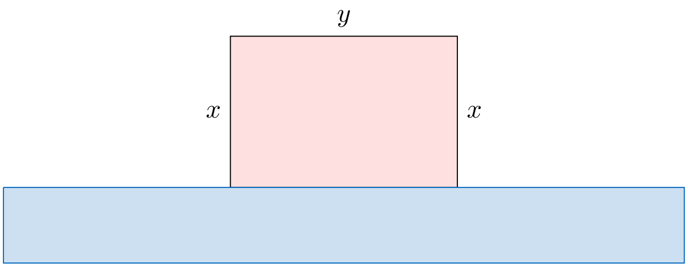
MODELLING AND OPTIMISATION

A OPTIMISATION

A.1 SOLVING OPTIMISATION PROBLEMS

Ex 1: A farmer fences off a rectangular field with a total of 4000 m of fencing. Since the field is located along a river, the farmer only needs to fence three of the four sides.

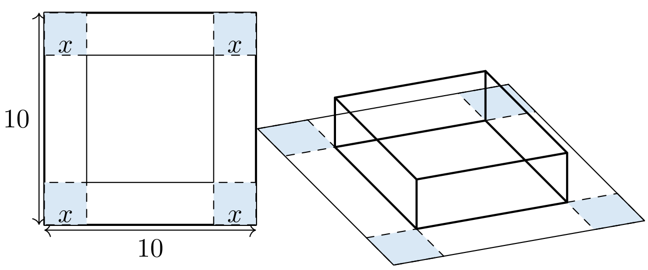


1. Let x be the length of each side perpendicular to the river, and y the side parallel to the river. Write an expression for the area A of the field in terms of x.
2. Determine the dimensions of the field which maximize the area (verification with the second derivative test is optional).

Answer: Assume $x > 0, y > 0$.

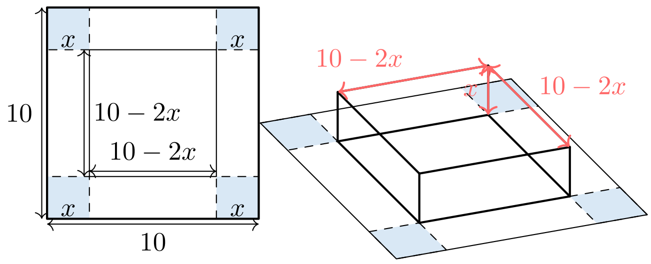
1. **Expression of A in terms of x** The fencing condition is
$$\begin{aligned} 2x + y &= 4000 \\ \implies y &= 4000 - 2x. \end{aligned}$$
The area is
$$\begin{aligned} A &= xy \\ &= x(4000 - 2x) \\ &= 4000x - 2x^2. \end{aligned}$$
2. **Maximization** Differentiate:
$$\frac{dA}{dx} = 4000 - 4x.$$
Solve $\frac{dA}{dx} = 0$:
$$\begin{aligned} 4000 - 4x &= 0 \\ x &= 1000. \end{aligned}$$
Then
$$\begin{aligned} y &= 4000 - 2(1000) \\ &= 2000. \end{aligned}$$
3. **Conclusion** The maximum area is obtained for dimensions:
$$\boxed{x = 1000\,\text{m},\ y = 2000\,\text{m}.}$$

Ex 2: A square sheet of paper 10 cm by 10 cm is made into an open box (no top) by cutting out squares of side x cm at each corner and folding up the sides.



1. Write an expression for the volume V of the box in terms of x.
2. Find the value of x that maximizes the volume (verification with the second derivative test is optional).

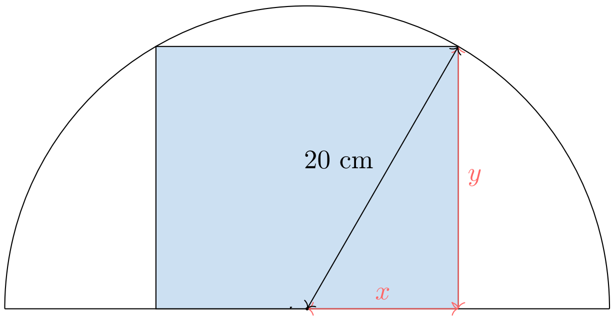
Answer:



For a real box, $0 < x < 5$ (so $10 - 2x > 0$).

