MODELLING WITH FUNCTIONS

A MODELLING CYCLE

A.1 IDENTIFY VARIABLES AND PARAMETERS

Ex 1: For the bacterial growth depending on time $N = N_0 e^{kt}$, identify:

- The **Independent Variable** (input).
  $\boxed{t}$
- The **Dependent Variable** (output).
  $\boxed{N}$
- The **Parameters** (the constants that define the specific situation).
  $\boxed{N_0, k}$

Answer:

- **Independent Variable:** t (time). This is the input value that varies freely.
- **Dependent Variable:** N (number of bacteria). This is the output value that depends on the time.
- **Parameters:** N_0 (initial population) and k (growth rate). These are constants that stay fixed for a specific bacterial culture (though e is a mathematical constant, N_0 and k are the model parameters).

Ex 2: For the height of an object dropped under gravity $h(t) = -\frac{g}{2}t^2 + h_0$, identify:

- The **Independent Variable** (input).
  $\boxed{t}$
- The **Dependent Variable** (output).
  $\boxed{h}$
- The **Parameters** (the constants that define the specific situation).
  $\boxed{g, h_0}$

Answer:

- **Independent Variable:** t (time). It is the variable that progresses forward.
- **Dependent Variable:** h (height). The height depends on how much time has passed.
- **Parameters:** g and h_0. g is the acceleration due to gravity (constant for the planet), and h_0 is the initial height (constant for the specific drop).

Ex 3: For the total amount with simple interest depending on time $A(t) = P(1 + rt)$, identify:

- The **Independent Variable** (input).
  $\boxed{t}$
- The **Dependent Variable** (output).
  $\boxed{A}$
- The **Parameters** (the constants that define the specific situation).
  $\boxed{P, r}$

Answer:

- **Independent Variable:** t (time). This is the duration of the investment which varies.
- **Dependent Variable:** A (total amount). The final amount depends on how long the money is invested.
- **Parameters:** P and r. P is the initial principal and r is the interest rate; these are fixed constants for a specific investment contract.

B LINEAR MODELS

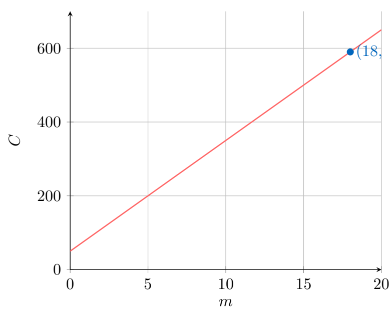
B.1 APPLYING LINEAR MODELS

Ex 4: A gym membership costs a joining fee of \$50 and a monthly subscription of \$30.

1. Write a linear model $C(m)$ for the total cost after m months.
2. Calculate the total cost after 2 years.
3. If a member has paid a total of \$590, how many months have they been a member?

Answer:

1. **Model:** $C(m) = 30m + 50$.
2. **Cost after 2 years (24 months):**
$$C(24) = 30(24) + 50 = 720 + 50 = \$770.$$
3. **Find m:**
$$590 = 30m+50 \implies 540 = 30m \implies m = \frac{540}{30} = 18 \text{ months.}$$



Ex 5: Water is flowing into a tank at a constant rate. Initially, the tank contains 200 litres. After 5 minutes, it contains 350 litres.

1. Find the rate of flow (gradient) in litres per minute.
2. Write the equation for Volume $V(t)$.
3. Interpret the physical meaning of the y-intercept.

Answer:

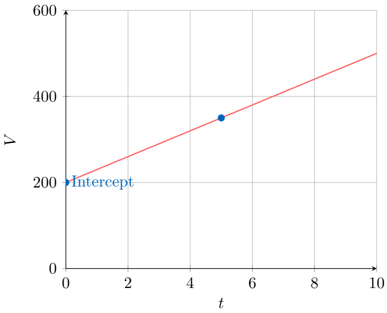
1. **Gradient m:**

$$m = \frac{\Delta V}{\Delta t} = \frac{350 - 200}{5 - 0} = \frac{150}{5} = 30 \text{ L/min.}$$

2. **Equation:**

$$V(t) = 30t + 200.$$

3. **Intercept:**
The y-intercept (200) represents the initial amount of water in the tank at $t = 0$.



Ex 6: A sports centre offers two membership plans for using the facilities:

- **Plan A:** A joining fee of \$50 plus \$10 per visit.
- **Plan B:** An unlimited pass with a fixed fee of \$125 (no cost per visit).

Let n be the number of visits.

1. Write a linear model $C_A(n)$ for the cost of Plan A and a model $C_B(n)$ for Plan B.
2. Calculate the cost of both plans for 5 visits.
3. Find the number of visits for which the total cost is the same for both plans.
4. Which subscription is the best option if you plan to visit the centre 10 times?

Answer:

1. **Models:**

$$C_A(n) = 10n + 50$$
$$C_B(n) = 125$$

2. **Cost for 5 visits ($n = 5$):**
   - Plan A: $C_A(5) = 10(5) + 50 = 50 + 50 = \$100$.
   - Plan B: $C_B(5) = \$125$.

   Plan A is cheaper for 5 visits.

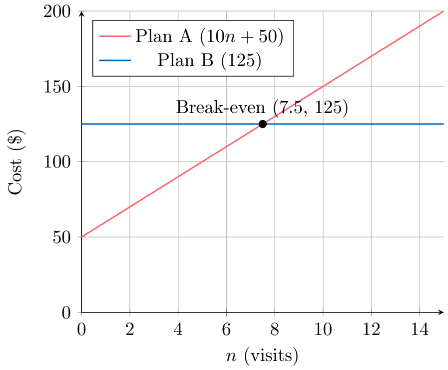
3. **Intersection:**
We equate the two costs:

$$10n + 50 = 125$$
$$10n = 75$$
$$n = 7.5$$

The costs are equal at 7.5 visits.

