MATRIX DIAGONALISATION

A EIGENVALUES AND EIGENVECTORS

A.1 CALCULATING EIGENVALUES

Ex 1: Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Answer: We solve the characteristic equation $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$:

$$\begin{aligned} \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} &= 0 \\ (2-\lambda)(2-\lambda) - (1)(1) &= 0 \\ (4 - 4\lambda + \lambda^2) - 1 &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \end{aligned}$$

We solve the quadratic equation using the discriminant $\Delta = b^2 - 4ac$ with $a = 1$, $b = -4$, and $c = 3$:

$$\Delta = (-4)^2 - 4(1)(3) = 16 - 12 = 4$$

Since $\Delta > 0$, there are two distinct real roots:

$$\lambda = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2}$$

$$\lambda_1 = \frac{4+2}{2} = 3 \quad \text{and} \quad \lambda_2 = \frac{4-2}{2} = 1$$

So, the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

Ex 2: Find the eigenvalues of the matrix $\mathbf{B} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$.

Answer: We solve the characteristic equation $\det(\mathbf{B} - \lambda \mathbf{I}) = 0$:

$$\begin{aligned} \det \begin{pmatrix} 4-\lambda & 3 \\ 2 & 5-\lambda \end{pmatrix} &= 0 \\ (4-\lambda)(5-\lambda) - (3)(2) &= 0 \\ (20 - 9\lambda + \lambda^2) - 6 &= 0 \\ \lambda^2 - 9\lambda + 14 &= 0 \end{aligned}$$

We solve using the discriminant Δ with $a = 1$, $b = -9$, and $c = 14$:

$$\Delta = (-9)^2 - 4(1)(14) = 81 - 56 = 25$$

The roots are:

$$\lambda = \frac{-(-9) \pm \sqrt{25}}{2(1)} = \frac{9 \pm 5}{2}$$

$$\lambda_1 = \frac{9+5}{2} = \frac{14}{2} = 7 \quad \text{and} \quad \lambda_2 = \frac{9-5}{2} = \frac{4}{2} = 2$$

So, the eigenvalues are $\lambda_1 = 7$ and $\lambda_2 = 2$.

Ex 3: Find the eigenvalues of the matrix $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Answer: We solve the characteristic equation $\det(\mathbf{C} - \lambda \mathbf{I}) = 0$:

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)(2-\lambda) - (2)(3) &= 0 \\ (2 - \lambda - 2\lambda + \lambda^2) - 6 &= 0 \\ \lambda^2 - 3\lambda - 4 &= 0 \end{aligned}$$

We solve using the discriminant Δ with $a = 1$, $b = -3$, and $c = -4$:

$$\Delta = (-3)^2 - 4(1)(-4) = 9 + 16 = 25$$

Since $\Delta > 0$, there are two distinct real roots:

$$\lambda = \frac{-(-3) \pm \sqrt{25}}{2(1)} = \frac{3 \pm 5}{2}$$

$$\lambda_1 = \frac{3+5}{2} = 4 \quad \text{and} \quad \lambda_2 = \frac{3-5}{2} = -1$$

So, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.

Ex 4: Find the eigenvalues of the matrix $\mathbf{D} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$.

Answer: We solve the characteristic equation $\det(\mathbf{D} - \lambda \mathbf{I}) = 0$:

$$\begin{aligned} \det \begin{pmatrix} 3-\lambda & -2 \\ -1 & 2-\lambda \end{pmatrix} &= 0 \\ (3-\lambda)(2-\lambda) - (-2)(-1) &= 0 \\ (6 - 3\lambda - 2\lambda + \lambda^2) - 2 &= 0 \\ \lambda^2 - 5\lambda + 4 &= 0 \end{aligned}$$

We solve using the discriminant Δ with $a = 1$, $b = -5$, and $c = 4$:

$$\Delta = (-5)^2 - 4(1)(4) = 25 - 16 = 9$$

The roots are:

$$\lambda = \frac{-(-5) \pm \sqrt{9}}{2(1)} = \frac{5 \pm 3}{2}$$

$$\lambda_1 = \frac{5+3}{2} = 4 \quad \text{and} \quad \lambda_2 = \frac{5-3}{2} = 1$$

So, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 1$.

Ex 5: Find the eigenvalues of the triangular matrix $\mathbf{E} = \begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}$.

