosses it—base down or point down? Here’s what it can do:

- Base down: 
- Point down: 

Since the cone’s shape might make one more likely, Isaac can’t guess. So, he tosses it 50 times and counts:

- Base down: 30 times.
- Point down: 20 times.

He estimates:

- $P(\text{"base down"}) = \frac{30}{50} = 0.6$ (60%).
- $P(\text{"point down"}) = \frac{20}{50} = 0.4$ (40%).

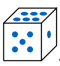
The more he tosses, the closer he gets to the real chances!

Definition **Experimental Probability**

The probability $P(E)$ of an event E can be estimated using the formula:

$$P(E) \approx \frac{\text{number of times the event occurs}}{\text{number of trials}}$$

B.6 EXPECTED VALUE

Discover: Suppose a fair six-sided die is rolled 120 times. Find the expected number of times the result will be a "six" = .

Answer: The sample space when rolling a fair six-sided die is $\{1, 2, 3, 4, 5, 6\}$, with each outcome equally likely. Thus, the probability of rolling a "six" in a single trial is:

$$P(\text{six}) = \frac{1}{6}$$

Since the die is rolled 120 times, the expected number of "sixes" is:

$$120 \times \frac{1}{6} = 20$$

So, we expect 20 of the 120 rolls to result in a "six."

Proposition **Expected Value**

When an experiment is performed n times (called trials) and the probability of an event occurring in each trial is p, the **expected value** (or expectation) of the number of occurrences of that event is:

$$n \times p$$

This represents the average number of times the event is expected to occur over n trials.

Ex: Jane enters a raffle where the chance of winning a prize with each ticket is $\frac{1}{10}$. She buys 20 raffle tickets. Find the expected number of prizes Jane is likely to win.

Answer: Let's calculate the expected value:

- The probability of winning with one ticket is:

$$p = P(\text{"winning with a ticket"}) = \frac{1}{10}$$

- The number of trials (tickets) is:

$$n = 20$$

- The expected number of prizes is:

$$\begin{aligned} n \times p &= 20 \times \frac{1}{10} \\ &= \frac{20}{10} \\ &= 2 \end{aligned}$$

Thus, Jane is expected to win 2 prizes on average.