4. **Best option for 10 visits:**
Since $10 > 7.5$, Plan B should be cheaper.
   - Plan A: $C_A(10) = 10(10) + 50 = \$150$.
   - Plan B: \$125.

   Plan B is the best option.



Ex 7: The length L of a spring (in cm) depends linearly on the mass m attached to it (in kg).

- When no mass is attached ($m = 0$), the spring is 15 cm long.
- When a 4 kg mass is attached, the spring is 23 cm long.

1. Find the equation of the linear model $L(m) = am + b$.
2. Interpret the meaning of the parameter a in this context.
3. Calculate the mass required to stretch the spring to 30 cm.

Answer:

1. **Find parameters a and b:** We have two points: $(0, 15)$ and $(4, 23)$.
   - The y-intercept is given directly: $b = 15$.
   - The gradient a is:

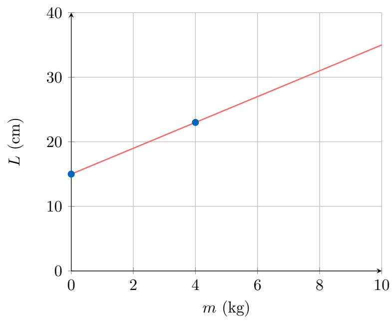
$$a = \frac{L_2 - L_1}{m_2 - m_1} = \frac{23 - 15}{4 - 0} = 2.$$

   The model is $\boldsymbol{L(m) = 2m + 15}$.

2. **Interpretation:** The parameter $a = 2$ represents the extension of the spring per unit of mass (2 cm per kg).

3. **Find mass for $L = 30$:**

$$30 = 2m + 15 \implies 15 = 2m \implies m = 7.5 \text{ kg.}$$



C QUADRATIC MODELS

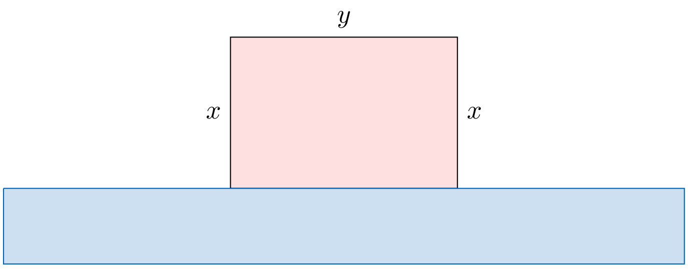
C.1 APPLYING QUADRATIC MODELS

Ex 8: A farmer has 100 metres of fencing to enclose a rectangular paddock along a riverbank (no fence is needed along the river).

1. If x is the width of the paddock, show that the area is given by $A(x) = 100x - 2x^2$.
2. Find the dimensions that maximise the area.
3. Calculate the maximum area.

Answer:

1. **Derive Area:**
   Let width $= x$.



$$\begin{aligned} 2x + y &= 100 \\ y &= 100 - 2x. \end{aligned}$$

$$\text{Area } = xy = x(100 - 2x) = 100x - 2x^2.$$

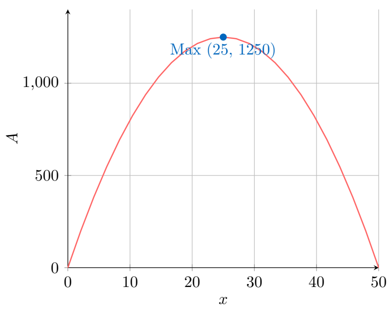
2. **Maximise:**
   This is a parabola opening downwards ($a = -2$). Max is at the vertex.

$$x = -\frac{b}{2a} = -\frac{100}{2(-2)} = \frac{100}{4} = 25 \text{ m}.$$

   Width $= 25$m, Length $= 100 - 2(25) = 50$m.

3. **Max Area:**

$$A(25) = 25 \times 50 = 1250 \text{ m}^2.$$



Ex 9: A ball is thrown upwards. Its height is modelled by $H(t) = -4.9t^2 + 19.6t + 1.5$.

1. What is the initial height of the ball?
2. At what time does the ball reach its maximum height?
3. When does the ball hit the ground?

Answer:

1. **Initial Height:** $H(0) = 1.5$ m.
2. **Time of Max Height:**

$$t = -\frac{b}{2a} = -\frac{19.6}{2(-4.9)} = 2 \text{ s}.$$

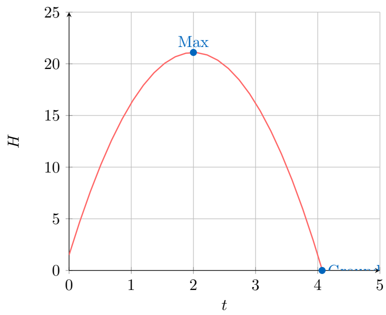
3. **Hit Ground:** $H(t) = 0$.

$$-4.9t^2 + 19.6t + 1.5 = 0$$

   Using quadratic formula:

$$\begin{aligned} t &= \frac{-19.6 \pm \sqrt{19.6^2 - 4(-4.9)(1.5)}}{2(-4.9)} \\ t &= \frac{-19.6 \pm 20.33}{-9.8} \end{aligned}$$

   $t \approx -0.07$ (reject) or $t \approx 4.07$ s.



Ex 10: The path of a projectile is modelled by a quadratic function $y = a(x - h)^2 + k$.

- The maximum height of 20 metres is reached at a horizontal distance of $x = 2$ metres.
- The projectile launches from the ground at the origin $(0, 0)$.

1. Identify the values of h and k from the vertex information.
2. Use the point $(0, 0)$ to calculate the value of the parameter a.
3. Write the full equation and find where the projectile lands.