Answer: We solve the characteristic equation $\det(\mathbf{E} - \lambda \mathbf{I}) = 0$:

$$\begin{aligned} \det \begin{pmatrix} 5-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix} &= 0 \\ (5-\lambda)(-3-\lambda) - (2)(0) &= 0 \\ (5-\lambda)(-3-\lambda) - 0 &= 0 \\ (5-\lambda)(-3-\lambda) &= 0 \end{aligned}$$

Since the equation is already in factored form, we can read the roots directly without calculating the discriminant:

$$5 - \lambda = 0 \implies \lambda = 5$$

$$-3 - \lambda = 0 \implies \lambda = -3$$

So, the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -3$.
Note: For a triangular matrix, the eigenvalues are the diagonal elements.

A.2 FINDING AN EIGENVALUE FROM AN EIGENVECTOR

Ex 6: Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ and the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1. Calculate the product $\mathbf{Ax}$.
2. Hence, determine the eigenvalue λ associated with the eigenvector $\mathbf{x}$.

Answer:

1. We calculate the matrix multiplication:
$$\begin{aligned}\mathbf{Ax} &= \begin{pmatrix}1 & 1\\4 & 1\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\\ &= \begin{pmatrix}1(1)+1(2)\\4(1)+1(2)\end{pmatrix}\\ &= \begin{pmatrix}3\\6\end{pmatrix}\end{aligned}$$
2. We look for a scalar λ such that $\mathbf{Ax} = \lambda\mathbf{x}$.
$$\begin{aligned}\begin{pmatrix}3\\6\end{pmatrix} &= \lambda\begin{pmatrix}1\\2\end{pmatrix}\\ 3\begin{pmatrix}1\\2\end{pmatrix} &= \lambda\begin{pmatrix}1\\2\end{pmatrix}\end{aligned}$$
Therefore, the eigenvalue is $\lambda = 3$.

Ex 7: Consider the matrix $\mathbf{B} = \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix}$ and the vector $\mathbf{u} = \begin{pmatrix}1\\1\end{pmatrix}$.

1. Calculate the product $\mathbf{Bu}$.
2. Hence, determine the eigenvalue λ associated with the eigenvector $\mathbf{u}$.

Answer:

1. We calculate the matrix multiplication:
$$\begin{aligned}\mathbf{Bu} &= \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}3(1)+(-1)(1)\\(-1)(1)+3(1)\end{pmatrix}\\ &= \begin{pmatrix}2\\2\end{pmatrix}\end{aligned}$$
2. We look for a scalar λ such that $\mathbf{Bu} = \lambda\mathbf{u}$.
$$\begin{aligned}\begin{pmatrix}2\\2\end{pmatrix} &= \lambda\begin{pmatrix}1\\1\end{pmatrix}\\ 2\begin{pmatrix}1\\1\end{pmatrix} &= \lambda\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}$$
Therefore, the eigenvalue is $\lambda = 2$.

Ex 8: Consider the matrix $\mathbf{C} = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}$ and the vector $\mathbf{v} = \begin{pmatrix}1\\3\end{pmatrix}$.

1. Calculate the product $\mathbf{Cv}$.
2. Hence, determine the eigenvalue λ associated with the eigenvector $\mathbf{v}$.

Answer:

1. We calculate the matrix multiplication:
$$\begin{aligned}\mathbf{Cv} &= \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}\\ &= \begin{pmatrix}2(1)+(-1)(3)\\3(1)+(-2)(3)\end{pmatrix}\\ &= \begin{pmatrix}-1\\-3\end{pmatrix}\end{aligned}$$
2. We look for a scalar λ such that $\mathbf{Cv} = \lambda\mathbf{v}$.
$$\begin{aligned}\begin{pmatrix}-1\\-3\end{pmatrix} &= \lambda\begin{pmatrix}1\\3\end{pmatrix}\\ -1\begin{pmatrix}1\\3\end{pmatrix} &= \lambda\begin{pmatrix}1\\3\end{pmatrix}\end{aligned}$$
Therefore, the eigenvalue is $\lambda = -1$.

A.3 CALCULATING EIGENVECTORS

Ex 9: Consider the matrix $\mathbf{A} = \begin{pmatrix}3 & 2\\2 & 0\end{pmatrix}$.
One of the eigenvalues of this matrix is $\lambda = 4$.
Find the eigenvector $\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix}$ corresponding to this eigenvalue.

Answer: **Find an eigenvector for $\lambda = 4$:**
$$\begin{aligned}(\mathbf{A} - 4\mathbf{I})\mathbf{x} &= \mathbf{0}\\ \left(\begin{pmatrix}3 & 2\\2 & 0\end{pmatrix} - 4\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}3-4 & 2\\2 & 0-4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}-1 & 2\\2 & -4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}-x+2y\\2x-4y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$
This gives $-x + 2y = 0$ and $2x - 4y = 0$, so $x = 2y$.
Letting $y = t$, we have $x = 2t$, so
$$\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}2t\\t\end{pmatrix} = t\begin{pmatrix}2\\1\end{pmatrix}, \quad t \neq 0.$$
Any vector of the form $t\begin{pmatrix}2\\1\end{pmatrix}$, $t \neq 0$, is an eigenvector corresponding to the eigenvalue 4.