1. **Volume in terms of x** Each corner square is $x \times x$, so the base dimensions are $10 - 2x$ by $10 - 2x$ and the height is x.
$$\begin{aligned} V(x) &= (10 - 2x)(10 - 2x)(x) \\ &= x(100 - 40x + 4x^2) \\ &= 100x - 40x^2 + 4x^3. \end{aligned}$$
2. **Maximization** Differentiate:
$$\frac{dV}{dx} = 100 - 80x + 12x^2.$$
Solve $\frac{dV}{dx} = 0$:
$$\begin{aligned} 100 - 80x + 12x^2 &= 0 \\ 12x^2 - 80x + 100 &= 0 \\ 3x^2 - 20x + 25 &= 0 \quad \text{(divide by 4)} \end{aligned}$$
Quadratic formula:
$$\begin{aligned} x &= \frac{20 \pm \sqrt{(-20)^2 - 4(3)(25)}}{2 \cdot 3} \\ &= \frac{20 \pm \sqrt{400 - 300}}{6} \\ &= \frac{20 \pm \sqrt{100}}{6} \\ &= \frac{20 \pm 10}{6}. \end{aligned}$$
So
$$x = \frac{30}{6} = 5, \quad \text{or} \quad x = \frac{10}{6} = \frac{5}{3}.$$
3. **Check which gives maximum** At $x = 5$, the base dimension is $10 - 2(5) = 0$, so $V = 0$ (not maximum). At $x = \frac{5}{3}$,
$$\begin{aligned} V &= (10 - 2\tfrac{5}{3})^2 \cdot \tfrac{5}{3} \\ &= \left(\tfrac{20}{3}\right)^2 \cdot \tfrac{5}{3} \\ &= \tfrac{400}{9} \cdot \tfrac{5}{3} \\ &= \tfrac{2000}{27}. \end{aligned}$$
This is the maximum.
4. **Conclusion** The maximum volume occurs when
$$\boxed{x = \tfrac{5}{3}\,\text{cm}}$$
giving maximum volume $\frac{2000}{27}$ cm^3.

Ex 3: Consider a semicircle of radius 20 cm. A rectangle is inscribed in the semicircle, with its base lying on the diameter.



1. Show that the area of the rectangle can be written as $A(x) = 2x\sqrt{400 - x^2}$, where x is half the base.
2. Find the value of x that maximizes $A(x)$, and determine the maximum area of the rectangle (verification with the second derivative test is optional).

Answer: Let $x \in [0, 20]$ denote half the base.

1. **Expression of the area**

$$\begin{aligned} x^2 + y^2 &= 20^2 \quad \text{(Pythagoras' theorem)} \\ y^2 &= 400 - x^2 \\ y &= \sqrt{400 - x^2} \end{aligned}$$

Hence the area:

$$A(x) = 2x \cdot y = 2x \cdot \sqrt{400 - x^2}.$$

2. **Maximization** Differentiate:

$$\begin{aligned} \frac{dA}{dx} &= 2\sqrt{400 - x^2} + 2x \cdot \frac{-2x}{2\sqrt{400 - x^2}} \\ &= \frac{2(400 - x^2) - 2x^2}{\sqrt{400 - x^2}} \\ &= \frac{800 - 4x^2}{\sqrt{400 - x^2}} \end{aligned}$$

Solve $\frac{dA}{dx} = 0$:

$$\begin{aligned} 800 - 4x^2 &= 0 \\ x^2 &= 200 \\ x &= 10\sqrt{2} \end{aligned}$$

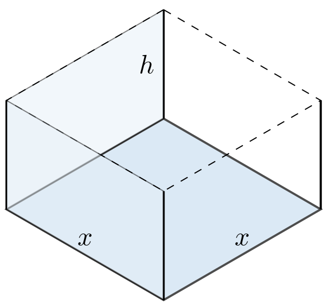
The height is

$$y = \sqrt{400 - (10\sqrt{2})^2} = \sqrt{400 - 200} = 10\sqrt{2}.$$

3. **Maximum area**

$$A_{\text{max}} = (2x)(y) = (20\sqrt{2})(10\sqrt{2}) = 400\,\text{cm}^2$$

Ex 4: A box has a volume of $1\,\text{m}^3$, a square base, and an open top. Find the dimensions of the box which minimize its surface area (verification with the second derivative test is optional).



