MATRICES

A STRUCTURE

A.1 DEFINITION

A.1.1 IDENTIFYING THE SIZE OF A MATRIX

Ex 1: What is the size of the following matrix?

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 7 \\ 1 & 9 & 4 \end{pmatrix}$$

Size: $\boxed{2} \times \boxed{3}$

Answer: To find the size of a matrix, we count the number of rows and the number of columns.
The matrix $\mathbf{A}$ has 2 rows and 3 columns.
Therefore, its size is 2×3.

Ex 2: What is the size of the following matrix?

$$\mathbf{B} = \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

Size: $\boxed{4} \times \boxed{1}$

Answer: The matrix $\mathbf{B}$ has 4 rows and 1 column.
Therefore, its size is 4×1. This is a column matrix.

Ex 3: What is the size of the following matrix?

$$\mathbf{C} = \begin{pmatrix} 10 & 20 & 30 & 40 & 50 \end{pmatrix}$$

Size: $\boxed{1} \times \boxed{5}$

Answer: The matrix $\mathbf{C}$ has 1 row and 5 columns.
Therefore, its size is 1×5. This is a row matrix.

Ex 4: What is the size of the following matrix?

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Size: $\boxed{3} \times \boxed{3}$

Answer: To find the size of a matrix, we count the number of rows and the number of columns.
The matrix $\mathbf{D}$ has 3 rows and 3 columns.
Therefore, its size is 3×3. This is a square matrix.

A.1.2 IDENTIFYING THE ENTRIES OF A MATRIX

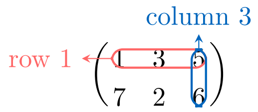
Ex 5: Consider the matrix $\mathbf{A}$ defined as:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 2 & 6 \end{pmatrix}$$

What is the value of the entry a_{13}?

$$a_{13} = \boxed{5}$$

Answer: The entry a_{ij} is located in the i-th row and j-th column.
To find a_{13}, we look for the entry in the 1st row and the 3rd column.
The entry at that position is 5.



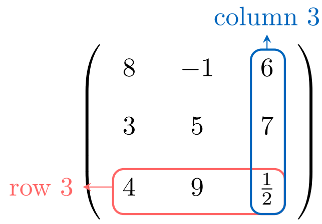
Ex 6: Consider the matrix $\mathbf{A}$ defined as:

$$\mathbf{A} = \begin{pmatrix} 8 & -1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & \frac{1}{2} \end{pmatrix}$$

What is the value of the entry a_{33}?

$$a_{33} = \boxed{\frac{1}{2}}$$

Answer: The entry a_{ij} is located in the i-th row and j-th column.
To find a_{33}, we look for the entry in the 3rd row and the 3rd column.
The entry at that position is $\frac{1}{2}$.



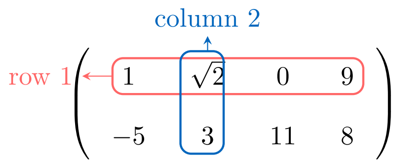
Ex 7: Consider the matrix $\mathbf{B}$ defined as:

$$\mathbf{B} = \begin{pmatrix} 1 & \sqrt{2} & 0 & 9 \\ -5 & 3 & 11 & 8 \end{pmatrix}$$

What is the value of the entry b_{12}?

$$b_{12} = \boxed{\sqrt{2}}$$

Answer: The entry b_{ij} is located in the i-th row and j-th column.
To find b_{12}, we look for the entry in the 1st row and the 2nd column.
The entry at that position is $\sqrt{2}$.



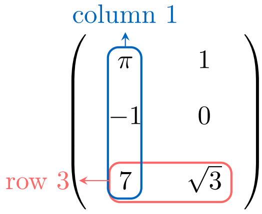
Ex 8: Consider the matrix $\mathbf{C}$ defined as:

$$\mathbf{C} = \begin{pmatrix} \pi & 1 \\ -1 & 0 \\ 7 & \sqrt{3} \end{pmatrix}$$

What is the value of the entry c_{31}?

$$c_{31} = \boxed{7}$$

Answer: The entry c_{ij} is located in the i-th row and j-th column. To find c_{31}, we look for the entry in the 3rd row and the 1st column.
The entry at that position is 7.



A.2 SPECIAL MATRICES

A.2.1 IDENTIFYING TYPES OF MATRICES

MCQ 9: Which of the following matrices is a square matrix?