Ex 10: Consider the matrix $\mathbf{B} = \begin{pmatrix}4 & 1\\2 & 3\end{pmatrix}$.
One of the eigenvalues of this matrix is $\lambda = 5$.
Find the eigenvector $\mathbf{x} = \begin{pmatrix}x\\y\end{pmatrix}$ corresponding to this eigenvalue.

Answer: **Find an eigenvector for $\lambda = 5$:**
$$\begin{aligned}(\mathbf{B} - 5\mathbf{I})\mathbf{x} &= \mathbf{0}\\ \left(\begin{pmatrix}4 & 1\\2 & 3\end{pmatrix} - 5\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}4-5 & 1\\2 & 3-5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}-1 & 1\\2 & -2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}-x+y\\2x-2y\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$

This gives $-x+y=0$ and $2x-2y=0$, so $y=x$.
Letting $x=t$, we have $y=t$, so

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}t\\t\end{pmatrix}=t\begin{pmatrix}1\\1\end{pmatrix},\quad t\neq 0.$$

Any vector of the form $t\begin{pmatrix}1\\1\end{pmatrix}$, $t\neq 0$, is an eigenvector corresponding to the eigenvalue 5.

Ex 11: Consider the matrix $\mathbf{C}=\begin{pmatrix}5 & -1\\2 & 2\end{pmatrix}$.
One of the eigenvalues of this matrix is $\lambda=3$.
Find the eigenvector $\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}$ corresponding to this eigenvalue.

Answer: **Find an eigenvector for $\lambda=3$:**

$$(\mathbf{C}-3\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\left(\begin{pmatrix}5 & -1\\2 & 2\end{pmatrix}-3\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}5-3 & -1\\2 & 2-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}2 & -1\\2 & -1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}2x-y\\2x-y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $2x-y=0$, so $y=2x$.
Letting $x=t$, we have $y=2t$, so

$$\mathbf{x}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}t\\2t\end{pmatrix}=t\begin{pmatrix}1\\2\end{pmatrix},\quad t\neq 0.$$

Any vector of the form $t\begin{pmatrix}1\\2\end{pmatrix}$, $t\neq 0$, is an eigenvector corresponding to the eigenvalue 3.

A.4 FINDING EIGENVALUES AND EIGENVECTORS

Ex 12: Consider the matrix $\mathbf{A}=\begin{pmatrix}7 & -4\\8 & -5\end{pmatrix}$.

1. Find the eigenvalues λ_1 and λ_2 of matrix $\mathbf{A}$.
2. Find the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Answer:

1. **Find Eigenvalues:**
We solve the characteristic equation $\det(\mathbf{A}-\lambda\mathbf{I})=0$:

$$\det\begin{pmatrix}7-\lambda & -4\\8 & -5-\lambda\end{pmatrix}=0$$
$$(7-\lambda)(-5-\lambda)-(-4)(8)=0$$
$$(-35-7\lambda+5\lambda+\lambda^2)+32=0$$
$$\lambda^2-2\lambda-3=0$$

We solve the quadratic equation using the discriminant $\Delta=b^2-4ac$ with $a=1$, $b=-2$, and $c=-3$:

$$\Delta=(-2)^2-4(1)(-3)=4+12=16$$

Since $\Delta>0$, there are two distinct real roots:

$$\lambda=\frac{-(-2)\pm\sqrt{16}}{2(1)}=\frac{2\pm 4}{2}$$
$$\lambda_1=\frac{2+4}{2}=3\quad\text{and}\quad\lambda_2=\frac{2-4}{2}=-1$$

So, the eigenvalues are $\lambda_1=3$ and $\lambda_2=-1$.

