MATRIX DIAGONALISATION

A EIGENVALUES AND EIGENVECTORS

A.1 CALCULATING EIGENVALUES

Ex 1: Find the eigenvalues of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Ex 2: Find the eigenvalues of the matrix $\mathbf{B} = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$.

Ex 3: Find the eigenvalues of the matrix $\mathbf{C} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Ex 4: Find the eigenvalues of the matrix $\mathbf{D} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$.

Ex 5: Find the eigenvalues of the triangular matrix $\mathbf{E} = \begin{pmatrix} 5 & 2 \\ 0 & -3 \end{pmatrix}$.

A.2 FINDING AN EIGENVALUE FROM AN EIGENVECTOR

Ex 6: Consider the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ and the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

1. Calculate the product $\mathbf{Ax}$.
2. Hence, determine the eigenvalue λ associated with the eigenvector $\mathbf{x}$.

Ex 7: Consider the matrix $\mathbf{B} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$ and the vector $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

1. Calculate the product $\mathbf{Bu}$.
2. Hence, determine the eigenvalue λ associated with the eigenvector $\mathbf{u}$.

Ex 8: Consider the matrix $\mathbf{C} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$ and the vector $\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

1. Calculate the product $\mathbf{Cv}$.
2. Hence, determine the eigenvalue λ associated with the eigenvector $\mathbf{v}$.

A.3 CALCULATING EIGENVECTORS

Ex 9: Consider the matrix $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ 2 & 0 \end{pmatrix}$.

One of the eigenvalues of this matrix is $\lambda = 4$.

Find the eigenvector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to this eigenvalue.

Ex 10: Consider the matrix $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.
One of the eigenvalues of this matrix is $\lambda = 5$.
Find the eigenvector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to this eigenvalue.

Ex 11: Consider the matrix $\mathbf{C} = \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$.
One of the eigenvalues of this matrix is $\lambda = 3$.
Find the eigenvector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ corresponding to this eigenvalue.

A.4 FINDING EIGENVALUES AND EIGENVECTORS

Ex 12: Consider the matrix $\mathbf{A} = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}$.

1. Find the eigenvalues λ_1 and λ_2 of matrix $\mathbf{A}$.
2. Find the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

Ex 13: Consider the matrix $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$.

1. Find the eigenvalues of matrix $\mathbf{B}$.
2. Find the corresponding eigenvectors.

Ex 14: Consider the matrix $\mathbf{C} = \begin{pmatrix} 5 & -1 \\ 2 & 2 \end{pmatrix}$.

1. Find the eigenvalues of matrix $\mathbf{C}$.
2. Find the corresponding eigenvectors.

B MATRIX DIAGONALISATION

B.1 VERIFYING MATRIX DIAGONALISATION

Ex 15: The matrix
$$\mathbf{A} = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}$$
has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$ with corresponding eigenvectors
$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$
Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

Ex 16: The matrix
$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$$
has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 1$ with corresponding eigenvectors
$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$
Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

Ex 17: The matrix
$$\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$$
has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 1$ with corresponding eigenvectors
$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
Show that $\mathbf{P} = (\mathbf{x}_1 \mid \mathbf{x}_2)$ diagonalises $\mathbf{A}$.

B.2 PERFORMING FULL MATRIX DIAGONALISATION

Ex 18: Consider the matrix $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 2 & -1 \end{pmatrix}$.

Find the diagonal matrix $\mathbf{D}$, the matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ such that $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

Ex 19: Consider the matrix $\mathbf{B} = \begin{pmatrix} 9 & -10 \\ 5 & -6 \end{pmatrix}$.

Find the diagonal matrix $\mathbf{D}$, the matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ such that $\mathbf{B} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

Ex 20: Consider the matrix $\mathbf{C} = \begin{pmatrix} 3 & 2 \\ 3 & -2 \end{pmatrix}$.

Find the diagonal matrix $\mathbf{D}$, the matrix $\mathbf{P}$ and its inverse $\mathbf{P}^{-1}$ such that $\mathbf{C} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$.

C MATRIX POWERS

C.1 CALCULATING MATRIX POWERS USING DIAGONALISATION

Ex 21: The matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

diagonalises

$$\mathbf{A} = \begin{pmatrix} 8 & -5 \\ 10 & -7 \end{pmatrix}$$

with

$$\mathbf{P}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

Calculate the matrix $\mathbf{A}^4$.

Ex 22: The matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

diagonalises

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix}$$

with

$$\mathbf{P}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Calculate the matrix $\mathbf{A}^6$.

Ex 23: The matrix

$$\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

diagonalises

$$\mathbf{A} = \begin{pmatrix} -9 & 12 \\ -8 & 11 \end{pmatrix}$$

with

$$\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{D} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

Calculate the matrix $\mathbf{A}^3$.

C.2 CALCULATING MATRIX POWERS USING DIAGONALISATION

Ex 24: Consider the matrix $\mathbf{A} = \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}$.

1. Find the eigenvalues λ_1 and λ_2 of matrix $\mathbf{A}$.
2. Find the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
3. Determine the matrices $\mathbf{D}$, $\mathbf{P}$, and $\mathbf{P}^{-1}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$.
4. Hence, calculate the matrix $\mathbf{A}^6$.

Ex 25: Consider the matrix $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$.

1. Find the eigenvalues λ_1 and λ_2 of matrix $\mathbf{B}$.
2. Find the corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$.
3. Determine the matrices $\mathbf{D}$, $\mathbf{P}$, and $\mathbf{P}^{-1}$ such that $\mathbf{B} = \mathbf{PDP}^{-1}$.
4. Hence, calculate the matrix $\mathbf{B}^4$.