☐ $\mathbf{A} = \begin{pmatrix} 1 & 5 & 9 \\ 0 & 3 & 7 \end{pmatrix}$

☐ $\mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

☒ $\mathbf{C} = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$

☐ $\mathbf{D} = \begin{pmatrix} 1 & 5 & 9 \end{pmatrix}$

Answer: A square matrix is defined as a matrix with the same number of rows and columns.

- Matrix $\mathbf{A}$ has 2 rows and 3 columns, so it is not square.
- Matrix $\mathbf{B}$ has 2 rows and 1 column, so it is not square.
- Matrix $\mathbf{C}$ has 2 rows and 2 columns, so it is a square matrix of order 2.
- Matrix $\mathbf{D}$ has 1 row and 3 columns, so it is not square.

MCQ 10: Which of the following matrices is a column matrix?

☐ $\mathbf{A} = \begin{pmatrix} 2 & 0 & 9 \end{pmatrix}$

☒ $\mathbf{B} = \begin{pmatrix} 7 \\ -1 \\ 4 \end{pmatrix}$

☐ $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

☐ $\mathbf{D} = \begin{pmatrix} 2 & 8 \\ 6 & 1 \end{pmatrix}$

Answer: A column matrix is defined as a matrix with only one column.

- Matrix $\mathbf{A}$ has 1 row and 3 columns. It is a row matrix.
- Matrix $\mathbf{B}$ has 3 rows and 1 column. It is a column matrix.
- Matrix $\mathbf{C}$ has 2 rows and 2 columns. It is a square matrix (and a zero matrix).
- Matrix $\mathbf{D}$ has 2 rows and 2 columns. It is a square matrix.

MCQ 11: Which of the following is the identity matrix of order 3?

☐ $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

☐ $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

☒ $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

☐ $\mathbf{D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Answer: The identity matrix of order n is a square matrix of size $n \times n$ with ones on the main diagonal and zeros everywhere else.

- Matrix $\mathbf{A}$ is the zero matrix of order 3.
- Matrix $\mathbf{B}$ is a square matrix of order 3, but its off-diagonal entries are not zero.
- Matrix $\mathbf{C}$ is a square matrix of order 3 with ones on the main diagonal and zeros elsewhere. This is the identity matrix $\mathbf{I}_3$.
- Matrix $\mathbf{D}$ is a square matrix of order 3, but the ones are on the anti-diagonal.

MCQ 12: Which of the following matrices is a row matrix?

☐ $\mathbf{A} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$

☐ $\mathbf{B} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$

☐ $\mathbf{C} = \begin{pmatrix} 5 & 10 & 15 \\ 20 & 25 & 30 \end{pmatrix}$

☒ $\mathbf{D} = \begin{pmatrix} 5 & 10 & 15 \end{pmatrix}$

Answer: A row matrix is defined as a matrix that has only one row.

- Matrix $\mathbf{A}$ has 3 rows and 1 column. It is a column matrix.
- Matrix $\mathbf{B}$ has 2 rows and 2 columns. It is a square matrix.
- Matrix $\mathbf{C}$ has 2 rows and 3 columns. It is a rectangular matrix but not a row matrix.
- Matrix $\mathbf{D}$ has 1 row and 3 columns. It is a row matrix.

MCQ 13: What type of special matrix is $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$?

☒ A row matrix

☐ A column matrix

☒ A zero matrix

☐ An identity matrix

Answer: We analyze the properties of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$.

- It has only one row, so it is a **row matrix**.
- It has three columns, not one, so it is not a column matrix.
- All of its entries are zero, so it is a **zero matrix** (specifically, $0_{1,3}$).
- It is not a square matrix, so it cannot be an identity matrix.

Therefore, the matrix is both a row matrix and a zero matrix.

MCQ 14: What type of special matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$?

☒ A square matrix

☐ A column matrix

☐ A zero matrix

☒ An identity matrix

Answer: We analyze the properties of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

- It has 3 rows and 3 columns, so it is a **square matrix**.
- It has more than one column, so it is not a column matrix.
- Not all of its entries are zero, so it is not a zero matrix.
- It is a square matrix of order 3 with ones on the main diagonal and zeros elsewhere. This is the definition of the **identity matrix**, $\mathbf{I}_3$.

Therefore, the matrix is both a square matrix and an identity matrix.