2. **Find Eigenvectors:**
   - **For $\lambda_1=3$:**

$$(\mathbf{A}-3\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\left(\begin{pmatrix}7 & -4\\8 & -5\end{pmatrix}-3\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}7-3 & -4\\8 & -5-3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}4 & -4\\8 & -8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}4x-4y\\8x-8y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $4x-4y=0$, so $y=x$.
Letting $x=t$, we have $y=t$, so

$$\mathbf{x}_1=\begin{pmatrix}t\\t\end{pmatrix}=t\begin{pmatrix}1\\1\end{pmatrix},\quad t\neq 0.$$

   - **For $\lambda_2=-1$:**

$$(\mathbf{A}-(-1)\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\left(\begin{pmatrix}7 & -4\\8 & -5\end{pmatrix}+1\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\right)\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}8 & -4\\8 & -4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}8x-4y\\8x-4y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $8x-4y=0$, or $2x=y$.
Letting $x=t$, we have $y=2t$, so

$$\mathbf{x}_2=\begin{pmatrix}t\\2t\end{pmatrix}=t\begin{pmatrix}1\\2\end{pmatrix},\quad t\neq 0.$$

Ex 13: Consider the matrix $\mathbf{B}=\begin{pmatrix}4 & 1\\1 & 4\end{pmatrix}$.

1. Find the eigenvalues of matrix $\mathbf{B}$.
2. Find the corresponding eigenvectors.

Answer:

1. **Find Eigenvalues:**
We solve the characteristic equation $\det(\mathbf{B}-\lambda\mathbf{I})=0$:

$$\det\begin{pmatrix}4-\lambda & 1\\1 & 4-\lambda\end{pmatrix}=0$$
$$(4-\lambda)(4-\lambda)-(1)(1)=0$$
$$(16-8\lambda+\lambda^2)-1=0$$
$$\lambda^2-8\lambda+15=0$$

We solve the quadratic equation using the discriminant $\Delta=b^2-4ac$ with $a=1$, $b=-8$, and $c=15$:

$$\Delta=(-8)^2-4(1)(15)=64-60=4$$

Since $\Delta>0$, there are two distinct real roots:

$$\lambda=\frac{-(-8)\pm\sqrt{4}}{2(1)}=\frac{8\pm 2}{2}$$
$$\lambda_1=\frac{8+2}{2}=5\quad\text{and}\quad\lambda_2=\frac{8-2}{2}=3$$

So, the eigenvalues are $\lambda_1=5$ and $\lambda_2=3$.

2. **Find Eigenvectors:**

- **For $\lambda_1 = 5$:**

$$(\mathbf{B} - 5\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\left(\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 4-5 & 1 \\ 1 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} -x+y \\ x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $-x+y=0$ and $x-y=0$, so $y=x$.
Letting $x=t$, we have $y=t$, so

$$\mathbf{x}_1 = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \neq 0.$$

- **For $\lambda_2 = 3$:**

$$(\mathbf{B} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\left(\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 4-3 & 1 \\ 1 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $x+y=0$, so $y=-x$.
Letting $x=t$, we have $y=-t$, so

$$\mathbf{x}_2 = \begin{pmatrix} t \\ -t \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad t \neq 0.$$

Ex 14: Consider the matrix $\mathbf{C} = \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$.

1. Find the eigenvalues of matrix $\mathbf{C}$.
2. Find the corresponding eigenvectors.

Answer:

1. **Find Eigenvalues:**
We solve the characteristic equation $\det(\mathbf{C} - \lambda\mathbf{I}) = 0$:

$$\det\begin{pmatrix} 5-\lambda & -1 \\ 2 & 2-\lambda \end{pmatrix} = 0$$
$$(5-\lambda)(2-\lambda) - (-1)(2) = 0$$
$$(10 - 5\lambda - 2\lambda + \lambda^2) + 2 = 0$$
$$\lambda^2 - 7\lambda + 12 = 0$$

We solve the quadratic equation. The discriminant is $\Delta = (-7)^2 - 4(1)(12) = 49 - 48 = 1$.

$$\lambda = \frac{-(-7) \pm \sqrt{1}}{2} = \frac{7 \pm 1}{2}$$

$$\lambda_1 = \frac{8}{2} = 4 \quad \text{and} \quad \lambda_2 = \frac{6}{2} = 3$$

So, the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 3$.

2. **Find Eigenvectors:**

- **For $\lambda_1 = 4$:**

$$(\mathbf{C} - 4\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\left(\begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x-y \\ 2x-2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $x-y=0$, so $x=y$.
Letting $x=t$, we have $y=t$, so

$$\mathbf{x}_1 = \begin{pmatrix} t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \neq 0.$$

- **For $\lambda_2 = 3$:**

$$(\mathbf{C} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\left(\begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2x-y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $2x-y=0$, so $y=2x$.
Letting $x=t$, we have $y=2t$, so

$$\mathbf{x}_2 = \begin{pmatrix} t \\ 2t \end{pmatrix} = t\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad t \neq 0.$$

B MATRIX DIAGONALISATION

B.1 VERIFYING MATRIX DIAGONALISATION

Ex 15: The matrix

$$\mathbf{A} = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

Answer: To show that $\mathbf{P}$ diagonalises $\mathbf{A}$, we must show that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

