PROBABILITY

Ever wondered if it'll rain tomorrow or if you'll win a game? That's probability! It's a math way to guess how likely things are to happen.

A ALGEBRA OF EVENTS

A.1 SAMPLE SPACES

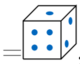
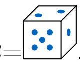
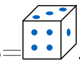
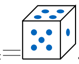
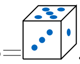
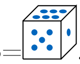
Definition Outcome

An **outcome** is one possible result of a random experiment.

Ex: What are the outcomes when you flip a coin? 

Answer: The outcomes are Heads (H)= and Tails (T)=.

Ex: What are the outcomes when you roll a six-sided die?

Answer: The outcomes are 1=,2=,3=,4=,5=,and 6=.

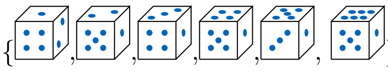
Definition Sample Space

The **sample space** is the set of all possible outcomes of a random experiment.

Ex: What's the sample space when you flip a coin?

Answer: The sample space is {Heads, Tails} = {}, or just {H, T} for short.

Ex: What's the sample space when you roll a six-sided die?

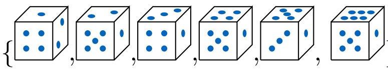
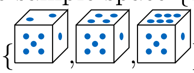
Answer: The sample space is {1, 2, 3, 4, 5, 6} = {}.

A.2 EVENTS

Definition Event

Un **événement** est un sous-ensemble des issues possibles de l'univers. We write it E.

Ex: In the experiment of rolling a die, find E the event of rolling an even number.

Answer: Among the outcomes of the sample space {1, 2, 3, 4, 5, 6} = {} ,the event of rolling an even number is $E = \{2, 4, 6\}$ = {}.

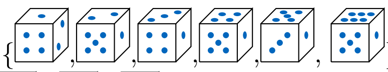
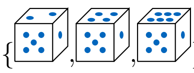
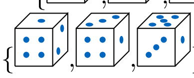
A.3 COMPLEMENTARY EVENTS

Ever wonder what happens if you look for everything **except** a certain event? That's where the complementary event comes in! It's just everything in the sample space that isn't in your event. We usually write it as E' ("E-prime").

Definition Complementary Event

The **complementary event** of an event E is all the outcomes in the sample space that are **not** in E. We write it E'.

Ex: In the experiment of rolling a die, let E the event of rolling an even number. Find E'.

Answer: The sample space is $\{1, 2, 3, 4, 5, 6\}$ = {} and $E = \{2, 4, 6\}$ = {}.
So E' is all the other numbers: $\{1, 3, 5\}$ = {}. These are the odd numbers.

A.4 MULTI-STEP RANDOM EXPERIMENTS

Method Representations of Multi-Step Random Experiments

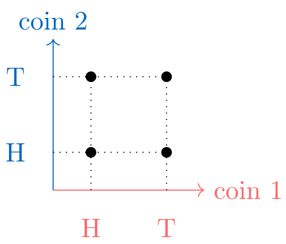
When an experiment involves more than one step, we can represent the sample space (the set of all possible outcomes) in several ways:

- using a **grid** (to visually map combinations along two axes),
- using a **table** (to organize outcomes in rows and columns),
- using a **tree diagram** (to show each sequential step), or
- by **listing** all possible outcomes.

Ex: For the random experiment of tossing two coins, display the sample space by:

1. using a grid
2. using a table
3. using a tree diagram
4. listing all possible outcomes

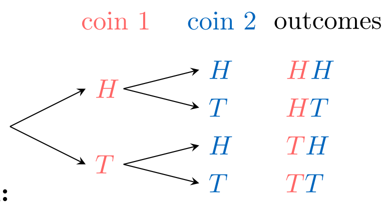
Answer:



1. **Grid:**

2. **Table:**

coin 2 / coin 1	H	T
H	HH	HT
T	TH	TT



3. **Tree diagram:**

4. **List:** $\{HH, HT, TH, TT\}$

A.5 E OR F

Definition E or F

The **union of two events** E and F, denoted as E **or** F or $E \cup F$, is the event that occurs if at least one of the events E or F happens (i.e., if E occurs, or F occurs, or both occur).

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an even number: $E = \{2, 4, 6\}$.
Let event F be the event of rolling a number less than 4: $F = \{1, 2, 3\}$.
Find the event E or F.

Answer: E or F includes all outcomes from both event sets.
So, E or $F = E \cup F = \{1, 2, 3, 4, 6\}$.

