PROBABILITY

Ever wondered if it'll rain tomorrow or if you'll win a game? That's probability! It's a math way to guess how likely things are to happen.

A ALGEBRA OF EVENTS

A.1 SAMPLE SPACES

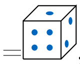
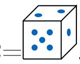
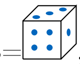
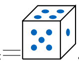
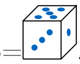
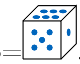
Definition Outcome

An **outcome** is one possible result of a random experiment.

Ex: What are the outcomes when you flip a coin? 

Answer: The outcomes are Heads (H)= and Tails (T)=.

Ex: What are the outcomes when you roll a six-sided die?

Answer: The outcomes are 1=,2=,3=,4=,5=,and 6=.

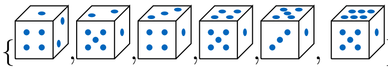
Definition Sample Space

The **sample space** is the set of all possible outcomes of a random experiment.

Ex: What's the sample space when you flip a coin?

Answer: The sample space is {Heads, Tails} = {}, or just {H, T} for short.

Ex: What's the sample space when you roll a six-sided die?

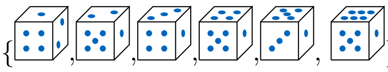
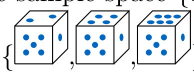
Answer: The sample space is {1, 2, 3, 4, 5, 6} = {}.

A.2 EVENTS

Definition Event

Un **événement** est un sous-ensemble des issues possibles de l'univers. We write it E.

Ex: In the experiment of rolling a die, find E the event of rolling an even number.

Answer: Among the outcomes of the sample space {1, 2, 3, 4, 5, 6} = {} ,the event of rolling an even number is $E = \{2, 4, 6\}$ = {}.

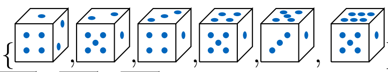
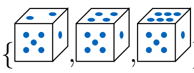
A.3 COMPLEMENTARY EVENTS

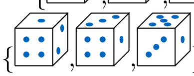
Ever wonder what happens if you look for everything **except** a certain event? That's where the complementary event comes in! It's just everything in the sample space that isn't in your event. We usually write it as E' ("E-prime").

Definition Complementary Event

The **complementary event** of an event E is all the outcomes in the sample space that are **not** in E. We write it E'.

Ex: In the experiment of rolling a die, let E the event of rolling an even number. Find E'.

Answer: The sample space is $\{1, 2, 3, 4, 5, 6\}$ = {} and $E = \{2, 4, 6\}$ = {}.

So E' is all the other numbers: $\{1, 3, 5\}$ = {}. These are the odd numbers.

A.4 MULTI-STEP RANDOM EXPERIMENTS

Discover: A **multi-step random experiment** is an experiment that involves a sequence of actions, where each action (or step) has its own set of possible outcomes. For example, tossing two coins is a multi-step experiment because it consists of two separate coin tosses:

- The first coin toss (step 1) can result in Heads (H) or Tails (T).
- The second coin toss (step 2) can also result in Heads (H) or Tails (T).

The overall outcome of the experiment is found by combining the results of each step. For instance, the outcome HT means the first coin landed on Heads and the second on Tails.
Using different representations—such as grids, tables, tree diagrams, or simply listing the outcomes—helps us organize and visualize all the possible combinations that result from multiple steps.

Method Representations of Multi-Step Random Experiments

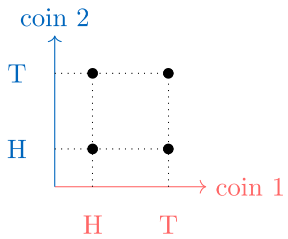
When an experiment involves more than one step, we can represent the sample space (the set of all possible outcomes) in several ways:

- using a **grid** (to visually map combinations along two axes),
- using a **table** (to organize outcomes in rows and columns),
- using a **tree diagram** (to show each sequential step), or
- by **listing** all possible outcomes.