1. **Form the matrix P:**

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

2. **Find the inverse $\mathbf{P}^{-1}$:**

$$\det(\mathbf{P}) = (1)(2) - (1)(1) = 1,$$

so

$$\mathbf{P}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

3. **Calculate $\mathbf{P}^{-1}\mathbf{AP}$:**

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{AP} &= \begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}7 & -4\\ 8 & -5\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\\ &= \begin{pmatrix}2 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}3 & -1\\ 3 & -2\end{pmatrix}\\ &= \begin{pmatrix}6-3 & -2+2\\ -3+3 & 1-2\end{pmatrix}\\ &= \begin{pmatrix}3 & 0\\ 0 & -1\end{pmatrix}.\end{aligned}$$

The result is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix}3 & 0\\ 0 & -1\end{pmatrix},$$

which contains the eigenvalues on the diagonal. Thus, $\mathbf{P}$ diagonalises $\mathbf{A}$.

Ex 16: The matrix

$$\mathbf{A} = \begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix}$$

has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix}1\\ 1\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix}1\\ -2\end{pmatrix}.$$

Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

Answer: To show that $\mathbf{P}$ diagonalises $\mathbf{A}$, we must show that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

1. **Form the matrix $\mathbf{P}$:**

$$\mathbf{P} = \begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix}.$$

2. **Find the inverse $\mathbf{P}^{-1}$:**

$$\det(\mathbf{P}) = (1)(-2) - (1)(1) = -3,$$

so

$$\mathbf{P}^{-1} = \frac{1}{-3}\begin{pmatrix}-2 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}.$$

3. **Calculate $\mathbf{P}^{-1}\mathbf{AP}$:**

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{AP} &= \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}3 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -2\end{pmatrix}\\ &= \frac{1}{3}\begin{pmatrix}2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}4 & 1\\ 4 & -2\end{pmatrix}\\ &= \frac{1}{3}\begin{pmatrix}12 & 0\\ 0 & 3\end{pmatrix}\\ &= \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

The result is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},$$

which contains the eigenvalues on the diagonal. Thus, $\mathbf{P}$ diagonalises $\mathbf{A}$.

Ex 17: The matrix

$$\mathbf{A} = \begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}$$

has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ with corresponding eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix}2\\ 1\end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

Answer: To show that $\mathbf{P}$ diagonalises $\mathbf{A}$, we must show that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix.

1. **Form the matrix $\mathbf{P}$:**

$$\mathbf{P} = \begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}.$$

2. **Find the inverse $\mathbf{P}^{-1}$:**

$$\det(\mathbf{P}) = (2)(1) - (1)(1) = 1,$$

so

$$\mathbf{P}^{-1} = \frac{1}{1}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix} = \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}.$$

3. **Calculate $\mathbf{P}^{-1}\mathbf{AP}$:**

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{AP} &= \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}3 & -2\\ 1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}\begin{pmatrix}4 & 1\\ 2 & 1\end{pmatrix}\\ &= \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

The result is the diagonal matrix

$$\mathbf{D} = \begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix},$$

which contains the eigenvalues on the diagonal. Thus, $\mathbf{P}$ diagonalises $\mathbf{A}$.

B.2 PERFORMING FULL MATRIX DIAGONALISATION

Ex 18: Consider the matrix $\mathbf{A} = \begin{pmatrix}4 & 3\\ 2 & -1\end{pmatrix}$.
Find the diagonal matrix $\mathbf{D}$, the matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.

Answer:

1. **Find Eigenvalues:**
   We solve the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$:

$$\begin{aligned}\det\begin{pmatrix}4-\lambda & 3\\ 2 & -1-\lambda\end{pmatrix} &= 0\\ (4-\lambda)(-1-\lambda) - (3)(2) &= 0\\ (-4 - 4\lambda + \lambda + \lambda^2) - 6 &= 0\\ \lambda^2 - 3\lambda - 10 &= 0\\ (\lambda - 5)(\lambda + 2) &= 0\end{aligned}$$

   The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = -2$.

2. **Find Eigenvectors:**
   - **For $\lambda_1 = 5$:**

$$\begin{aligned}(\mathbf{A} - 5\mathbf{I})\mathbf{x} &= \mathbf{0}\\ \begin{pmatrix}4-5 & 3\\ 2 & -1-5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\\ \begin{pmatrix}-1 & 3\\ 2 & -6\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} &= \begin{pmatrix}0\\ 0\end{pmatrix}\end{aligned}$$

This gives $-x+3y=0$, which simplifies to $x=3y$. Letting $y=t$, we get $x=3t$.

$$\mathbf{x}=\begin{pmatrix}3t\\t\end{pmatrix}=t\begin{pmatrix}3\\1\end{pmatrix},\quad t\neq 0$$

We choose $t=1$ to get $\mathbf{x}_1=\begin{pmatrix}3\\1\end{pmatrix}$.