A.2.2 CONSTRUCTING SPECIAL MATRICES

Ex 15: Write the identity matrix of order 2, denoted $\mathbf{I}_2$.

$$\mathbf{I}_2 = \begin{pmatrix} \boxed{1} & \boxed{0} \\ \boxed{0} & \boxed{1} \end{pmatrix}$$

Answer: The identity matrix of order n, denoted $\mathbf{I}_n$, is a square matrix of size $n \times n$ with ones on the main diagonal (from the top-left to the bottom-right) and zeros in all other positions.
For order 2, this gives:

$$\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Ex 16: Find the opposite matrix of $\mathbf{A} = \begin{pmatrix} -1 & 7 \\ 0 & -2 \end{pmatrix}$.

$$-\mathbf{A} = \begin{pmatrix} \boxed{1} & \boxed{-7} \\ \boxed{0} & \boxed{2} \end{pmatrix}$$

Answer: The opposite of a matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is found by taking the additive inverse of each of its entries.

$$-\mathbf{A} = \begin{pmatrix} -(-1) & -(7) \\ -(0) & -(-2) \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 0 & 2 \end{pmatrix}$$

Ex 17: Find the opposite matrix of $\mathbf{A} = \begin{pmatrix} 9 & -2 & 0 & -11 \end{pmatrix}$.

$$-\mathbf{A} = \begin{pmatrix} \boxed{-9} & \boxed{2} & \boxed{0} & \boxed{11} \end{pmatrix}$$

Answer: The opposite of a matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is found by taking the additive inverse of each of its entries.

$$-\mathbf{A} = \begin{pmatrix} -(9) & -(-2) & -(0) & -(-11) \end{pmatrix} = \begin{pmatrix} -9 & 2 & 0 & 11 \end{pmatrix}$$

A.3 EQUALITY

A.3.1 IDENTIFYING EQUAL MATRICES

MCQ 18: Which of the following matrices is equal to matrix $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix}$?

☐ $\mathbf{B} = \begin{pmatrix} 4 & 9 \\ 0 & 1 \end{pmatrix}$

☒ $\mathbf{C} = \begin{pmatrix} 2^2 & 0 \\ 3^2 & 1^2 \end{pmatrix}$

☐ $\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 9 & 1 & 0 \end{pmatrix}$

☐ $\mathbf{E} = \begin{pmatrix} 4 \\ 9 \\ 0 \\ 1 \end{pmatrix}$

Answer: For two matrices to be equal, they must have the same size and their corresponding entries must be equal. Matrix $\mathbf{A}$ has a size of 2×2.

- Matrix $\mathbf{B}$ has the same size and entries as $\mathbf{A}$, but the entries b_{12} and b_{21} are swapped. They are not equal.
- Matrix $\mathbf{C}$ simplifies to $\begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix}$. It has the same size and the same corresponding entries as $\mathbf{A}$. They are equal.
- Matrix $\mathbf{D}$ has a size of 2×3, which is different from $\mathbf{A}$. They are not equal.
- Matrix $\mathbf{E}$ has a size of 4×1, which is different from $\mathbf{A}$. They are not equal.

MCQ 19: Which of the following matrices is equal to the row matrix $\mathbf{A} = \begin{pmatrix} \sqrt{9} & 5 & 2^3 \end{pmatrix}$?

☐ $\mathbf{B} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$

☐ $\mathbf{C} = \begin{pmatrix} 8 & 5 & 3 \end{pmatrix}$

☒ $\mathbf{D} = \begin{pmatrix} 3 & 5 & 8 \end{pmatrix}$

☐ $\mathbf{E} = \begin{pmatrix} 3 & 5 & 8 \\ 3 & 5 & 8 \end{pmatrix}$

Answer: First, we must simplify the matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} \sqrt{9} & 5 & 2^3 \end{pmatrix} = \begin{pmatrix} 3 & 5 & 8 \end{pmatrix}$$

Matrix $\mathbf{A}$ has a size of 1×3. For another matrix to be equal to $\mathbf{A}$, it must have the same size and the same entries in the same positions.

- Matrix $\mathbf{B}$ is a column matrix of size 3×1. It has a different size, so it is not equal to $\mathbf{A}$.
- Matrix $\mathbf{C}$ has the same size (1×3) and the same numbers as $\mathbf{A}$, but the entries are in a different order. They are not equal.