Answer:

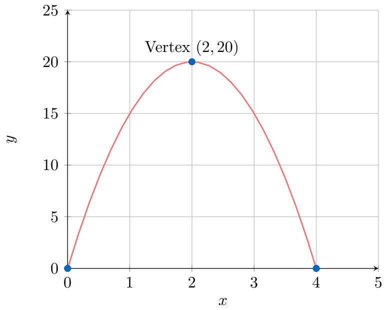
1. **Vertex** (h, k)**:**
   The maximum is at $(2, 20)$, so $h = 2$ and $k = 20$.

$$y = a(x - 2)^2 + 20.$$

2. **Find** a**:**
   Substitute $(0, 0)$:

$$\begin{aligned} 0 &= a(0 - 2)^2 + 20 \\ 0 &= 4a + 20 \\ 4a &= -20 \\ a &= -5. \end{aligned}$$

3. **Equation and Landing:**
   Equation: $\boldsymbol{y = -5(x-2)^2 + 20}$.
   To find the landing point ($y = 0$), we use symmetry. If it starts at $x = 0$ and peaks at $x = 2$, it lands at $x = 4$.
   Check: $-5(4-2)^2 + 20 = -5(4) + 20 = 0$.
   It lands at 4 metres.



D CUBIC MODELS

D.1 APPLYING CUBIC MODELS

Ex 11: The temperature T (in °C) of a chemical reaction t minutes after it begins is modelled by the cubic function:

$$T(t) = -t^3 + 9t^2 + 22, \quad t \geq 0$$

1. Find the initial temperature of the reaction.
2. Calculate the temperature after 4 minutes.
3. Determine how long it takes for the temperature to return to its initial value.

Answer:

1. **Initial Temperature** $(t = 0)$**:**

$$T(0) = -(0)^3 + 9(0)^2 + 22 = 22°\text{C}.$$

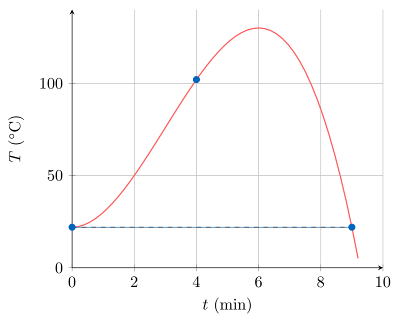
2. **Temperature at** $t = 4$**:**

$$T(4) = -(4)^3 + 9(4)^2 + 22 = -64 + 9(16) + 22 = -64 + 144 + 22 = 102°\text{C}.$$

3. **Return to Initial Value:** We want to find t (other than 0) such that $T(t) = 22$.

$$\begin{aligned} -t^3 + 9t^2 + 22 &= 22 \\ -t^3 + 9t^2 &= 0 \\ t^2(-t + 9) &= 0 \end{aligned}$$

   Solutions are $t = 0$ (start) and $t = 9$. It takes **9 minutes**.



Ex 12: An open box is made by cutting squares of side length x cm from the corners of a rectangular sheet of cardboard measuring 20 cm by 14 cm, and folding up the sides. The volume V of the box is given by:

$$V(x) = 4x^3 - 68x^2 + 280x$$

1. Calculate the volume of the box if a square of size $x = 2$ cm is cut.
2. Determine the physical domain of the function (i.e., what is the maximum possible value for x?).

Answer:

1. **Volume for** $x = 2$**:**

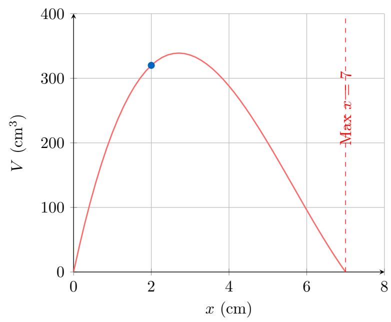
$$\begin{aligned} V(2) &= 4(2)^3 - 68(2)^2 + 280(2) \\ &= 4(8) - 68(4) + 560 \\ &= 32 - 272 + 560 \\ &= 320 \text{ cm}^3. \end{aligned}$$

   (Alternatively, using geometry: Length $= 20 - 2(2) = 16$, Width $= 14 - 2(2) = 10$, Height $= 2$. Vol $= 16 \times 10 \times 2 = 320$).

2. **Domain:**
   You cannot cut a square larger than half the shortest side of the cardboard. The shortest side is 14 cm.

$$2x < 14 \implies x < 7.$$

   The physical domain is $0 < x < 7$.



E EXPONENTIAL MODELS

E.1 APPLYING EXPONENTIAL MODELS

Ex 13: The population of a city is growing exponentially according to $P(t) = 50\,000e^{0.04t}$, where t is in years.

1. Find the population after 10 years.
2. How long will it take for the population to reach 100,000?

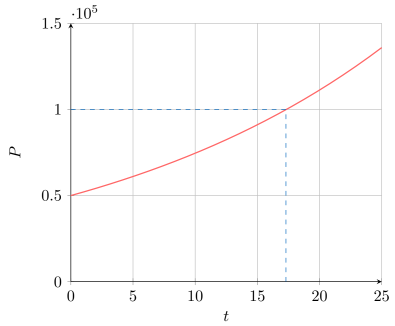
Answer:

1. **Population after 10 years:**

$$P(10) = 50000e^{0.04(10)} \approx 74\,591.$$

2. **Time to reach 100,000:**

$$\begin{aligned} 100\,000 &= 50\,000e^{0.04t} \\ 2 &= e^{0.04t} \\ \ln(2) &= 0.04t \\ t &= \frac{\ln(2)}{0.04} \approx 17.3 \text{ years.} \end{aligned}$$



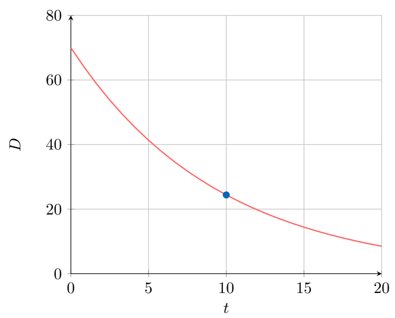
Ex 14: A cup of coffee cools down. The difference between the coffee temperature and the room temperature is given by $D(t) = 70(0.9)^t$, where t is in minutes.