Ex: For the random experiment of tossing two coins, display the sample space by:

1. using a grid
2. using a table
3. using a tree diagram
4. listing all possible outcomes

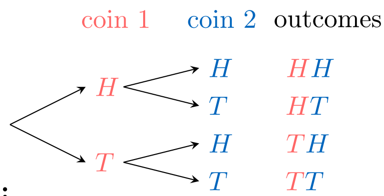
Answer:



1. **Grid:**

2. **Table:**

coin 1 \ coin 2	H	T
H	HH	HT
T	TH	TT



3. **Tree diagram:**

4. **List:** $\{HH, HT, TH, TT\}$

A.5 E OR F

Definition E or F

The **union of two events** E and F, denoted as E **or** F or $E \cup F$, is the event that occurs if at least one of the events E or F happens (i.e., if E occurs, or F occurs, or both occur).

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an even number: $E = \{2, 4, 6\}$.
Let event F be the event of rolling a number less than 4: $F = \{1, 2, 3\}$.
Find the event E or F.

Answer: E or F includes all outcomes from both event sets.
So, E or $F = E \cup F = \{1, 2, 3, 4, 6\}$.

A.6 E AND F

Definition E and F

The **intersection of two events** E and F, denoted as E **and** F or $E \cap F$, is the set of all outcomes that are common to both E and F. The event E and F occurs if and only if both E and F happen simultaneously.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an odd number: $E = \{1, 3, 5\}$.
Let event F be the event of rolling a number less than 4: $F = \{1, 2, 3\}$.
Find the event E and F.

Answer: E and F includes all outcomes that are common to both E and F.
So, E and $F = E \cap F = \{1, 3\}$.

A.7 MUTUALLY EXCLUSIVE

Definition Mutually Exclusive

Two events E and F are said to be **mutually exclusive** if they cannot occur at the same time. In other words, the occurrence of event E excludes the possibility of event F occurring, and vice versa. This is mathematically represented as:

$$E \text{ and } F = E \cap F = \emptyset,$$

where $\emptyset$ denotes the empty set, indicating that there are no outcomes common to both E and F.

Ex: Consider the roll of a standard six-sided die.
Let event E be the event of rolling an odd number: $E = \{1, 3, 5\}$.
Let event F be the event of rolling an even number: $F = \{2, 4, 6\}$.
Show that E and F are mutually exclusive.

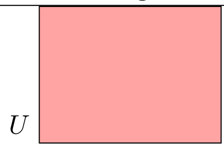
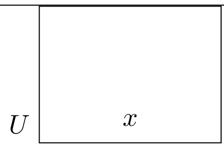
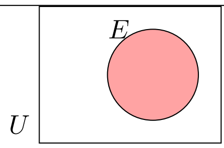
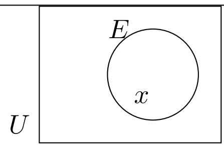
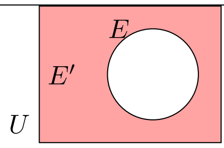
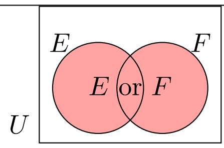
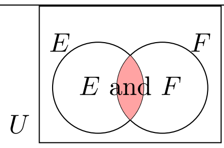
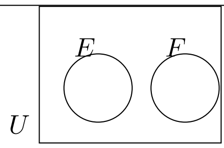
Answer: There are no outcomes common to both E and F:

$$E \cap F = \emptyset.$$

Thus, the events E and F are mutually exclusive.

A.8 VENN DIAGRAM

Definition Correspondence Between Set Theory and Probabilistic Vocabulary

Notation	Set Vocabulary	Probabilistic Vocabulary	Venn Diagram
U	Universal set	Sample space	
x	Element of U	Outcome	
$\emptyset$	Empty set	Impossible event	
E	Subset of U	Event	
$x \in E$	x is an element of E	x is an outcome of E	
E'	Complement of E in U	Complement of E in U	
E or F	Union of E and F: $E \cup F$	E or F	
E and F	Intersection of E and F: $E \cap F$	E and F	
$E \cap F = \emptyset$	E and F are disjoint	E and F are mutually exclusive	