- Matrix $\mathbf{D}$ has the same size (1×3) and its entries are identical to $\mathbf{A}$'s entries in the corresponding positions. They are equal.

- Matrix $\mathbf{E}$ has a size of 2×3, which is different from $\mathbf{A}$. They are not equal.

MCQ 20: Let $\mathbf{A} = \begin{pmatrix} \frac{10}{2} & 0 \\ 1 & -3 \end{pmatrix}$. Which of the following statements is true?

- ☐ $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 0 & -3 \end{pmatrix}$
- ☐ $\mathbf{A} = \begin{pmatrix} 5 & 0 \end{pmatrix}$
- ☐ $\mathbf{A} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ -3 \end{pmatrix}$
- ☒ $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ \sin(\frac{\pi}{2}) & -3 \end{pmatrix}$

Answer: First, let's simplify the matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} \frac{10}{2} & 0 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 1 & -3 \end{pmatrix}$$

Now we evaluate each option.

- Option 1: The matrix $\begin{pmatrix} 5 & 1 \\ 0 & -3 \end{pmatrix}$ has its off-diagonal entries swapped compared to $\mathbf{A}$. The statement is false.

- Option 2: The matrix $\begin{pmatrix} 5 & 0 \end{pmatrix}$ is a 1×2 matrix, which has a different size from $\mathbf{A}$. The statement is false.

- Option 3: This is a 4×1 column matrix. It has a different size from $\mathbf{A}$. The statement is false.

- Option 4: We evaluate the entry $\sin(\frac{\pi}{2})$, which is equal to 1. The matrix is $\begin{pmatrix} 5 & 0 \\ 1 & -3 \end{pmatrix}$. This matrix has the same size and identical corresponding entries as $\mathbf{A}$. The statement is true.

A.3.2 SOLVING FOR UNKNOWNS USING MATRIX EQUALITY

Ex 21: Find the values of x and y that make the two matrices equal:

$$\begin{pmatrix} x & 7 \\ -2 & y+1 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$$

$$x = \boxed{5} \quad \text{and} \quad y = \boxed{2}$$

Answer: For two matrices to be equal, their corresponding entries must be equal. By comparing the entries at each position, we can set up equations.

- From the position (1,1): $x = 5$.

- From the position (2,2): $y + 1 = 3$.

Solving the second equation for y:

$$y + 1 = 3 \implies y = 3 - 1 \implies y = 2$$

So, the values are $x = 5$ and $y = 2$.

Ex 22: Find the values of a and b such that the following matrices are equal:

$$\begin{pmatrix} a+b & 5 \\ 1 & a-b \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 1 & 4 \end{pmatrix}$$

$$a = \boxed{6} \quad \text{and} \quad b = \boxed{2}$$

Answer: By the definition of matrix equality, the corresponding entries must be equal. This gives us a system of two linear equations:

$$\begin{cases} a + b = 8 & (1) \\ a - b = 4 & (2) \end{cases}$$

We can solve this system. Adding equation (1) and (2) together:

$$(a + b) + (a - b) = 8 + 4 \implies 2a = 12 \implies a = 6$$

Now, substitute the value of a back into equation (1):

$$6 + b = 8 \implies b = 8 - 6 \implies b = 2$$

The solution is $a = 6$ and $b = 2$.

Ex 23: Find the values of x and y for which the following matrix equality holds:

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} y & -x \\ -x & y \end{pmatrix}$$

$$x = \boxed{0} \quad \text{and} \quad y = \boxed{0}$$

Answer: By equating the corresponding entries of the two matrices, we obtain a system of equations.

- From position (1,1): $x = y$.

- From position (1,2): $y = -x$.

- From position (2,1): $y = -x$.

- From position (2,2): $x = y$.

The system we need to solve is:

$$\begin{cases} x = y \\ y = -x \end{cases}$$

Substitute the first equation into the second one:

$$x = -x$$

This simplifies to $2x = 0$, which means $x = 0$.
Since $x = y$, it follows that $y = 0$.
The only solution is $x = 0$ and $y = 0$.

B MATRIX OPERATIONS

B.1 MATRIX ADDITION

B.1.1 VERIFYING THE CONDITION FOR ADDITION

MCQ 24: Which of the following matrix sums is possible?

- ☐ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}5 & 6\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\4\\5\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}+\begin{pmatrix}7 & 8\\9 & 10\end{pmatrix}$
- ☒ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}5 & 6\\7 & 8\end{pmatrix}$

Answer: Matrix addition is only defined for matrices of the same size. We must check the size of the matrices in each option.