- **For $\lambda_2=-2$:**

$$(\mathbf{A}-(-2)\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\begin{pmatrix}4+2 & 3\\2 & -1+2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}6 & 3\\2 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $2x+y=0$, which simplifies to $y=-2x$. Letting $x=t$, we get $y=-2t$.

$$\mathbf{x}=\begin{pmatrix}t\\-2t\end{pmatrix}=t\begin{pmatrix}1\\-2\end{pmatrix},\quad t\neq 0$$

We choose $t=1$ to get $\mathbf{x}_2=\begin{pmatrix}1\\-2\end{pmatrix}$.

3. **Construct Matrices:**

$$\mathbf{D}=\begin{pmatrix}5 & 0\\0 & -2\end{pmatrix}\quad\text{and}\quad\mathbf{P}=\begin{pmatrix}3 & 1\\1 & -2\end{pmatrix}$$

Find $\mathbf{P}^{-1}$:

$$\det(\mathbf{P})=(3)(-2)-(1)(1)=-7$$
$$\mathbf{P}^{-1}=\frac{1}{-7}\begin{pmatrix}-2 & -1\\-1 & 3\end{pmatrix}=\frac{1}{7}\begin{pmatrix}2 & 1\\1 & -3\end{pmatrix}$$

Ex 19: Consider the matrix $\mathbf{B}=\begin{pmatrix}9 & -10\\5 & -6\end{pmatrix}$.

Find the diagonal matrix $\mathbf{D}$, the matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ such that $\mathbf{B}=\mathbf{PDP}^{-1}$.

Answer:

1. **Find Eigenvalues:**
We solve the characteristic equation $\det(\mathbf{B}-\lambda\mathbf{I})=0$:

$$\det\begin{pmatrix}9-\lambda & -10\\5 & -6-\lambda\end{pmatrix}=0$$
$$(9-\lambda)(-6-\lambda)-(-10)(5)=0$$
$$(-54-9\lambda+6\lambda+\lambda^2)+50=0$$
$$\lambda^2-3\lambda-4=0$$
$$(\lambda-4)(\lambda+1)=0$$

The eigenvalues are $\lambda_1=4$ and $\lambda_2=-1$.

2. **Find Eigenvectors:**

- **For $\lambda_1=4$:**

$$(\mathbf{B}-4\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\begin{pmatrix}5 & -10\\5 & -10\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $5x-10y=0$, which simplifies to $x=2y$. Letting $y=t$, we get $x=2t$.

$$\mathbf{x}=\begin{pmatrix}2t\\t\end{pmatrix}=t\begin{pmatrix}2\\1\end{pmatrix},\quad t\neq 0$$

We choose $t=1$ to get $\mathbf{x}_1=\begin{pmatrix}2\\1\end{pmatrix}$.

- **For $\lambda_2=-1$:**

$$(\mathbf{B}-(-1)\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\begin{pmatrix}10 & -10\\5 & -5\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $5x-5y=0$, which simplifies to $x=y$. Letting $y=t$, we get $x=t$.

$$\mathbf{x}=\begin{pmatrix}t\\t\end{pmatrix}=t\begin{pmatrix}1\\1\end{pmatrix},\quad t\neq 0$$

We choose $t=1$ to get $\mathbf{x}_2=\begin{pmatrix}1\\1\end{pmatrix}$.

3. **Construct Matrices:**

$$\mathbf{D}=\begin{pmatrix}4 & 0\\0 & -1\end{pmatrix}\quad\text{and}\quad\mathbf{P}=\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$$

Find $\mathbf{P}^{-1}$:

$$\det(\mathbf{P})=(2)(1)-(1)(1)=1$$
$$\mathbf{P}^{-1}=\frac{1}{1}\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}=\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix}$$

Ex 20: Consider the matrix $\mathbf{C}=\begin{pmatrix}3 & 2\\3 & -2\end{pmatrix}$.

Find the diagonal matrix $\mathbf{D}$, the matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ such that $\mathbf{C}=\mathbf{PDP}^{-1}$.