A.6 E AND F

Definition E and F

The **intersection of two events** E and F, denoted as **E and F** or $E \cap F$, is the set of all outcomes that are common to both E and F. The event E and F occurs if and only if both E and F happen simultaneously.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an odd number: $E = \{1, 3, 5\}$.
Let event F be the event of rolling a number less than 4: $F = \{1, 2, 3\}$.
Find the event E and F.

Answer: E and F includes all outcomes that are common to both E and F.
So, E and $F = E \cap F = \{1, 3\}$.

A.7 MUTUALLY EXCLUSIVE

Definition Mutually Exclusive

Two events E and F are said to be **mutually exclusive** if they cannot occur at the same time. In other words, the occurrence of event E excludes the possibility of event F occurring, and vice versa. This is mathematically represented as:

$$E \text{ and } F = E \cap F = \emptyset,$$

where $\emptyset$ denotes the empty set, indicating that there are no outcomes common to both E and F.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an odd number: $E = \{1, 3, 5\}$.
Let event F be the event of rolling an even number: $F = \{2, 4, 6\}$.
Show that E and F are mutually exclusive.

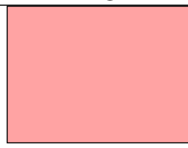
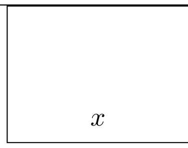
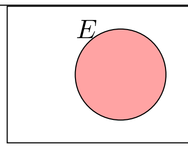
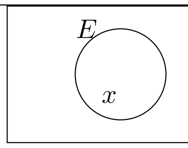
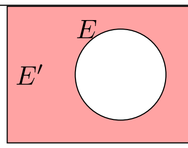
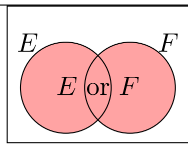
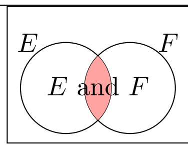
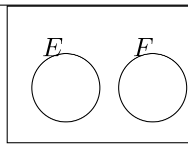
Answer: There are no outcomes common to both E and F:

$$E \cap F = \emptyset.$$

Thus, the events E and F are mutually exclusive.

A.8 VENN DIAGRAM

Definition Correspondence Between Set Theory and Probabilistic Vocabulary

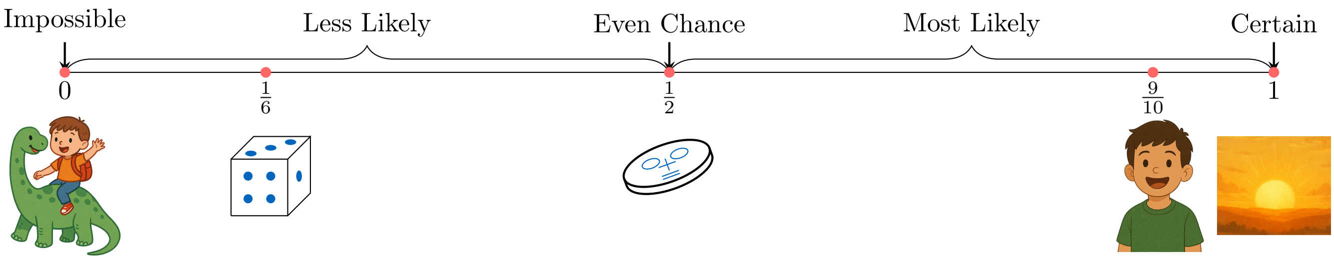
Notation	Set Vocabulary	Probabilistic Vocabulary	Venn Diagram
U	Universal set	Sample space	U 
x	Element of U	Outcome	U 
$\emptyset$	Empty set	Impossible event	
E	Subset of U	Event	U 
$x \in E$	x is an element of E	x is an outcome of E	U 
E'	Complement of E in U	Complement of E in U	U 
E or F	Union of E and F: $E \cup F$	E or F	U 
E and F	Intersection of E and F: $E \cap F$	E and F	U 
$E \cap F = \emptyset$	E and F are disjoint	E and F are mutually exclusive	U 

B PROBABILITY

B.1 PROBABILITY AXIOMS

Definition Probability

The **probability** of an event E, written $P(E)$, is a number that tells us how likely the event is to happen. It's always between 0 (impossible) and 1 (certain).