- Option 1: A 2×2 matrix and a 1×2 matrix. The sizes are different.
- Option 2: A 2×1 matrix and a 3×1 matrix. The sizes are different.
- Option 3: A 2×3 matrix and a 2×2 matrix. The sizes are different.
- Option 4: A 2×2 matrix and another 2×2 matrix. The sizes are the same, so the addition is possible.

MCQ 25: Which of the following matrix sums is possible?

- ☐ $\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4 & 5 & 6\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\end{pmatrix}+\begin{pmatrix}4 & 5\end{pmatrix}$
- ☒ $\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}4\\5\\6\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}5\\6\end{pmatrix}$

Answer: Matrix addition is only defined for matrices of the same size. We must check the size of the matrices in each option.

- Option 1: A 3×1 matrix and a 1×3 matrix. The sizes are different.
- Option 2: A 1×3 matrix and a 1×2 matrix. The sizes are different.
- Option 3: A 3×1 matrix and another 3×1 matrix. The sizes are the same, so the addition is possible.
- Option 4: A 2×2 matrix and a 2×1 matrix. The sizes are different.

MCQ 26: Which of the following matrix sums is possible?

- ☒ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}5 & 6\\7 & 8\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}+\begin{pmatrix}5 & 6\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\2\end{pmatrix}+\begin{pmatrix}3\\4\\5\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}+\begin{pmatrix}7 & 8\\9 & 10\end{pmatrix}$

Answer: Matrix addition is only defined for matrices of the same size. We must check the size of the matrices in each option.

- Option 1: A 2×2 matrix and another 2×2 matrix. The sizes are the same, so the addition is possible.
- Option 2: A 2×2 matrix and a 1×2 matrix. The sizes are different.
- Option 3: A 2×1 matrix and a 3×1 matrix. The sizes are different.
- Option 4: A 2×3 matrix and a 2×2 matrix. The sizes are different.

B.1.2 CALCULATING MATRIX SUMS

Ex 27: Calculate the sum of the following matrices:

$$\begin{pmatrix}5 & -1\\2 & 8\end{pmatrix}+\begin{pmatrix}-3 & 1\\4 & -2\end{pmatrix}=\begin{pmatrix}\boxed{2} & \boxed{0}\\\boxed{6} & \boxed{6}\end{pmatrix}$$

Answer: To add two matrices, we add their corresponding entries.

$$\begin{aligned}\begin{pmatrix}5 & -1\\2 & 8\end{pmatrix}+\begin{pmatrix}-3 & 1\\4 & -2\end{pmatrix}&=\begin{pmatrix}5+(-3) & -1+1\\2+4 & 8+(-2)\end{pmatrix}\\&=\begin{pmatrix}2 & 0\\6 & 6\end{pmatrix}\end{aligned}$$

Ex 28: Calculate the sum of the following column matrices:

$$\begin{pmatrix}4\\-2\\7\end{pmatrix}+\begin{pmatrix}-1\\5\\3\end{pmatrix}=\begin{pmatrix}\boxed{3}\\\boxed{3}\\\boxed{10}\end{pmatrix}$$

Answer: To add two column matrices, add their corresponding entries.

$$\begin{aligned}\begin{pmatrix}4\\-2\\7\end{pmatrix}+\begin{pmatrix}-1\\5\\3\end{pmatrix}&=\begin{pmatrix}4+(-1)\\-2+5\\7+3\end{pmatrix}\\&=\begin{pmatrix}3\\3\\10\end{pmatrix}\end{aligned}$$

Ex 29: Calculate the sum of the following row matrices:

$$\begin{pmatrix}10 & 0 & -5\end{pmatrix}+\begin{pmatrix}2 & 4 & 5\end{pmatrix}=\begin{pmatrix}\boxed{12} & \boxed{4} & \boxed{0}\end{pmatrix}$$

Answer: To add two matrices, we add their corresponding entries.

$$\begin{aligned}\begin{pmatrix}10 & 0 & -5\end{pmatrix}+\begin{pmatrix}2 & 4 & 5\end{pmatrix}&=\begin{pmatrix}10+2 & 0+4 & -5+5\end{pmatrix}\\&=\begin{pmatrix}12 & 4 & 0\end{pmatrix}\end{aligned}$$