Answer:

1. **Model the problem**
Let the side of the square base be x (m) and the height be h (m). The volume constraint is

$$x^2 h = 1.$$

The surface area (no top) is

$$S = x^2 + 4xh.$$

2. **Substitute the constraint**
From $x^2 h = 1$, we get

$$h = \frac{1}{x^2}.$$

Substituting into S:

$$\begin{aligned} S(x) &= x^2 + 4x \cdot \frac{1}{x^2} \\ &= x^2 + \frac{4}{x}. \end{aligned}$$

3. **Differentiate and find critical points**

$$S'(x) = 2x - \frac{4}{x^2}.$$

Set $S'(x) = 0$:

$$\begin{aligned} 2x - \frac{4}{x^2} &= 0 \\ 2x^3 - 4 &= 0 \\ x^3 &= 2 \\ x &= \sqrt[3]{2}. \end{aligned}$$

4. **Find the corresponding height**
From $h = \frac{1}{x^2}$:

$$\begin{aligned} h &= \frac{1}{(\sqrt[3]{2})^2} \\ &= \frac{1}{\sqrt[3]{4}}. \end{aligned}$$

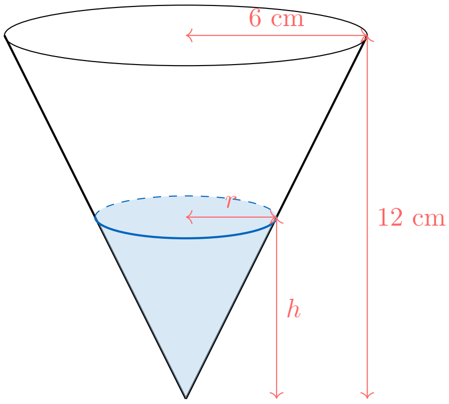
5. **Conclude**
The dimensions of the box that minimize the surface area are:

$$\text{Base side } = \sqrt[3]{2} \text{ m}, \quad \text{Height } = \frac{1}{\sqrt[3]{4}} \text{ m}.$$

B RATES OF CHANGE

B.0.1 SOLVING RELATED RATES PROBLEMS

Ex 5: Water is poured into an inverted right circular cone. The cone has a height of 12 cm and a radius of 6 cm at the top. The water is being poured in at a constant rate of 4 cm^3 per second. Let r be the radius of the water's surface and h be the height of the water at time t.



1. Show that the radius of the water's surface is always half of its height, i.e., $r = \frac{h}{2}$.

2. Find the rate at which the height of the water is increasing when the water is 8 cm deep.

Answer:

1. **Show that $r = \frac{h}{2}$**
   By similar triangles, the ratio of the radius to the height of the water is the same as the ratio for the entire cone:

$$\begin{aligned}\frac{r}{h} &= \frac{\text{Radius of cone}}{\text{Height of cone}}\\ &= \frac{6}{12}\\ &= \frac{1}{2}\end{aligned}$$

   Therefore,

$$r = \frac{h}{2}.$$

2. **Find the rate of change of the height**

   - **Model**: We are given $\frac{dV}{dt} = 4\ \text{cm}^3/\text{s}$. We want $\frac{dh}{dt}$ when $h = 8$ cm.
   - **Equation**: Volume of the water in the cone:

$$\begin{aligned}V &= \frac{1}{3}\pi r^2 h\\ &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h\\ &= \frac{1}{3}\pi\cdot\frac{h^2}{4}\cdot h\\ &= \frac{\pi h^3}{12}\end{aligned}$$

   - **Differentiate**: Differentiate with respect to t:

$$\begin{aligned}\frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{\pi h^3}{12}\right)\\ \frac{dV}{dt} &= \frac{\pi}{12}\cdot 3h^2\frac{dh}{dt}\\ &= \frac{\pi h^2}{4}\frac{dh}{dt}\end{aligned}$$

