MATRICES

Matrices are rectangular arrays of numbers that provide a powerful tool for solving systems of linear equations and representing linear transformations. Just as complex numbers extend the real number line, matrices can be seen as an extension of the concept of number to higher dimensions. This chapter introduces the algebraic foundations of matrices, their operations, and their fundamental role in various mathematical and scientific applications. Unless stated otherwise, matrix entries are real numbers.

A STRUCTURE

A.1 DEFINITION

Definition Matrix

A **matrix** $\mathbf{A}$ of size $n \times p$ is a rectangular array of numbers with n rows and p columns:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}.$$

a_{ij} is called the **entry** or **element** of $\mathbf{A}$ at position (i, j) (in the i-th row and j-th column).

Ex: $\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 2 & 6 \end{pmatrix}$ has 2 rows and 3 columns, so its size is 2×3.
a_{13} is equal to 5. It is the entry in row 1 and column 3.

$$\begin{array}{cc} & \begin{array}{ccc} & & \text{column } 3 \end{array} \\ \text{row } 1 & \begin{pmatrix} 1 & 3 & 5 \\ 7 & 2 & 6 \end{pmatrix} \end{array}$$

A.2 SPECIAL MATRICES

Definition Column Matrix

A matrix with one column, $\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$, is called a **column matrix** or **column vector**.

Ex: $C = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ has 3 rows and 1 column. C is a column matrix of size 3×1.

Definition Row Matrix

A matrix with one row, $\begin{pmatrix} r_1 & r_2 & \ldots & r_p \end{pmatrix}$, is called a **row matrix** or **row vector**.

Ex: $R = \begin{pmatrix} 1 & 3 & -1 \end{pmatrix}$ has 1 row and 3 columns. R is a row matrix of size 1×3.

Definition Square Matrix

A matrix with the same number of rows and columns is called a **square matrix**. A square matrix of size $n \times n$ is said to be of order n:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Ex: $\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ has 2 rows and 2 columns. $\mathbf{A}$ is a square matrix of order 2.

Definition Zero Matrix

A matrix where all entries are zero is called a **zero matrix** or **null matrix**.
It is denoted $0_{n,p}$ if it has n rows and p columns, or simply 0 if the size is clear from the context.

Ex: $0_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Definition Identity Matrix

The **identity matrix**, denoted I_n, is a square matrix of order n with ones on the main diagonal and zeros elsewhere:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Ex: $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity matrix of order 3.

Definition Opposite Matrix

The **opposite matrix** of $\mathbf{A}$, denoted $-\mathbf{A}$, is the matrix where each entry is the additive inverse of the corresponding entry in $\mathbf{A}$:

$$-\mathbf{A} = \begin{pmatrix} -a_{11} & \cdots & -a_{1p} \\ \vdots & & \vdots \\ -a_{n1} & \cdots & -a_{np} \end{pmatrix}.$$

Ex: Find the opposite matrix of $\mathbf{A} = \begin{pmatrix} 1 & 3 & -5 \\ 7 & -2 & 6 \end{pmatrix}$.

Answer: $-\mathbf{A} = \begin{pmatrix} -1 & -3 & 5 \\ -7 & 2 & -6 \end{pmatrix}$

A.3 EQUALITY

Definition Equality of Matrices

Two matrices A and B are equal, written as $A = B$, if they have:

- the same number of rows,
- the same number of columns,
- and their corresponding entries are equal.

Ex:

$$\begin{pmatrix} 1^2 & 2^2 \\ 3^2 & 4^2 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

B MATRIX OPERATIONS

B.1 MATRIX ADDITION

Definition Matrix Addition

Let $\mathbf{A}$ and B be two matrices of the same size $n \times p$.
The **sum** $\mathbf{A} + B$ is the matrix obtained by adding the corresponding entries of $\mathbf{A}$ and B:

$$\begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{np} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & \cdots & a_{1p}+b_{1p} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{np}+b_{np} \end{pmatrix}.$$

Ex: $$\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} + \begin{pmatrix} -2 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0-2 & 1-1 & 2+1 \\ 3+0 & 4+1 & 5+0 \end{pmatrix}$$
$$= \begin{pmatrix} -2 & 0 & 3 \\ 3 & 5 & 5 \end{pmatrix}$$

Proposition Properties of Matrix Addition

Let $\mathbf{A}, B, C$ be three matrices of the same size.

- **Commutativity**: $\mathbf{A} + B = B + \mathbf{A}$
- **Associativity**: $(\mathbf{A} + B) + C = \mathbf{A} + (B + C)$
- **Identity Element**: The zero matrix 0 is the identity element for addition: $\mathbf{A} + 0 = 0 + \mathbf{A} = \mathbf{A}$
- **Inverse Element**: $\mathbf{A} + (-\mathbf{A}) = 0$.