Ex: The probability of an event "even chance" can be represented as:

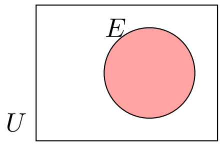
- **Fraction**: $\frac{1}{2}$
- **Decimal**: To convert the fraction to a decimal, divide the numerator by the denominator: $1 \div 2 = 0.5$.
- **Percentage**: To convert the decimal to a percentage, multiply by 100%: $0.5 \times 100\% = 50\%$.

Definition Probability Axioms

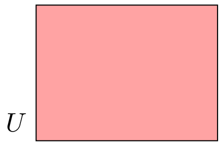
P is a **probability** if:

- $0 \leqslant P(E) \leqslant 1$, for any event E,
- $P(U) = 1$,
- If E and F are mutually exclusive events, then $P(E \text{ or } F) = P(E) + P(F)$.

- The probability of an event E, $P(E)$, is represented by the shaded area of the event in a Venn diagram:

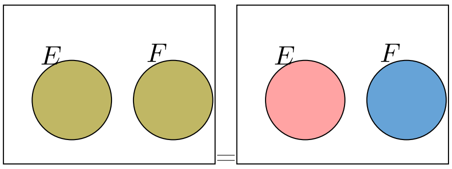


- The first axiom states that the probability of an event E is a value between 0 and 1, inclusive.
- The second axiom states that the probability of the sample space U is equal to 1, i.e., 100%. This is because the sample space U includes all possible outcomes of a random experiment, so the event U always occurs, and $P(U) = 1$. In a Venn diagram, this is represented as the entire shaded area of the sample space:



$$P(U) = 1$$

- The third axiom states that if two events are mutually exclusive (i.e., they cannot occur simultaneously), then the probability of their union is the sum of their individual probabilities. In a Venn diagram, for two mutually exclusive events (with no overlap), the total area of their union is the sum of the individual areas:



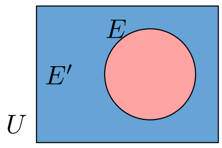
$$P(E \text{ or } F) = P(E) + P(F)$$

B.2 PROBABILITY RULES

Proposition Complement Rule

For any event E with complementary event E',

$$P(E) + P(E') = 1$$



Also,

$$P(E') = 1 - P(E)$$

Ex: Farid has a 0.8 (80%) chance of finishing his homework on time tonight (event E). What's the chance he **doesn't** finish on time?

Answer: The complementary event E' is "Farid does **not** finish his homework on time." By the complement rule:

$$\begin{aligned} P(E') &= 1 - P(E) \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

So, there's a 0.2 (or 20%) chance he doesn't finish on time!

Proposition Addition Law of Probability

For any events E and F,

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Ex: A local high school is holding a talent show where students can participate in singing, dancing, or both. The probability that a randomly selected student participates in singing is 0.4, the probability that a student participates in dancing is 0.3, and the probability that a student participates in both singing and dancing is 0.1. Find the probability that a randomly selected student participates in either singing or dancing.

Answer:

- Let S be the event "participates in singing", and D the event "participates in dancing". We are given $P(S) = 0.4$, $P(D) = 0.3$, and $P(S \text{ and } D) = 0.1$.
- The probability that a student participates in either singing or dancing is $P(S \text{ or } D)$.
- By the addition law of probability:

$$\begin{aligned} P(S \text{ or } D) &= P(S) + P(D) - P(S \text{ and } D) \\ &= 0.4 + 0.3 - 0.1 \\ &= 0.6 \end{aligned}$$

- Therefore, the probability is $\boxed{0.6}$.

B.3 EQUALLY LIKELY

Definition Equally Likely

When all outcomes are **equally likely**, the probability of an event E is:

$$\begin{aligned} P(E) &= \frac{\textbf{number of outcomes in the event}}{\textbf{number of outcomes in the sample space}} \\ &= \frac{n(E)}{n(U)} \end{aligned}$$

Ex: What's the probability of rolling an even number with a fair six-sided die?

Answer:

- Sample space = $\{1, 2, 3, 4, 5, 6\}$ (6 outcomes).
- $E = \{2, 4, 6\}$ (3 outcomes).
- $$\begin{aligned} P(E) &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

So, there's a $\frac{1}{2}$ chance (or 50%) of rolling an even number!

B.4 PROBABILITY OF INDEPENDENT EVENTS

Independent events are situations where what happens in one event does **not** affect what happens in the other. For example: rolling two dice at the same time. The result of the first die doesn't change the chances for the second die—they are independent!

