MATRIX DIAGONALISATION

In this chapter, we study vectors $\mathbf{x}$ whose direction is unchanged by a matrix $\mathbf{A}$. After multiplying by $\mathbf{A}$, the new vector $\mathbf{Ax}$ lies on the same line through the origin as the original one (it may be longer, shorter, or even point in the opposite direction). Such a vector is called an *eigenvector*, and the number λ that tells how the length (and possibly the direction) changes is called the *eigenvalue*.

A EIGENVALUES AND EIGENVECTORS

Definition Eigenvalues and Eigenvectors

Let $\mathbf{A}$ be a square matrix. An **eigenvector** of $\mathbf{A}$ is a non-zero vector $\mathbf{x}$ such that:

$$\mathbf{Ax} = \lambda\mathbf{x}$$

where λ is a scalar called the **eigenvalue** associated with the eigenvector $\mathbf{x}$.

Method Finding Eigenvalues and Eigenvectors

To find the eigenvalues and eigenvectors of a matrix $\mathbf{A}$:

1. **Find the eigenvalues (λ):** Solve the characteristic equation

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0,$$

where $\mathbf{I}$ is the identity matrix. The solutions are the eigenvalues.

2. **Find the eigenvectors ($\mathbf{x}$):** For each eigenvalue λ, substitute it into

$$\mathbf{Ax} = \lambda\mathbf{x} \text{ or } (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

and solve for the non-zero vector $\mathbf{x}$.

Ex: Find the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}.$$

Answer:

- **Find eigenvalues:**

$$\begin{aligned}
\det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\
\det\left(\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) &= 0 \\
\det\left(\begin{pmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix}\right) &= 0 \\
(4-\lambda)(3-\lambda) - (2)(1) &= 0 \\
\lambda^2 - 7\lambda + 12 - 2 &= 0 \\
\lambda^2 - 7\lambda + 10 &= 0 \\
(\lambda - 5)(\lambda - 2) &= 0
\end{aligned}$$

The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 2$.

- **Find an eigenvector for $\lambda_1 = 5$:**

$$\begin{aligned}
(\mathbf{A} - 5\mathbf{I})\mathbf{x} &= \mathbf{0} \\
\left(\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} 4-5 & 2 \\ 1 & 3-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\begin{pmatrix} -x + 2y \\ x - 2y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\end{aligned}$$

This gives $-x+2y=0$ and $x-2y=0$, so $x=2y$.
Letting $y=t$, we have $x=2t$, so

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2t\\t\end{pmatrix}=t\begin{pmatrix}2\\1\end{pmatrix},\quad t\neq 0.$$

Any vector of the form $t\begin{pmatrix}2\\1\end{pmatrix}$, $t\neq 0$, is an eigenvector corresponding to the eigenvalue 5.

- **Find an eigenvector for $\lambda_2=2$:**

$$\begin{aligned}(\mathbf{A}-2\mathbf{I})\mathbf{x}&=\mathbf{0}\\ \left(\begin{pmatrix}4&2\\1&3\end{pmatrix}-2\begin{pmatrix}1&0\\0&1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}&=\begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}4-2&2\\1&3-2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}&=\begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}2&2\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}&=\begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}2x+2y\\x+y\end{pmatrix}&=\begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$

This gives $2x+2y=0$ and $x+y=0$, so $y=-x$.
Letting $x=t$, we have $y=-t$, so

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}t\\-t\end{pmatrix}=t\begin{pmatrix}1\\-1\end{pmatrix},\quad t\neq 0.$$

Any vector of the form $t\begin{pmatrix}1\\-1\end{pmatrix}$, $t\neq 0$, is an eigenvector corresponding to the eigenvalue 2.

B MATRIX DIAGONALISATION

Definition Diagonal Matrix

A square matrix is said to be **diagonal** if the elements **not** on its leading diagonal are zero.
A 2×2 diagonal matrix has the form:

$$\mathbf{D}=\begin{pmatrix}a&0\\0&b\end{pmatrix}$$

Definition Diagonalisable Matrix

A square matrix $\mathbf{A}$ is **diagonalisable** if there exists an invertible matrix $\mathbf{P}$ such that

$$\mathbf{D}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

is a diagonal matrix. We say that **P diagonalises A**.
Equivalently, we can write

$$\mathbf{A}=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

From the definition of diagonalisation, we start with the relationship:

$$\mathbf{D}=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$$

We can rearrange this equation to make $\mathbf{A}$ the subject. This specific form is particularly useful for calculating powers of matrices (as we will see in the next section).
To isolate $\mathbf{A}$, we multiply by $\mathbf{P}$ on the left and by $\mathbf{P}^{-1}$ on the right:

$$\begin{aligned}\mathbf{D}&=\mathbf{P}^{-1}\mathbf{A}\mathbf{P}\\ \mathbf{P}\mathbf{D}&=\underbrace{\mathbf{P}\mathbf{P}^{-1}}_{\mathbf{I}}\mathbf{A}\mathbf{P}\quad\text{(multiply by }\mathbf{P}\text{ on the left)}\\ \mathbf{P}\mathbf{D}&=\mathbf{A}\mathbf{P}\\ \mathbf{P}\mathbf{D}\mathbf{P}^{-1}&=\mathbf{A}\underbrace{\mathbf{P}\mathbf{P}^{-1}}_{\mathbf{I}}\quad\text{(multiply by }\mathbf{P}^{-1}\text{ on the right)}\\ \mathbf{A}&=\mathbf{P}\mathbf{D}\mathbf{P}^{-1}\end{aligned}$$