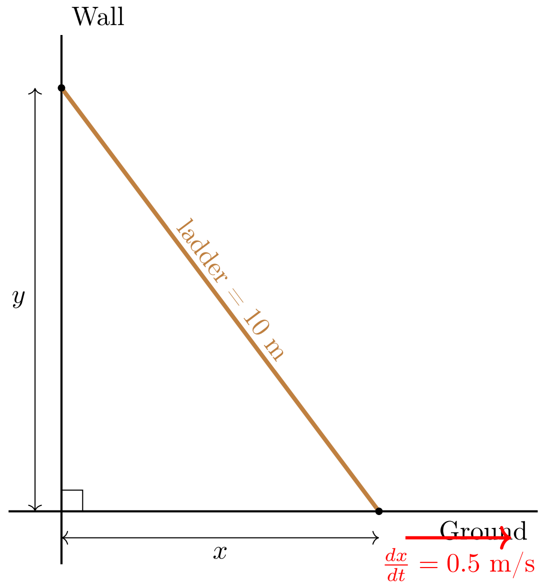
   - **Substitute and Solve**: With $\frac{dV}{dt} = 4$ and $h = 8$:

$$\begin{aligned}4 &= \frac{\pi(8)^2}{4}\frac{dh}{dt}\\ 4 &= \frac{64\pi}{4}\frac{dh}{dt}\\ 4 &= 16\pi\frac{dh}{dt}\\ \frac{dh}{dt} &= \frac{4}{16\pi}\\ &= \frac{1}{4\pi}\\ &\approx 0.0796\ \text{cm/s}\end{aligned}$$

Ex 6: A ladder 10 m long is leaning against a vertical wall. The bottom of the ladder is pulled away from the wall at a rate of 0.5 m/s. Let x be the distance from the bottom of the ladder to the wall, and y the height of the top of the ladder on the wall.

1. Show that x and y are related by the equation $x^2+y^2 = 100$.

2. Find the rate at which the top of the ladder is sliding down the wall when the bottom of the ladder is 6 m away from the wall.



Answer:

1. **Relation between x and y**
   By Pythagoras' theorem:

$$x^2 + y^2 = 10^2 = 100.$$

2. **Rate of change of y**

   - **Model**: Differentiate the equation with respect to time t:

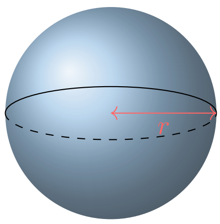
$$\begin{aligned}\frac{d}{dt}\left(x^2+y^2\right) &= \frac{d}{dt}(100)\\ 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0\\ x\frac{dx}{dt} + y\frac{dy}{dt} &= 0\end{aligned}$$

- **Solve**: When $x = 6$, $\frac{dx}{dt} = 0.5$, and $y = \sqrt{100 - 36} = 8$.

$$\begin{aligned} 6(0.5) + 8\frac{dy}{dt} &= 0 \\ 3 + 8\frac{dy}{dt} &= 0 \\ \frac{dy}{dt} &= -\frac{3}{8} \\ &= -0.375 \text{ m/s} \end{aligned}$$

So the top of the ladder is sliding down at 0.375 m/s.

Ex 7: A spherical balloon is being inflated so that its volume increases at a constant rate of 100 cm^3/s. Let r be the radius of the balloon at time t.



1. Write the formula for the volume of the balloon in terms of r.
2. Find the rate at which the radius is increasing when $r = 5$ cm.

Answer:

1. **Volume formula**

$$V = \tfrac{4}{3}\pi r^3.$$

2. **Rate of change of r**

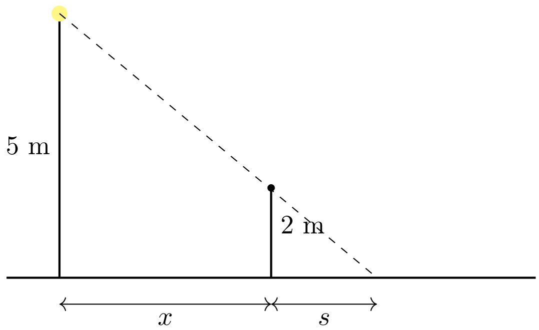
- **Differentiate**:

$$\begin{aligned} \frac{dV}{dt} &= \frac{d}{dt}\left(\tfrac{4}{3}\pi r^3\right) \\ &= 4\pi r^2 \frac{dr}{dt} \end{aligned}$$

- **Substitute and Solve**: With $\frac{dV}{dt} = 100$ and $r = 5$:

$$\begin{aligned} 100 &= 4\pi(5^2)\frac{dr}{dt} \\ 100 &= 100\pi\frac{dr}{dt} \\ \frac{dr}{dt} &= \frac{100}{100\pi} \\ &= \frac{1}{\pi} \\ &\approx 0.318 \text{ cm/s} \end{aligned}$$

Ex 8: A street lamp is 5 m tall. A person 2 m tall walks away from the lamp at a speed of 1.2 m/s. Let x be the distance of the person from the base of the lamp, and s the length of their shadow.