1. Find the temperature difference at $t = 0$.
2. Find the temperature difference after 10 minutes.
3. If the room temperature is 20°C, what is the actual temperature of the coffee after 10 minutes?

Answer:

1. **At** $t = 0$**:** $D(0) = 70(1) = 70°\text{C}$.
2. **At** $t = 10$**:** $D(10) = 70(0.9)^{10} \approx 24.4°\text{C}$.
3. **Actual Temperature:**

$$T_{\text{coffee}} = T_{\text{room}} + D(t) = 20 + 24.4 = 44.4°\text{C}$$



Ex 15: The value of a high-end laptop computer depreciates exponentially over time. Its value $V(t)$ (in dollars) t years after purchase is modelled by $V(t) = Ae^{kt}$.

- The laptop was purchased new for \$2000.
- After 2 years, its value had dropped to \$1200.

1. Identify the initial value parameter A.
2. Use the value at $t = 2$ to calculate the rate parameter k (round to 3 decimal places).
3. Write the specific equation for the model.
4. Estimate the value of the laptop after 5 years.

Answer:

1. **Parameter** A**:**
At $t = 0$, $V(0) = 2000$. The equation is:

$$\begin{aligned} Ae^{k(0)} &= 2000 \\ A &= 2000 \end{aligned}$$

2. **Parameter** k**:**
At $t = 2$, $V(2) = 1200$. The equation is:

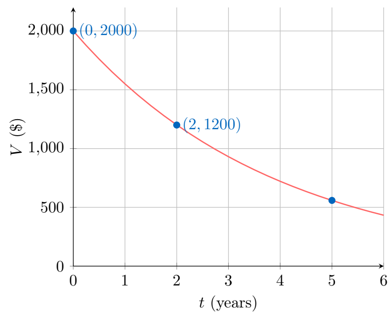
$$\begin{aligned} 1200 &= 2000e^{k(2)} \\ \frac{1200}{2000} &= e^{2k} \\ 0.6 &= e^{2k} \\ \ln(0.6) &= 2k \\ k &= \frac{\ln(0.6)}{2} \approx \frac{-0.5108}{2} \approx -0.255. \end{aligned}$$

3. **Model Equation:**

$$V(t) = 2000e^{-0.255t}.$$

4. **Value at** $t = 5$**:**

$$V(5) = 2000e^{-0.255(5)} = 2000e^{-1.275} \approx 2000(0.2794) \approx \$559.$$



F DIRECT/INVERSE VARIATION MODELS

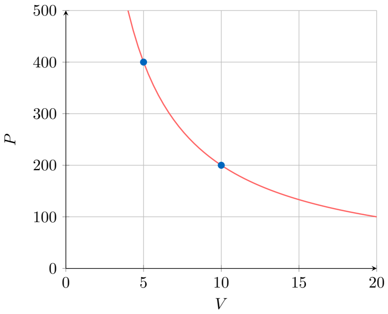
F.1 APPLYING DIRECT/INVERSE VARIATION MODELS

Ex 16: Boyle's Law states that for a fixed amount of gas at constant temperature, pressure P varies inversely with volume V. When $V = 10$ L, $P = 200$ kPa.

1. Find the equation relating P and V.
2. Calculate the pressure when the volume is compressed to 5 L.

Answer:

1. **Equation:**
$P = \frac{k}{V}$.
Substitute values: $200 = \frac{k}{10} \implies k = 2000$.
Equation: $P = \frac{2000}{V}$.
2. **Prediction:**
When $V = 5$: $P = \frac{2000}{5} = 400$ kPa.

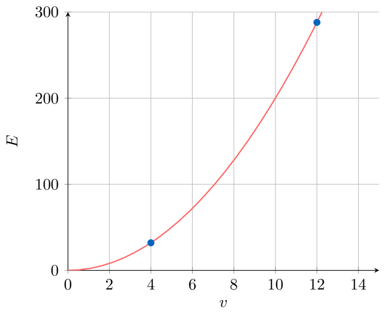


Ex 17: The kinetic energy E of an object varies directly with the square of its velocity v. An object moving at 4 m/s has 32 Joules of energy.

1. Find the equation connecting E and v.
2. Calculate the energy if the velocity is tripled (to 12 m/s).

Answer:

1. **Equation:**
   $E = kv^2$.
   Substitute values:
   $32 = k(4^2) \implies 32 = 16k \implies k = 2$.
   Equation: $E = 2v^2$.
2. **Calculate E:**
   When $v = 12$: $E = 2(12)^2 = 288$ Joules.

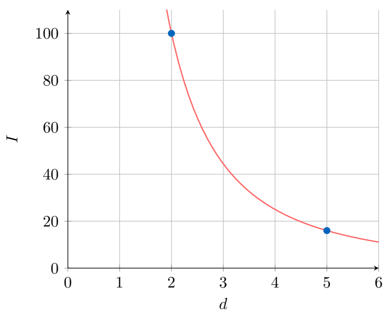


Ex 18: The intensity I of light from a bulb varies inversely with the square of the distance d from the bulb. At a distance of 2 meters, the intensity is 100 units.

1. Find the equation relating I and d.
2. Calculate the intensity at a distance of 5 meters.

Answer:

1. **Equation:**
   $I = \frac{k}{d^2}$.
   Substitute values: $100 = \frac{k}{2^2} \implies 100 = \frac{k}{4} \implies k = 400$.
   Equation: $I = \frac{400}{d^2}$.
2. **Prediction:**
   When $d = 5$: $I = \frac{400}{5^2} = \frac{400}{25} = 16$ units.