Ex 30: Calculate the sum of the following matrices:

$$\begin{pmatrix} \frac{1}{2} & 3 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} \boxed{1} & \boxed{1} \\ \boxed{0} & \boxed{5} \end{pmatrix}$$

Answer: To add two matrices, we add their corresponding entries.

$$\begin{aligned} \begin{pmatrix} \frac{1}{2} & 3 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -2 \\ 1 & 5 \end{pmatrix} &= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & 3 + (-2) \\ -1 + 1 & 0 + 5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix} \end{aligned}$$

B.1.3 CALCULATING MATRIX DIFFERENCES

Ex 31: Calculate the difference of the following matrices:

$$\begin{pmatrix} 10 & 8 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} \boxed{8} & \boxed{9} \\ \boxed{0} & \boxed{-1} \end{pmatrix}$$

Answer: To subtract two matrices, we subtract their corresponding entries.

$$\begin{aligned} \begin{pmatrix} 10 & 8 \\ 5 & 6 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 5 & 7 \end{pmatrix} &= \begin{pmatrix} 10 - 2 & 8 - (-1) \\ 5 - 5 & 6 - 7 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 9 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

Ex 32: Calculate the difference of the following column matrices:

$$\begin{pmatrix} 9 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} \boxed{4} \\ \boxed{3} \\ \boxed{-7} \end{pmatrix}$$

Answer: To subtract two column matrices, subtract their corresponding entries.

$$\begin{aligned} \begin{pmatrix} 9 \\ 1 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 9 - 5 \\ 1 - (-2) \\ -4 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix} \end{aligned}$$

Ex 33: Calculate the difference of the following row matrices:

$$\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} \boxed{0} & \boxed{-4} & \boxed{6} \end{pmatrix}$$

Answer: To subtract two matrices, we subtract their corresponding entries.

$$\begin{aligned} \begin{pmatrix} 1 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 2 & -3 \end{pmatrix} &= \begin{pmatrix} 1 - 1 & -2 - 2 & 3 - (-3) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -4 & 6 \end{pmatrix} \end{aligned}$$

B.1.4 EVALUATING MATRIX EXPRESSIONS

Ex 34: For $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$, find $\mathbf{A} - (\mathbf{B} + \mathbf{C})$.

Answer: We first calculate the sum inside the parentheses, $\mathbf{B} + \mathbf{C}$, and then subtract the result from $\mathbf{A}$.

$$\begin{aligned} \mathbf{A} - (\mathbf{B} + \mathbf{C}) &= \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} - \left(\begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} - \begin{pmatrix} -3 + 4 & 1 + 2 \\ 4 - 2 & -2 + 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} - \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 - 1 & -1 - 3 \\ 2 - 2 & 8 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -4 \\ 0 & 7 \end{pmatrix} \end{aligned}$$

Ex 35: For $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$, find $-(\mathbf{A} - \mathbf{B})$.

Answer: We first calculate the difference inside the parentheses, $\mathbf{A} - \mathbf{B}$, and then find the opposite of the resulting matrix.

$$\begin{aligned} -(\mathbf{A} - \mathbf{B}) &= -\left(\begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} - \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix} \right) \\ &= -\begin{pmatrix} 5 - (-3) & -1 - 1 \\ 2 - 4 & 8 - (-2) \end{pmatrix} \\ &= -\begin{pmatrix} 8 & -2 \\ -2 & 10 \end{pmatrix} \\ &= \begin{pmatrix} -8 & 2 \\ 2 & -10 \end{pmatrix} \end{aligned}$$

Note that this is also equal to $\mathbf{B} - \mathbf{A}$.

Ex 36: For $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$, find $\mathbf{A} + (\mathbf{B} - \mathbf{C})$.

Answer: We first calculate the difference inside the parentheses, $\mathbf{B} - \mathbf{C}$, and then add the result to $\mathbf{A}$.

$$\begin{aligned} \mathbf{A} + (\mathbf{B} - \mathbf{C}) &= \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} + \left(\begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix} \right) \\ &= \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} + \begin{pmatrix} -3 - 4 & 1 - 2 \\ 4 - (-2) & -2 - 3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix} + \begin{pmatrix} -7 & -1 \\ 6 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 5 - 7 & -1 - 1 \\ 2 + 6 & 8 - 5 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -2 \\ 8 & 3 \end{pmatrix} \end{aligned}$$

Ex 37: For $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$, find $\mathbf{A} - (\mathbf{B} - \mathbf{C})$.