1. Show that $\frac{5}{x+s} = \frac{2}{s}$.
2. Find the rate at which the length of the shadow is increasing when the person is 4 m away from the lamp.

Answer:

1. **Relation between x and s**
From similar triangles:

$$\begin{aligned} \frac{5}{x+s} &= \frac{2}{s} \\ 5s &= 2(x+s) \\ 5s &= 2x + 2s \\ 3s &= 2x \\ s &= \tfrac{2}{3}x \end{aligned}$$

2. **Rate of change of s**

- **Differentiate**: With $s = \frac{2}{3}x$,

$$\frac{ds}{dt} = \frac{2}{3}\frac{dx}{dt}$$

- **Substitute**: With $\frac{dx}{dt} = 1.2$,

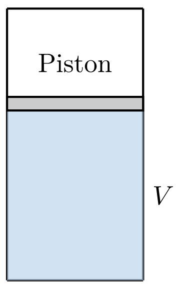
$$\begin{aligned} \frac{ds}{dt} &= \tfrac{2}{3}(1.2) \\ &= 0.8 \text{ m/s} \end{aligned}$$

So the shadow is lengthening at 0.8 m/s.

Ex 9: A gas is contained in a cylinder with a movable piston. The pressure P, volume V, and temperature T of the gas satisfy the ideal gas law:

$$PV = nRT,$$

where n and R are constants. The gas is heated so that the temperature increases at a constant rate of 2 K/s. At a certain instant, $T = 300$ K, $V = 0.01$ m^3, and $P = 2.5 \times 10^5$ Pa. Assume the piston can move so that the pressure P remains constant.



Cylinder

1. Using the ideal gas law, express the volume V as a function of the temperature T (since P is constant).

2. Find the rate of change of the volume $\frac{dV}{dt}$ when $T = 300$ K.

Answer:

1. **Volume as a function of T**
From the ideal gas law with constant P:

$$PV = nRT$$
$$V = \frac{nR}{P}T$$

2. **Rate of change of V**

- **Differentiate**:

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{nR}{P}T\right)$$
$$\frac{dV}{dt} = \frac{nR}{P}\frac{dT}{dt} \qquad \text{(with } P \text{ constant)}$$

- **Find** nR/P: Using $V = 0.01$ m^3, $T = 300$ K, $P = 2.5 \times 10^5$ Pa:

$$\frac{nR}{P} = \frac{V}{T}$$
$$= \frac{0.01}{300}$$
$$\approx 3.33 \times 10^{-5}$$

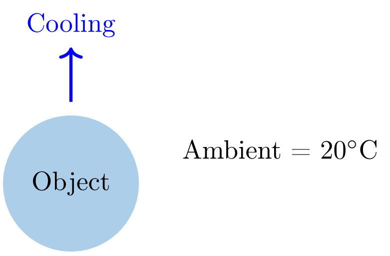
- **Substitute**: With $\frac{dT}{dt} = 2$ K/s,

$$\frac{dV}{dt} \approx \left(3.33 \times 10^{-5}\right)(2)$$
$$\approx 6.67 \times 10^{-5} \text{ m}^3/\text{s}$$

Ex 10: A hot object is placed in a room where the ambient temperature is 20°C. Its temperature T decreases according to Newton's law of cooling:

$$\frac{dT}{dt} = -k(T - 20),$$

where k is a positive constant. At a certain moment, the object's temperature is 80°C and is cooling at a rate of 2°C per minute.



1. Use the data to determine the constant k.

2. Using Newton's law of cooling, find the value of $\frac{dT}{dt}$ when $T = 50$°C.

Answer:

1. **Finding k**
At $T = 80$, $\frac{dT}{dt} = -2$:

$$\frac{dT}{dt} = -k(T - 20)$$
$$-2 = -k(80 - 20)$$
$$-2 = -60k$$
$$k = \frac{2}{60}$$
$$= \frac{1}{30}$$

2. **Cooling rate at $T = 50$**
With $k = \frac{1}{30}$:

$$\frac{dT}{dt} = -\tfrac{1}{30}(50 - 20)$$
$$= -\tfrac{30}{30}$$
$$= -1\,^\circ\text{C/min}$$