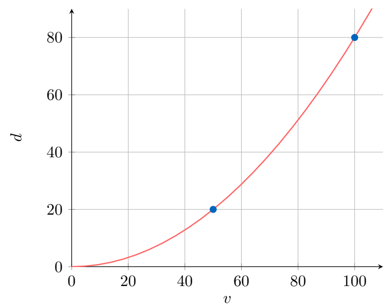


Ex 19: The braking distance d of a car varies directly with the square of its speed v. A car travelling at 50 km/h requires 20 meters to stop.

1. Find the equation connecting d and v.
2. Calculate the braking distance if the car is travelling at 100 km/h.

Answer:

1. **Equation:**
   $d = kv^2$.
   Substitute values:
   $20 = k(50^2) \implies 20 = 2500k \implies k = \frac{20}{2500} = 0.008$.
   Equation: $d = 0.008v^2$.
2. **Calculate d:**
   When $v = 100$: $d = 0.008(100)^2 = 0.008(10000) = 80$ meters.



G SINUSOIDAL MODELS

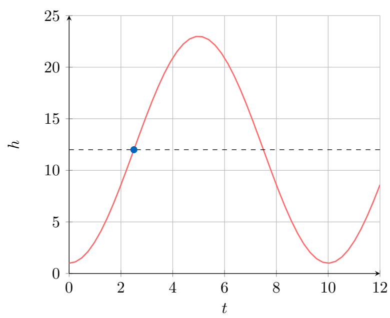
G.1 APPLYING SINUSOIDAL MODELS

Ex 20: A person on a Ferris wheel is at a height $h(t) = 12 - 11\cos(\frac{\pi}{5}t)$ metres, where t is in minutes.

1. Find the minimum and maximum height.
2. Find the period of the ride.
3. Find the height after 2.5 minutes.

Answer:

1. **Min/Max:**
   Amplitude $= 11$, Midline $= 12$.
   Min $= 12 - 11 = 1$ m. Max $= 12 + 11 = 23$ m.
2. **Period:**
   $T = \frac{2\pi}{b} = \frac{2\pi}{\pi/5} = 10$ minutes.
3. **Height at 2.5:**
   $h(2.5) = 12 - 11\cos(\frac{\pi}{5}(2.5)) = 12 - 11\cos(\frac{\pi}{2})$.
   Since $\cos(\pi/2) = 0$, $h(2.5) = 12$ m.



Ex 21: The depth of the tide $d(t)$ in a harbour is modelled by $d(t) = A\cos(B(t-C)) + D$.

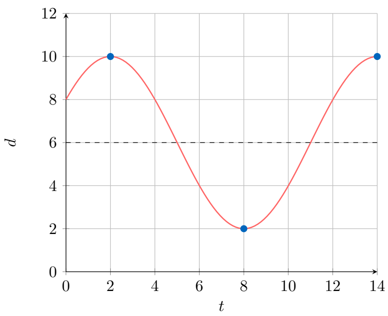
- High tide is 10 m and occurs at 02:00 ($t = 2$).
- Low tide is 2 m and occurs at 08:00 ($t = 8$).

Find the values of the parameters $A, B, C,$ and D.

Answer:

1. **Vertical Shift (D):**
   Midline between max (10) and min (2).
   $$D = \frac{10+2}{2} = 6.$$
2. **Amplitude (A):**
   Distance from midline to max.
   $$A = 10 - 6 = 4.$$
3. **Period and B:**
   Time from high to low is half a period ($8 - 2 = 6$ hours).
   Full Period $T = 12$ hours.
   $$B = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6}.$$
4. **Phase Shift (C):**
   A positive cosine function starts at a maximum. The first maximum is at $t = 2$.
   $$C = 2.$$

   Final Equation:
   $$d(t) = 4\cos\left(\frac{\pi}{6}(t-2)\right) + 6$$

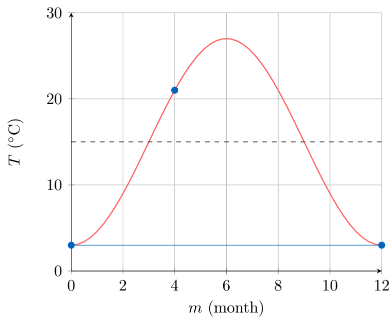


Ex 22: The average daily temperature T in a city is modelled by the function $T(m) = 15 - 12\cos(\frac{\pi}{6}m)$, where m is the month number ($m = 0$ is January).

1. Find the average yearly temperature (the midline).
2. Calculate the temperature in month $m = 4$ (May).
3. Find the minimum temperature and the month it occurs.

Answer:

1. **Midline:**
   This is the constant term D.
   $$\text{Average} = 15°\text{C}.$$
2. **Temperature at $m = 4$:**
   $$T(4) = 15 - 12\cos\left(\frac{\pi}{6}(4)\right) = 15 - 12\cos\left(\frac{2\pi}{3}\right).$$
   Since $\cos(2\pi/3) = -0.5$:
   $$T(4) = 15 - 12(-0.5) = 15 + 6 = 21°\text{C}.$$
3. **Minimum:**
   Cosine ranges from -1 to 1. Since we subtract the cosine, the minimum occurs when the cosine part is maximum (1).
   $$T_{min} = 15 - 12(1) = 3°\text{C}.$$
   This occurs when the angle is 0 (or 2π).
   $$\frac{\pi}{6}m = 0 \implies m = 0 \text{ (January)}.$$



H LOGARITHMIC MODELS

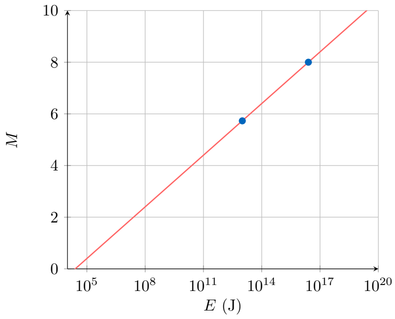
H.1 APPLYING LOGARITHMIC MODELS

Ex 23: The magnitude M of an earthquake on the Richter scale is given by $M = \frac{2}{3}\log_{10}\left(\frac{E}{E_0}\right)$, where E is the energy released (in joules) and $E_0 = 10^{4.4}$ is a reference energy.