Definition Independent Events

Two events, A and B, are **independent** if the chance of both happening is just the product of their individual chances. Mathematically:

$$P(A \textbf{ and } B) = P(A) \times P(B)$$

Ex: Imagine you do two totally separate actions:

1. Flipping a fair coin (heads or tails).

2. Rolling a fair six-sided die (1, 2, 3, 4, 5, or 6).

What's the probability of getting tails **and** rolling a number greater than 4 (like a 5 or 6)?

Answer: Let's break it down:

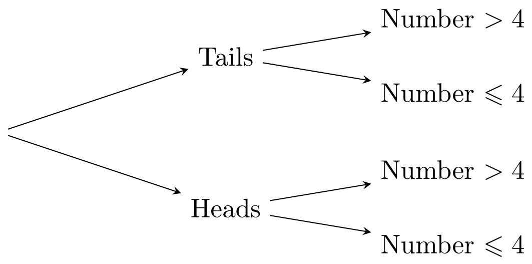
- These events are independent, so we multiply their probabilities.
- For the coin: You've got two options—heads or tails—and they're equally likely. So, $P(\text{"tails"}) = \frac{1}{2}$.
- For the die: There are six sides, and "greater than 4" means 5 or 6. That's 2 out of 6 possibilities, so $P(\text{"number} > 4\text{”}) = \frac{2}{6} = \frac{1}{3}$.
- Now, combine them:

$$\begin{aligned} P(\text{"tails" and "number} > 4\text{”}) &= P(\text{"tails"}) \times P(\text{"number} > 4\text{”}) \\ &= \frac{1}{2} \times \frac{1}{3} \\ &= \frac{1}{6} \end{aligned}$$

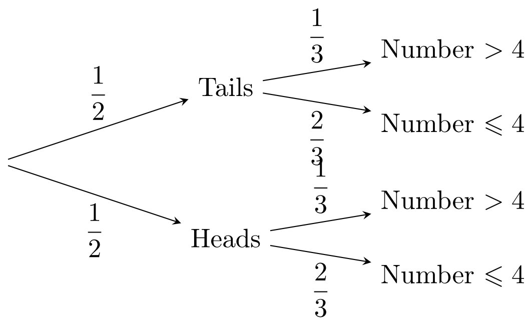
- Result: There's a $\frac{1}{6}$ chance of landing tails and rolling a 5 or 6.

Method Finding the Probability of Two Independent Events using a Probability Tree Diagram

1. **Draw the tree:** Start with two branches for the coin: "Heads" and "Tails." From each, draw two branches for the die: "Number $>$ 4" (5 or 6) and "Number $\leqslant$ 4" (1, 2, 3, 4).

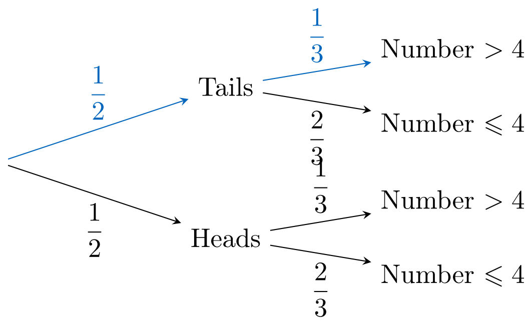


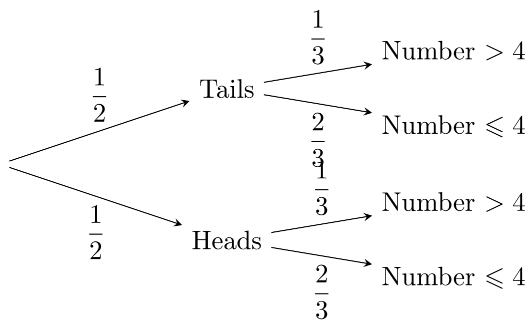
2. **Label probabilities:** Each coin branch is $\frac{1}{2}$. For the die, "Number $>$ 4" is $\frac{1}{3}$ (2 out of 6), "Number $\leqslant$ 4" is $\frac{4}{6} = \frac{2}{3}$.



3. **Multiply the probabilities along the path:** For "Tails" and "Number > 4":

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$





B.5 EXPERIMENTAL PROBABILITY

Theorem **Law of Large Numbers**

The probability of an event E can be estimated using the formula:

$$P(E) \approx \frac{\text{number of times the event occurs}}{\text{number of trials}}$$

Here, "trials" refer to the number of times the experiment is repeated.

Remark More trials yield better estimates.