B PROBABILITY

B.1 PROBABILITY AXIOMS

Discover: When you flip a coin, there are two possible outcomes: heads or tails. The chance of getting heads is the same as getting tails—it's 1 out of 2! In math, we write:

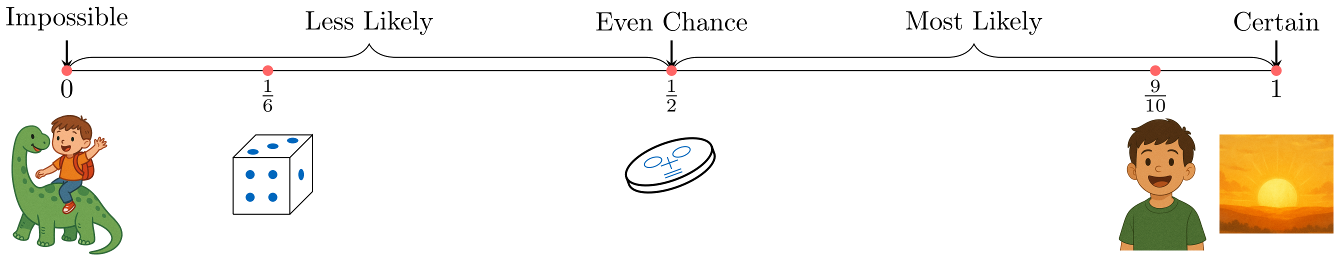
$$P\ (\text{"Getting Heads"}) = \frac{1}{2}$$

The probability → P; of getting heads → ("Getting Heads"); is equal → $=$; 1 chance out of 2 → $\frac{1}{2}$

This means heads will happen about half the time!

Definition Probability

The **probability** of an event E, written $P(E)$, is a number that tells us how likely the event is to happen. It's always between 0 (impossible) and 1 (certain).



Ex: The probability of an event "even chance" can be represented as:

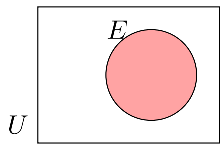
- **Fraction**: $\frac{1}{2}$
- **Decimal**: To convert the fraction to a decimal, divide the numerator by the denominator: $1 \div 2 = 0.5$.
- **Percentage**: To convert the decimal to a percentage, multiply by 100%: $0.5 \times 100\% = 50\%$.

Definition Probability Axioms

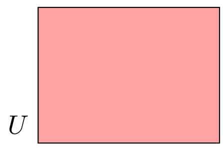
P is a **probability** if:

- $0 \leqslant P(E) \leqslant 1$, for any event E,
- $P(U) = 1$,
- If E and F are mutually exclusive events, then $P(E \text{ or } F) = P(E) + P(F)$.

- The probability of an event E, $P(E)$, is represented by the shaded area of the event in a Venn diagram:

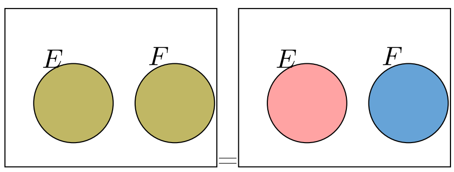


- The first axiom states that the probability of an event E is a value between 0 and 1, inclusive.
- The second axiom states that the probability of the sample space U is equal to 1, i.e., 100%. This is because the sample space U includes all possible outcomes of a random experiment, so the event U always occurs, and $P(U) = 1$. In a Venn diagram, this is represented as the entire shaded area of the sample space:



$P(U) = 1$

- The third axiom states that if two events are mutually exclusive (i.e., they cannot occur simultaneously), then the probability of their union is the sum of their individual probabilities. In a Venn diagram, for two mutually exclusive events (with no overlap), the total area of their union is the sum of the individual areas:



$P(E \text{ or } F) = P(E) + P(F)$

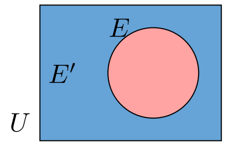
B.2 PROBABILITY RULES

Discover: Ever wondered how to quickly find the chance that something **doesn't** happen? There's a shortcut for that! It's called the complement rule.