1. Find the magnitude of an earthquake that releases 10^{13} joules of energy.
2. How much energy is released by an earthquake of magnitude 8.0?

Answer:

1. **Magnitude ($E = 10^{13}$):**
   $$M = \frac{2}{3}\log_{10}\left(\frac{10^{13}}{10^{4.4}}\right) = \frac{2}{3}\log_{10}(10^{13-4.4}) = \frac{2}{3}\log_{10}(10^{8.6}).$$
   $$M = \frac{2}{3}(8.6) \approx 5.73.$$
2. **Energy ($M = 8$):**
   $$8 = \frac{2}{3}\log_{10}\left(\frac{E}{10^{4.4}}\right) \implies 12 = \log_{10}\left(\frac{E}{10^{4.4}}\right).$$
   $$10^{12} = \frac{E}{10^{4.4}} \implies E = 10^{12} \times 10^{4.4} = 10^{16.4} \text{ Joules}.$$



Ex 24: The acidity of a solution is measured by pH, defined as pH $= -\log_{10}[H^+]$, where $[H^+]$ is the hydrogen ion concentration in moles per litre.

1. Calculate the pH of a solution with $[H^+] = 2.5\times10^{-4}$ mol/L.
2. Find the hydrogen ion concentration of a solution with a pH of 3.2.

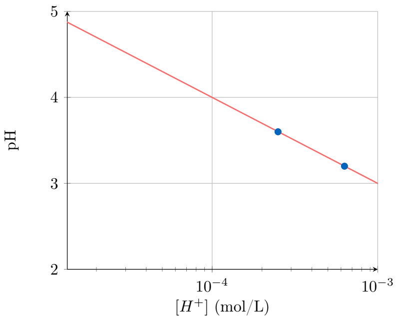
Answer:

1. **Calculate pH:**

$$\text{pH} = -\log_{10}(2.5\times10^{-4}) = -(\log_{10}(2.5) + \log_{10}(10^{-4}))$$
$$\text{pH} = -(0.398 - 4) = 3.602.$$

2. **Calculate $[H^+]$:**

$$3.2 = -\log_{10}[H^+] \implies -3.2 = \log_{10}[H^+].$$
$$[H^+] = 10^{-3.2} \approx 6.31\times10^{-4} \text{ mol/L}.$$



Ex 25: The loudness of a sound, L (in decibels, dB), is related to its intensity I (in W/m^2) by the formula:

$$L = 10\log_{10}\left(\frac{I}{10^{-12}}\right)$$

1. Find the loudness, in dB, of a vacuum cleaner with an intensity of 10^{-4} W/m^2.
2. Determine the intensity of a sound that has a loudness of 100 dB.

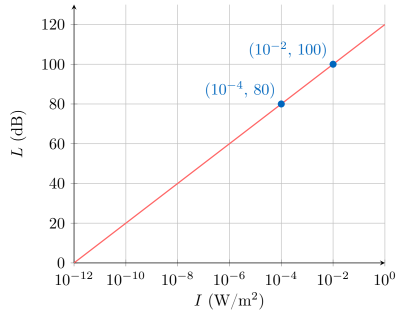
Answer:

1. **Loudness for $I = 10^{-4}$:**

$$\begin{aligned} L &= 10\log_{10}\left(\frac{10^{-4}}{10^{-12}}\right) \\ &= 10\log_{10}(10^8) \\ &= 10(8) \\ &= 80 \text{ dB}. \end{aligned}$$

2. **Intensity for $L = 100$:**

$$\begin{aligned} 100 &= 10\log_{10}\left(\frac{I}{10^{-12}}\right) \\ 10 &= \log_{10}\left(\frac{I}{10^{-12}}\right) \\ 10^{10} &= \frac{I}{10^{-12}} \\ I &= 10^{10}\times10^{-12} \\ I &= 10^{-2} \text{ (or 0.01) W/m}^2. \end{aligned}$$



Ex 26: A student's typing speed N (in words per minute) increases with practice time t (in weeks) according to the model:

$$N(t) = 30 + 15\ln t, \quad t \geq 1.$$

1. Calculate the typing speed after 1 week.
2. Calculate the typing speed after 10 weeks.
3. How many weeks of practice are needed to reach a speed of 60 words per minute?

Answer:

1. **Speed at $t = 1$:**

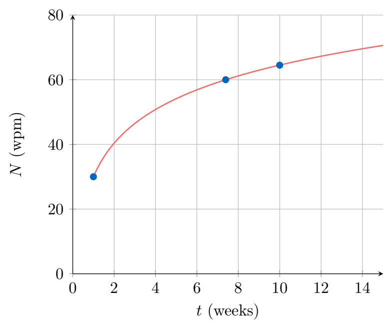
$$N(1) = 30 + 15\ln(1) = 30 + 0 = 30 \text{ wpm}.$$

2. **Speed at $t = 10$:**

$$N(10) = 30+15\ln(10) \approx 30+15(2.30) \approx 30+34.5 = 64.5 \text{ wpm}.$$

3. **Time to reach 60 wpm:**

$$\begin{aligned} 60 &= 30 + 15\ln t \\ 30 &= 15\ln t \\ \ln t &= 2 \\ t &= e^2 \approx 7.39 \text{ weeks}. \end{aligned}$$



I LOGISTIC MODELS

I.1 APPLYING LOGISTIC MODELS

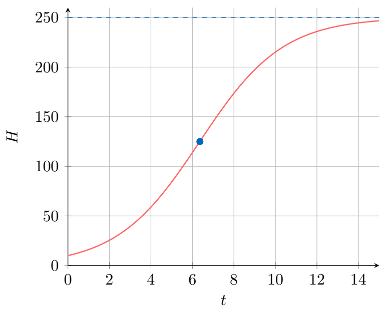
Ex 27: A sunflower grows according to the logistic model $H(t) = \frac{250}{1+24e^{-0.5t}}$ cm, where t is weeks.