Answer:

1. **Find Eigenvalues:**
We solve the characteristic equation $\det(\mathbf{C}-\lambda\mathbf{I})=0$:

$$\det\begin{pmatrix}3-\lambda & 2\\3 & -2-\lambda\end{pmatrix}=0$$
$$(3-\lambda)(-2-\lambda)-(2)(3)=0$$
$$(-6-3\lambda+2\lambda+\lambda^2)-6=0$$
$$\lambda^2-\lambda-12=0$$
$$(\lambda-4)(\lambda+3)=0$$

The eigenvalues are $\lambda_1=4$ and $\lambda_2=-3$.

2. **Find Eigenvectors:**

- **For $\lambda_1=4$:**

$$(\mathbf{C}-4\mathbf{I})\mathbf{x}=\mathbf{0}$$
$$\begin{pmatrix}3-4 & 2\\3 & -2-4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$
$$\begin{pmatrix}-1 & 2\\3 & -6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

This gives $-x+2y=0$, which simplifies to $x=2y$. Letting $y=t$, we get $x=2t$.

$$\mathbf{x}=\begin{pmatrix}2t\\t\end{pmatrix}=t\begin{pmatrix}2\\1\end{pmatrix},\quad t\neq 0$$

We choose $t=1$ to get $\mathbf{x}_1=\begin{pmatrix}2\\1\end{pmatrix}$.

- **For $\lambda_2 = -3$:**

$$(\mathbf{C} - (-3)\mathbf{I})\mathbf{x} = \mathbf{0}$$
$$\begin{pmatrix} 3+3 & 2 \\ 3 & -2+3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives $3x + y = 0$, which simplifies to $y = -3x$. Letting $x = t$, we get $y = -3t$.

$$\mathbf{x} = \begin{pmatrix} t \\ -3t \end{pmatrix} = t\begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad t \neq 0$$

We choose $t = 1$ to get $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$.

3. **Construct Matrices:**

$$\mathbf{D} = \begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{and} \quad \mathbf{P} = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$$

Find $\mathbf{P}^{-1}$:

$$\det(\mathbf{P}) = (2)(-3) - (1)(1) = -7$$
$$\mathbf{P}^{-1} = \frac{1}{-7}\begin{pmatrix} -3 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$$

C MATRIX POWERS

C.1 CALCULATING MATRIX POWERS USING DIAGONALISATION

Ex 21: The matrix
$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$
diagonalises
$$\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 10 & -7 \end{pmatrix}$$
with
$$\mathbf{P}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$
Calculate the matrix $\mathbf{A}^4$.

Answer:

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{P}\mathbf{D}^4\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3^4 & 0 \\ 0 & (-2)^4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 81 & 0 \\ 0 & 16 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1(81) & 1(16) \\ 1(81) & 2(16) \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 81 & 16 \\ 81 & 32 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 81(2)+16(-1) & 81(-1)+16(1) \\ 81(2)+32(-1) & 81(-1)+32(1) \end{pmatrix} \\ &= \begin{pmatrix} 162-16 & -81+16 \\ 162-32 & -81+32 \end{pmatrix} \\ &= \begin{pmatrix} 146 & -65 \\ 130 & -49 \end{pmatrix}.\end{aligned}$$

Ex 22: The matrix
$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$
diagonalises
$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$
with
$$\mathbf{P}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$
Calculate the matrix $\mathbf{A}^6$.

Answer:

$$\begin{aligned}\mathbf{A}^6 &= \mathbf{P}\mathbf{D}^6\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2^6 & 0 \\ 0 & 1^6 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 64 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1(64) & 2(1) \\ 1(64) & 1(1) \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 64 & 2 \\ 64 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 64(-1)+2(1) & 64(2)+2(-1) \\ 64(-1)+1(1) & 64(2)+1(-1) \end{pmatrix} \\ &= \begin{pmatrix} -64+2 & 128-2 \\ -64+1 & 128-1 \end{pmatrix} \\ &= \begin{pmatrix} -62 & 126 \\ -63 & 127 \end{pmatrix}.\end{aligned}$$

Ex 23: The matrix
$$\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$
diagonalises
$$\mathbf{A} = \begin{pmatrix} -9 & 12 \\ -8 & 11 \end{pmatrix}$$
with
$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$
Calculate the matrix $\mathbf{A}^3$.

Answer:

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{P}\mathbf{D}^3\mathbf{P}^{-1} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} (-1)^3 & 0 \\ 0 & 3^3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 27 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3(-1) & 1(27) \\ 2(-1) & 1(27) \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 27 \\ -2 & 27 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -3(1)+27(-2) & -3(-1)+27(3) \\ -2(1)+27(-2) & -2(-1)+27(3) \end{pmatrix} \\ &= \begin{pmatrix} -3-54 & 3+81 \\ -2-54 & 2+81 \end{pmatrix} \\ &= \begin{pmatrix} -57 & 84 \\ -56 & 83 \end{pmatrix}.\end{aligned}$$

C.2 CALCULATING MATRIX POWERS USING DIAGONALISATION

Ex 24: Consider the matrix $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$.