As with real numbers, subtraction is defined as adding the opposite matrix, which is equivalent to subtracting the corresponding entries one by one.

Definition Matrix Subtraction

Let $\mathbf{A}$ and B be two matrices of the same size $n \times p$.
The **difference** $\mathbf{A} - B$ is the matrix obtained by subtracting the corresponding entries of B from $\mathbf{A}$:

$$\mathbf{A} - B = \mathbf{A} + (-B).$$

Ex: $$\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0-(-2) & 1-(-1) \\ 3-0 & 4-1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

B.2 SCALAR MULTIPLICATION

Definition Scalar Multiplication

Let $\mathbf{A}$ be a matrix of size $n \times p$ and λ be a scalar.
The **product** $\lambda\mathbf{A}$ is the matrix obtained by multiplying each entry of $\mathbf{A}$ by λ:

$$\lambda \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{np} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \cdots & \lambda a_{1p} \\ \vdots & & \vdots \\ \lambda a_{n1} & \cdots & \lambda a_{np} \end{pmatrix}.$$

Ex: $$2\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 2\times 0 & 2\times 1 & 2\times 2 \\ 2\times 3 & 2\times 4 & 2\times 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 2 & 4 \\ 6 & 8 & 10 \end{pmatrix}$$

Proposition Properties of Scalar Multiplication

Let $\mathbf{A}, B$ be two matrices of the same size and λ, μ be two scalars.
Then,

- $\lambda(\mathbf{A} + B) = \lambda\mathbf{A} + \lambda B$

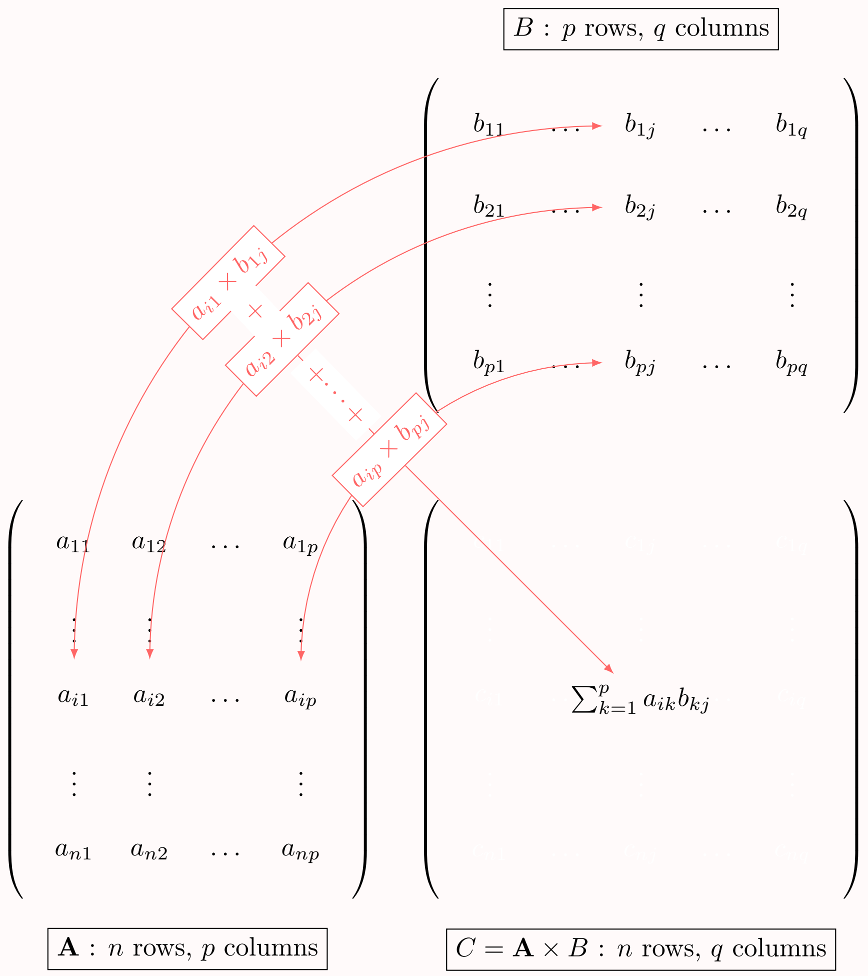
- $(\lambda + \mu)\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}$
- $(\lambda\mu)\mathbf{A} = \lambda(\mu\mathbf{A})$
- $1 \cdot \mathbf{A} = \mathbf{A}$

B.3 MATRIX MULTIPLICATION

Definition Matrix Multiplication

Let $\mathbf{A}$ be a matrix of size $n \times p$ and B be a matrix of size $p \times q$.
The **matrix product** $\mathbf{A} \times B$ is a matrix C of size $n \times q$ where the entry c_{ij} is the dot product of the i-th row of $\mathbf{A}$ and the j-th column of B:

$$c_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}, \qquad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant q.$$



The product is only defined if the number of columns in $\mathbf{A}$ is equal to the number of rows in B. We often write $\mathbf{A}B$ for $\mathbf{A} \times B$.