1. What is the maximum possible height (carrying capacity)?
2. Calculate the height at $t = 0$.
3. How many weeks does it take to reach 125 cm?

Answer:

1. **Max Height:** Numerator $L = 250$ cm.
2. **Initial Height:**
$H(0) = \frac{250}{1+24} = \frac{250}{25} = 10$ cm.
3. **Reach 125 cm:**

$$\begin{aligned} 125 &= \frac{250}{1+24e^{-0.5t}} \\ 1 + 24e^{-0.5t} &= \frac{250}{125} \\ 1 + 24e^{-0.5t} &= 2 \\ 24e^{-0.5t} &= 1 \\ e^{-0.5t} &= \frac{1}{24} \\ -0.5t &= \ln\left(\frac{1}{24}\right) \\ t &= \frac{\ln(1/24)}{-0.5} \approx 6.36 \text{ weeks.} \end{aligned}$$



Ex 28: The population of rabbits on an island is modelled by the function $P(t) = \frac{1000}{1+19e^{-0.4t}}$, where t is the time in months.

1. Find the initial population of rabbits.
2. State the carrying capacity of the island.
3. Calculate the time required for the population to reach 800 rabbits.

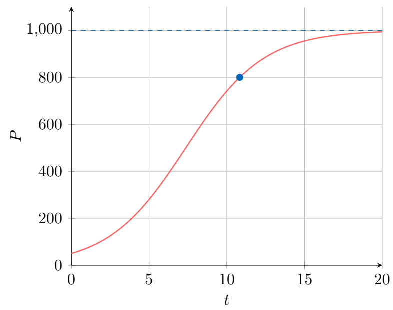
Answer:

1. **Initial Population** ($t = 0$):

$$P(0) = \frac{1000}{1+19e^0} = \frac{1000}{20} = 50 \text{ rabbits.}$$

2. **Carrying Capacity:** This is the numerator: $L = 1000$ rabbits.
3. **Reach 800 rabbits:**

$$\begin{aligned} 800 &= \frac{1000}{1+19e^{-0.4t}} \\ 1 + 19e^{-0.4t} &= \frac{1000}{800} = 1.25 \\ 19e^{-0.4t} &= 0.25 \\ e^{-0.4t} &= \frac{0.25}{19} \\ -0.4t &= \ln\left(\frac{0.25}{19}\right) \\ t &= \frac{\ln(0.25/19)}{-0.4} \approx 10.83 \text{ months.} \end{aligned}$$



Ex 29: A rumour spreads through a school of 800 students. The number of students who have heard the rumour after t days is given by $N(t) = \frac{800}{1+79e^{-0.6t}}$.

1. How many students start the rumour?
2. Find the time when half of the students (400) have heard the rumour.
3. Calculate when 90% of the school will have heard the rumour.

Answer:

1. **Initial students:**

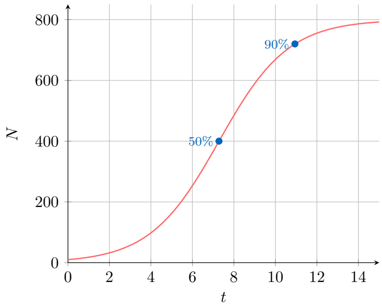
$$N(0) = \frac{800}{1+79} = \frac{800}{80} = 10 \text{ students.}$$

2. **Half the school** ($N = 400$)**:**

$$\begin{aligned} 400 &= \frac{800}{1+79e^{-0.6t}} \\ 1 + 79e^{-0.6t} &= 2 \\ 79e^{-0.6t} &= 1 \\ -0.6t &= \ln(1/79) \\ t &= \frac{\ln(1/79)}{-0.6} \approx 7.28 \text{ days.} \end{aligned}$$

3. **90% of school** ($N = 0.9 \times 800 = 720$)**:**

$$\begin{aligned} 720 &= \frac{800}{1+79e^{-0.6t}} \\ 1 + 79e^{-0.6t} &= \frac{800}{720} = \frac{10}{9} \\ 79e^{-0.6t} &= \frac{1}{9} \\ e^{-0.6t} &= \frac{1}{711} \\ t &= \frac{\ln(1/711)}{-0.6} \approx 10.94 \text{ days.} \end{aligned}$$



Ex 30: A population of fruit flies in a laboratory experiment is modelled by the logistic function:

$$P(t) = \frac{L}{1 + Ce^{-kt}}$$

where t is the time in days.

- The carrying capacity of the environment is known to be 500 flies.
- Initially ($t = 0$), there were 50 flies.
- After 2 days ($t = 2$), the population grew to 150 flies.

1. State the value of L.
2. Use the initial population to find the value of C.
3. Use the population at $t = 2$ to find the value of k.
4. Write the complete equation for $P(t)$.