1. Find the eigenvalues λ_1 and λ_2 of matrix $\mathbf{A}$.
2. Find the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
3. Determine the matrices $\mathbf{D}$, $\mathbf{P}$, and $\mathbf{P}^{-1}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.
4. Hence, calculate the matrix $\mathbf{A}^6$.

Answer:

1. **Find Eigenvalues:**

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= 0 \\ (5-\lambda)(-1-\lambda) - (-2)(4) &= 0 \\ -5 - 5\lambda + \lambda + \lambda^2 + 8 &= 0 \\ \lambda^2 - 4\lambda + 3 &= 0 \\ (\lambda - 3)(\lambda - 1) &= 0 \end{aligned}$$

The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

2. **Find Eigenvectors:**

- **For $\lambda_1 = 3$:**

$$\begin{pmatrix} 5-3 & -2 \\ 4 & -1-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$2x - 2y = 0 \implies x = y$. Let $x = 1$, then $y = 1$.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- **For $\lambda_2 = 1$:**

$$\begin{pmatrix} 5-1 & -2 \\ 4 & -1-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{pmatrix} 4 & -2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$4x - 2y = 0 \implies y = 2x$. Let $x = 1$, then $y = 2$.

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

3. **Find D, P, and $\mathbf{P}^{-1}$:**

$$\mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Calculate $\mathbf{P}^{-1}$:

$$\det(\mathbf{P}) = (1)(2) - (1)(1) = 1$$

$$\mathbf{P}^{-1} = \frac{1}{1}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

4. **Calculate $\mathbf{A}^6$:**

$$\begin{aligned} \mathbf{A}^6 &= \mathbf{PD}^6\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3^6 & 0 \\ 0 & 1^6 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 729 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 729 & 1 \\ 729 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1458-1 & -729+1 \\ 1458-2 & -729+2 \end{pmatrix} \\ &= \begin{pmatrix} 1457 & -728 \\ 1456 & -727 \end{pmatrix} \end{aligned}$$

Ex 25: Consider the matrix $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$.

1. Find the eigenvalues λ_1 and λ_2 of matrix $\mathbf{B}$.
2. Find the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
3. Determine the matrices $\mathbf{D}$, $\mathbf{P}$, and $\mathbf{P}^{-1}$ such that $\mathbf{B} = \mathbf{PDP}^{-1}$.
4. Hence, calculate the matrix $\mathbf{B}^4$.

Answer:

1. **Find Eigenvalues:**

$$\begin{aligned} \det(\mathbf{B} - \lambda\mathbf{I}) &= 0 \\ (4-\lambda)(4-\lambda) - (1)(1) &= 0 \\ 16 - 8\lambda + \lambda^2 - 1 &= 0 \\ \lambda^2 - 8\lambda + 15 &= 0 \\ (\lambda - 5)(\lambda - 3) &= 0 \end{aligned}$$

The eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 3$.

2. **Find Eigenvectors:**

- **For $\lambda_1 = 5$:**

$$\begin{pmatrix} 4-5 & 1 \\ 1 & 4-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$-x + y = 0 \implies y = x$. Let $x = 1$, then $y = 1$.

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

- **For $\lambda_2 = 3$:**

$$\begin{pmatrix} 4-3 & 1 \\ 1 & 4-3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0} \implies \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

$x + y = 0 \implies y = -x$. Let $x = 1$, then $y = -1$.

$$\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

3. **Find D, P, and $\mathbf{P}^{-1}$:**

$$\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 3 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Calculate $\mathbf{P}^{-1}$:

$$\det(\mathbf{P}) = (1)(-1) - (1)(1) = -2$$

$$\mathbf{P}^{-1} = \frac{1}{-2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

4. **Calculate $\mathbf{B}^4$:**

$$\begin{aligned} \mathbf{B}^4 &= \mathbf{PD}^4\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 5^4 & 0 \\ 0 & 3^4 \end{pmatrix}\left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 625 & 0 \\ 0 & 81 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 625 & 81 \\ 625 & -81 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 625+81 & 625-81 \\ 625-81 & 625+81 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 706 & 544 \\ 544 & 706 \end{pmatrix} \\ &= \begin{pmatrix} 353 & 272 \\ 272 & 353 \end{pmatrix} \end{aligned}$$