Proposition Diagonalising a 2×2 Matrix

If $\mathbf{A}$ is a 2×2 matrix with **distinct real eigenvalues** λ_1, λ_2 and corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2$, then the matrix $\mathbf{P}$ formed by these eigenvectors diagonalises $\mathbf{A}$:

$$\text{If } \mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2) \text{ then } \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Here, $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ is the matrix whose first column is $\mathbf{x}_1$ and whose second column is $\mathbf{x}_2$.

Ex: The matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

Answer: To show that $\mathbf{P}$ diagonalises $\mathbf{A}$, we must show that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

1. **Form the matrix P:**
$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$

2. **Find the inverse $\mathbf{P}^{-1}$:**
$$\det(\mathbf{P}) = (2)(-1) - (1)(1) = -3,$$
so
$$\mathbf{P}^{-1} = \frac{1}{-3}\begin{pmatrix} -1 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}.$$

3. **Calculate $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$:**
$$\begin{aligned}
\mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 10 & 2 \\ 5 & -2 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 15 & 0 \\ 0 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.
\end{aligned}$$

The result is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix},$$

which contains the eigenvalues on the diagonal. Thus, $\mathbf{P}$ diagonalises $\mathbf{A}$.

C MATRIX POWERS

Calculating powers of a matrix $\mathbf{A}$ (such as $\mathbf{A}^{10}$) by repeated multiplication is tedious. However, if we diagonalise the matrix first, the process becomes much simpler.
If $\mathbf{A}$ is diagonalisable, we can write

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where $\mathbf{D}$ is diagonal. To calculate $\mathbf{A}^n$:

$$\begin{aligned}
\mathbf{A}^n &= \underbrace{\mathbf{A}\mathbf{A}\ldots\mathbf{A}}_{n \text{ times}} \\
&= \underbrace{(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})\ldots(\mathbf{P}\mathbf{D}\mathbf{P}^{-1})}_{n \text{ times}} && \text{(substitute } \mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}\text{)} \\
&= \mathbf{P}\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P})\ldots\mathbf{D}\mathbf{P}^{-1} && \text{(regroup terms)} \\
&= \mathbf{P}\mathbf{D}\mathbf{I}\mathbf{D}\mathbf{I}\ldots\mathbf{D}\mathbf{P}^{-1} && \text{(since } \mathbf{P}^{-1}\mathbf{P} = \mathbf{I}\text{)} \\
&= \mathbf{P}(\mathbf{D}\mathbf{D}\ldots\mathbf{D})\mathbf{P}^{-1} \\
&= \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}.
\end{aligned}$$

It is easy to raise a diagonal matrix to a power. If

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

then

$$\begin{aligned}
\mathbf{D}^2 &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}, \\
\mathbf{D}^3 &= \mathbf{D}^2\mathbf{D} = \begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^3 & 0 \\ 0 & \lambda_2^3 \end{pmatrix}, \\
&\vdots \\
\mathbf{D}^n &= \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}.
\end{aligned}$$

Method Calculating Matrix Powers

To calculate a high power $\mathbf{A}^n$ of a diagonalisable matrix $\mathbf{A}$:

1. Find the eigenvalues and eigenvectors of $\mathbf{A}$.
2. Form $\mathbf{P}$ from the eigenvectors and $\mathbf{D}$ from the eigenvalues so that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.
3. Compute $\mathbf{D}^n$ by raising each diagonal entry to the power n.
4. Use
$$\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}.$$

Ex: The matrix

$$\mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

diagonalises

$$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$$

with

$$\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Calculate the matrix $\mathbf{A}^5$.

Answer:

$$\begin{aligned}
\mathbf{A}^5 &= \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1} \\
&= \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^5 & 0 \\ 0 & 2^5 \end{pmatrix}\left(\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\right) \\
&= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3125 & 0 \\ 0 & 32 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 6250 & 32 \\ 3125 & -32 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 6250(1)+32(1) & 6250(1)+32(-2) \\ 3125(1)-32(1) & 3125(1)-32(-2) \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 6282 & 6186 \\ 3093 & 3189 \end{pmatrix} \\
&= \begin{pmatrix} 2094 & 2062 \\ 1031 & 1063 \end{pmatrix}.
\end{aligned}$$