Answer:

1. **Carrying Capacity (L):** Given as the maximum limit.

$$L = 500.$$

2. **Find C (using $t = 0, P = 50$):**

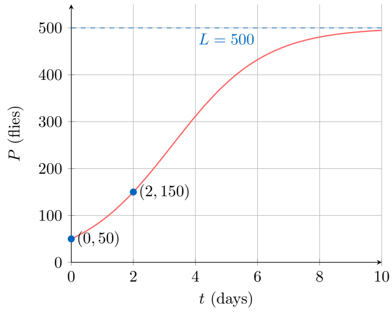
$$\begin{aligned} 50 &= \frac{500}{1 + Ce^0} \\ 50 &= \frac{500}{1 + C} \\ 1 + C &= \frac{500}{50} \\ 1 + C &= 10 \\ C &= 9. \end{aligned}$$

3. **Find k (using $t = 2, P = 150$):** Substitute $L = 500$, $C = 9$, $t = 2$, and $P = 150$:

$$\begin{aligned} 150 &= \frac{500}{1 + 9e^{-k(2)}} \\ 1 + 9e^{-2k} &= \frac{500}{150} \\ 1 + 9e^{-2k} &= \frac{10}{3} \\ 9e^{-2k} &= \frac{10}{3} - 1 \\ 9e^{-2k} &= \frac{7}{3} \\ e^{-2k} &= \frac{7}{27} \\ -2k &= \ln\left(\frac{7}{27}\right) \\ k &= \frac{\ln(7/27)}{-2} \approx 0.675. \end{aligned}$$

4. **Complete Equation:**

$$P(t) = \frac{500}{1 + 9e^{-0.675t}}.$$



J PIECEWISE LINEAR MODELS

J.1 APPLYING PIECEWISE LINEAR MODELS

Ex 31: A taxi charges \$5 for the first 2 km, and then \$2 per km for every additional km.

1. Write a piecewise function $C(d)$ for the cost of a trip of d km.
2. Calculate the cost of a 10 km trip.

Answer:

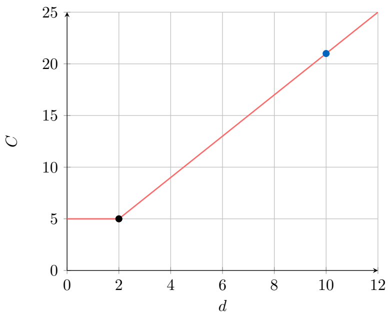
1. **Function:**

$$C(d) = \begin{cases} 5 & 0 \le d \le 2 \\ 5 + 2(d - 2) & d > 2 \end{cases}$$

Simplifying $d > 2$: $5 + 2d - 4 = 2d + 1$.

2. **Cost for 10 km:**
Since $10 > 2$, use the second rule.

$$C(10) = 5 + 2(10 - 2) = \$21$$



Ex 32: A delivery company charges a flat rate of \$10 for packages weighing up to 5 kg. For packages weighing more than 5 kg, they charge the flat rate plus \$3 for every additional kg.

1. Write a piecewise function $C(w)$ for the cost of shipping a package of weight w kg.
2. Calculate the cost to ship a package weighing 12 kg.

Answer:

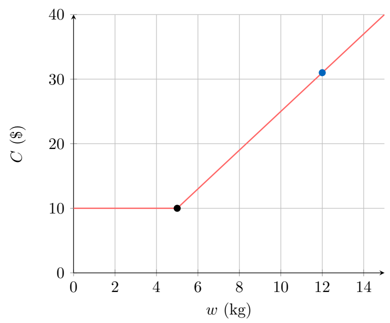
1. **Function:**

$$C(w) = \begin{cases} 10 & 0 < w \leq 5 \\ 10 + 3(w-5) & w > 5 \end{cases}$$

Simplifying for $w > 5$: $C(w) = 10 + 3w - 15 = 3w - 5$.

2. **Cost for 12 kg:**
Since $12 > 5$, use the second rule.

$$C(12) = 10 + 3(12-5) = 10 + 3(7) = 10 + 21 = \$31$$



Ex 33: A photo printing shop charges \$0.50 per photo for the first 50 photos. For any order exceeding 50 photos, the remaining photos are charged at a discounted rate of \$0.30 per photo.

1. Write a piecewise function $P(n)$ for the total price of printing n photos.
2. Calculate the price for printing 100 photos.

Answer:

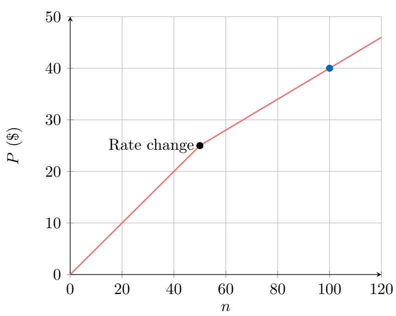
1. **Function:**

$$P(n) = \begin{cases} 0.50n & 0 \leq n \leq 50 \\ 25 + 0.30(n-50) & n > 50 \end{cases}$$

Note: The 25 comes from the first 50 photos ($50 \times 0.50 = 25$).

2. **Price for 100 photos:**
Since $100 > 50$, use the second rule.

$$P(100) = 25 + 0.30(100-50) = 25 + 0.30(50) = 25 + 15 = \$40$$



Ex 34: A car park charges \$3 per hour for the first 4 hours of parking. If a car stays longer than 4 hours, the rate drops to \$2 for every additional hour.

1. Write a piecewise function $C(t)$ for the total cost of parking for t hours.
2. Calculate the cost of parking for 7 hours.

Answer:

1. **Function:**

$$C(t) = \begin{cases} 3t & 0 \leq t \leq 4 \\ 12 + 2(t-4) & t > 4 \end{cases}$$

Note: The 12 comes from the cost of the first 4 hours ($4 \times 3 = 12$). Simplifying for $t > 4$: $C(t) = 12 + 2t - 8 = 2t + 4$.

2. **Cost for 7 hours:**
Since $7 > 4$, use the second rule.

$$C(7) = 12 + 2(7-4) = 12 + 2(3) = 12 + 6 = \$18$$