Proposition Complement Rule

For any event E with complementary event E',

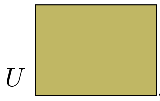
$$P(E) + P(E') = 1$$

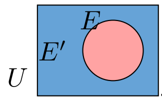


Also,

$$P(E') = 1 - P(E)$$

Proof

The green area, $P(U)$, 

,is the sum of the red area, $P(E)$, and the blue area, $P(E')$, 

.
So,

$$P(E) + P(E') = P(U)$$

Since $P(U) = 1$, we have:

$$P(E) + P(E') = 1$$

Ex: Farid has a 0.8 (80%) chance of finishing his homework on time tonight (event E). What's the chance he **doesn't** finish on time?

Answer: The complementary event E' is "Farid does **not** finish his homework on time." By the complement rule:

$$\begin{aligned} P(E') &= 1 - P(E) \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

So, there's a 0.2 (or 20%) chance he doesn't finish on time!

Proposition Addition Law of Probability

For any events E and F,

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Ex: A local high school is holding a talent show where students can participate in singing, dancing, or both. The probability that a randomly selected student participates in singing is 0.4, the probability that a student participates in dancing is 0.3, and the probability that a student participates in both singing and dancing is 0.1. Find the probability that a randomly selected student participates in either singing or dancing.

Answer:

- Let S be the event "participates in singing", and D the event "participates in dancing". We are given $P(S) = 0.4$, $P(D) = 0.3$, and $P(S \text{ and } D) = 0.1$.
- The probability that a student participates in either singing or dancing is $P(S \text{ or } D)$.
- By the addition law of probability:

$$\begin{aligned} P(S \text{ or } D) &= P(S) + P(D) - P(S \text{ and } D) \\ &= 0.4 + 0.3 - 0.1 \\ &= 0.6 \end{aligned}$$

- Therefore, the probability is $\boxed{0.6}$.

B.3 EQUALLY LIKELY

Discover: Have you ever flipped a fair coin or rolled a fair die? In these experiments, each outcome is just as likely as the others. We call these equally likely outcomes.

Definition Equally Likely

When all outcomes are **equally likely**, the probability of an event E is:

$$P(E) = \frac{\textbf{number of outcomes in the event}}{\textbf{number of outcomes in the sample space}} = \frac{n(E)}{n(U)}$$

Ex: What's the probability of rolling an even number with a fair six-sided die?

Answer:

- Sample space $= \{1, 2, 3, 4, 5, 6\}$ (6 outcomes).
- $E = \{2, 4, 6\}$ (3 outcomes).
- $$P(E) = \frac{3}{6} = \frac{1}{2}$$

So, there's a $\frac{1}{2}$ chance (or 50%) of rolling an even number!

B.4 PROBABILITY OF INDEPENDENT EVENTS

Independent events are situations where what happens in one event does **not** affect what happens in the other. For example: rolling two dice at the same time. The result of the first die doesn't change the chances for the second die—they are independent!

Definition Independent Events

Two events, A and B, are **independent** if the chance of both happening is just the product of their individual chances. Mathematically:

$$P(A \textbf{ and } B) = P(A) \times P(B)$$

Ex: Imagine you do two totally separate actions:

1. Flipping a fair coin (heads or tails).
2. Rolling a fair six-sided die (1, 2, 3, 4, 5, or 6).

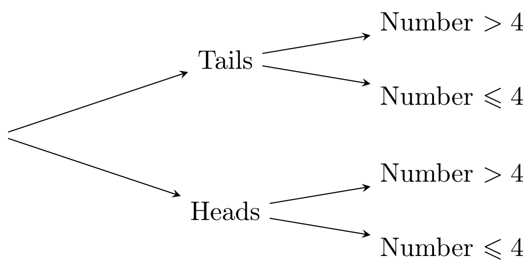
What's the probability of getting tails **and** rolling a number greater than 4 (like a 5 or 6)?