Ex:
$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 3 \times 3 & 2 \times 2 + 3 \times 4 \\ 1 \times 1 + 0 \times 3 & 1 \times 2 + 0 \times 4 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 1 & 2 \end{pmatrix}$$

In general, matrix multiplication is not commutative ($\mathbf{A}B \neq B\mathbf{A}$). For example, if $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then $\mathbf{A} \times B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ while $B \times \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.
Furthermore, the product of two non-zero matrices can be the zero matrix. For example,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proposition Properties of Matrix Multiplication

Assume all the following products are defined (sizes are compatible).

1. **Associativity**:
$$(\mathbf{A}B)C = \mathbf{A}(BC)$$
2. **Distributivity over addition**:
$$(\mathbf{A}+B)C = \mathbf{A}C + BC \quad \text{and} \quad \mathbf{A}(B+C) = \mathbf{A}B + \mathbf{A}C$$
3. **Identity Element**: The identity matrix is the neutral element for multiplication. For any matrix $\mathbf{A}$ of size $n \times p$, we have:
$$I_n\mathbf{A} = \mathbf{A}I_p = \mathbf{A}$$

C INVERTIBLE MATRICES

C.1 DEFINITION

We know how to solve an equation $ax = b$ with real coefficients. If $a \neq 0$, the unique solution is $x = a^{-1}b$. To find the solution, we consider the existence of the inverse of a. For matrices, solving $\mathbf{A}\mathbf{x} = B$ raises the same question about the existence of the inverse of $\mathbf{A}$.

Definition Inverse of a Matrix

Let $\mathbf{A}$ be a square matrix of order n.
We say that $\mathbf{A}$ is **invertible** if there exists a square matrix B of order n such that

$$\mathbf{A}B = B\mathbf{A} = I_n.$$

If it exists, the matrix B is unique and called the **inverse** of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$.
An invertible matrix is also called a **non-singular matrix**. A square matrix that is not invertible is called a **singular matrix**.

Ex: The inverse of the identity matrix is itself, because $I_nI_n = I_n$.

C.2 FINDING THE INVERSE OF A 2X2 MATRIX

Definition Determinant of a 2x2 Matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a square matrix of order 2.
The **determinant** of $\mathbf{A}$ is $ad - bc$. It is denoted $\det(\mathbf{A})$.

Ex: Calculate the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$.

Answer: For the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$, we have $a = 5$, $b = 2$, $c = 3$, and $d = 4$.

$$\begin{aligned} \det(\mathbf{A}) &= ad - bc \\ &= (5)(4) - (2)(3) \\ &= 20 - 6 \\ &= 14 \end{aligned}$$

Proposition Invertibility of a 2x2 Matrix

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a square matrix of order 2.
$\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$.
In that case,

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

D APPLICATIONS

D.1 SOLVING SYSTEMS OF LINEAR EQUATIONS

Proposition Matrix Representation

A linear system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

is equivalent to the equation $\mathbf{A}\mathbf{x} = B$ where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is a square matrix of order n, and

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

are column matrices (column vectors).

Ex: Write the system $\begin{cases} 2x + 5y &= 2 \\ x + 3y &= 5 \end{cases}$ in matrix form.

Answer: In matrix form, the system is

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Proposition

If $\mathbf{A}$ is invertible, then the equation $\mathbf{A}\mathbf{x} = B$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}B$.

Ex: Solve the system

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Answer:

- **Find the inverse**: Since $\det(\mathbf{A}) = (2)(3) - (5)(1) = 6 - 5 = 1 \neq 0$, the matrix $\mathbf{A}$ is invertible.

$$\mathbf{A}^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}.$$

- **Find the solution**:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} (3)(2) + (-5)(5) \\ (-1)(2) + (2)(5) \end{pmatrix} \\ &= \begin{pmatrix} -19 \\ 8 \end{pmatrix}. \end{aligned}$$

  The solution is $x = -19$ and $y = 8$.