Answer: Let's break it down:

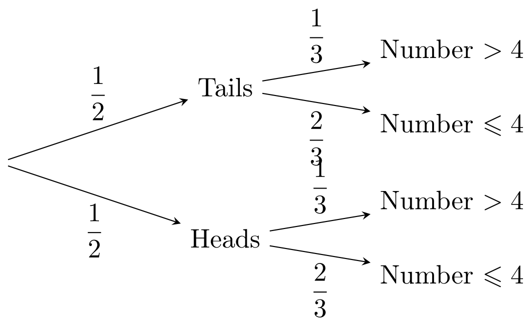
- These events are independent, so we multiply their probabilities.
- For the coin: You've got two options—heads or tails—and they're equally likely. So, $P(\text{"tails"}) = \frac{1}{2}$.
- For the die: There are six sides, and "greater than 4" means 5 or 6. That's 2 out of 6 possibilities, so $P(\text{"number} > 4\text{"}) = \frac{2}{6} = \frac{1}{3}$.
- Now, combine them:
$$P(\text{"tails" and "number} > 4\text{"}) = P(\text{"tails"}) \times P(\text{"number} > 4\text{"}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$
- Result: There's a $\frac{1}{6}$ chance of landing tails and rolling a 5 or 6.

Method **Finding the Probability of Two Independent Events using a Probability Tree Diagram**

1. **Draw the tree:** Start with two branches for the coin: "Heads" and "Tails." From each, draw two branches for the die: "Number > 4" (5 or 6) and "Number $\leqslant 4$" (1, 2, 3, 4).

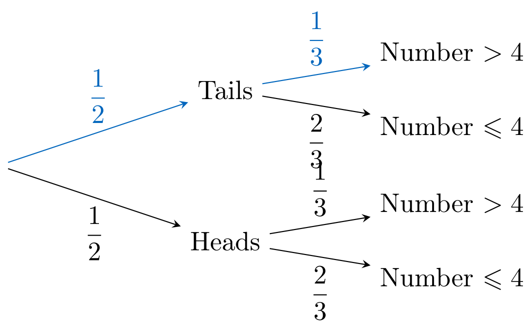


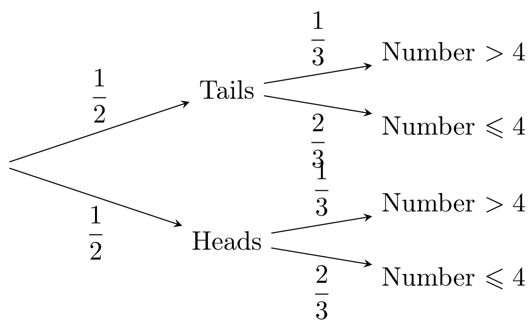
2. **Label probabilities:** Each coin branch is $\frac{1}{2}$. For the die, "Number > 4" is $\frac{1}{3}$ (2 out of 6), "Number $\leqslant 4$" is $\frac{4}{6} = \frac{2}{3}$.



3. **Multiply the probabilities along the path:** For "Tails" and "Number > 4":

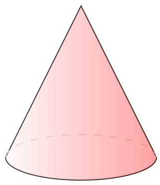
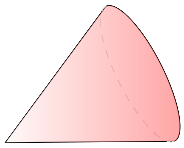
$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$





B.5 EXPERIMENTAL PROBABILITY

Discover: Isaac wishes to determine how a cone lands when tossed—base down or point down? The possible outcomes are as follows:

- Base down: 
- Point down: 

Due to the cone's shape potentially favoring one outcome, Isaac cannot predict the probabilities. He conducts 50 trials and records the results:

- Base down: 30 times.
- Point down: 20 times.

The chance of landing base down is approximately 30 times over 50 times. So, he estimates:

- $P(\text{"base down"}) = \frac{30}{50} = 0.6$ (60%).
- $P(\text{"point down"}) = \frac{20}{50} = 0.4$ (40%).

The more trials he conducts, the closer his estimates approach the true probabilities.

Theorem Law of Large Numbers

The probability of an event E can be estimated using the formula:

$$P(E) \approx \frac{\text{number of times the event occurs}}{\text{number of trials}}$$

Here, "trials" refer to the number of times the experiment is repeated.

Remark More trials yield better estimates.