Answer: We first calculate the difference inside the parentheses,

$\mathbf{B}-\mathbf{C}$, and then subtract the result from $\mathbf{A}$.

$$\begin{aligned}\mathbf{A}-(\mathbf{B}-\mathbf{C}) &= \begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix} - \left(\begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix} - \begin{pmatrix}4 & 2\\ -2 & 3\end{pmatrix}\right)\\ &= \begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix} - \begin{pmatrix}-3-4 & 1-2\\ 4-(-2) & -2-3\end{pmatrix}\\ &= \begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix} - \begin{pmatrix}-7 & -1\\ 6 & -5\end{pmatrix}\\ &= \begin{pmatrix}5-(-7) & -1-(-1)\\ 2-6 & 8-(-5)\end{pmatrix}\\ &= \begin{pmatrix}12 & 0\\ -4 & 13\end{pmatrix}\end{aligned}$$

B.1.5 PROVING THE PROPERTIES OF ADDITION

Ex 38: For two square matrices of order 2, $\mathbf{A}=\begin{pmatrix}x & y\\ z & w\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix}$, prove that $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$.

Answer: To prove the commutative property of matrix addition, we start with the left side, apply the definition of matrix addition, use the properties of real numbers for each entry, and then rearrange to get the right side.

$$\begin{aligned}\mathbf{A}+\mathbf{B} &= \begin{pmatrix}x & y\\ z & w\end{pmatrix} + \begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix}\\ &= \begin{pmatrix}x+x' & y+y'\\ z+z' & w+w'\end{pmatrix} && \text{(Matrix addition)}\\ &= \begin{pmatrix}x'+x & y'+y\\ z'+z & w'+w\end{pmatrix} && \text{(Commutativity of real number addition)}\\ &= \begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix} + \begin{pmatrix}x & y\\ z & w\end{pmatrix} && \text{(Matrix addition)}\\ &= \mathbf{B}+\mathbf{A}\end{aligned}$$

Ex 39: For three square matrices of order 2, $\mathbf{A}=\begin{pmatrix}x & y\\ z & w\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}x'' & y''\\ z'' & w''\end{pmatrix}$, prove that $\mathbf{A}+(\mathbf{B}+\mathbf{C})=(\mathbf{A}+\mathbf{B})+\mathbf{C}$.

Answer: To prove the associative property of matrix addition, we will evaluate the left side of the equation, use the associativity of real number addition on the entries, and then regroup the terms to show it is equal to the right side.

$$\begin{aligned}&\mathbf{A}+(\mathbf{B}+\mathbf{C})\\ &= \begin{pmatrix}x & y\\ z & w\end{pmatrix} + \left(\begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix} + \begin{pmatrix}x'' & y''\\ z'' & w''\end{pmatrix}\right)\\ &= \begin{pmatrix}x & y\\ z & w\end{pmatrix} + \begin{pmatrix}x'+x'' & y'+y''\\ z'+z'' & w'+w''\end{pmatrix} && \text{(Matrix addition for } \mathbf{B}+\mathbf{C})\\ &= \begin{pmatrix}x+(x'+x'') & y+(y'+y'')\\ z+(z'+z'') & w+(w'+w'')\end{pmatrix} && \text{(Matrix addition for } \mathbf{A}+(\ldots))\\ &= \begin{pmatrix}(x+x')+x'' & (y+y')+y''\\ (z+z')+z'' & (w+w')+w''\end{pmatrix} && \text{(Associativity of real number addition)}\\ &= \begin{pmatrix}x+x' & y+y'\\ z+z' & w+w'\end{pmatrix} + \begin{pmatrix}x'' & y''\\ z'' & w''\end{pmatrix} && \text{(Matrix addition)}\\ &= \left(\begin{pmatrix}x & y\\ z & w\end{pmatrix} + \begin{pmatrix}x' & y'\\ z' & w'\end{pmatrix}\right) + \begin{pmatrix}x'' & y''\\ z'' & w''\end{pmatrix}\\ &= (\mathbf{A}+\mathbf{B})+\mathbf{C}\end{aligned}$$

Ex 40: For a square matrix of order 2, $\mathbf{A}=\begin{pmatrix}x & y\\ z & w\end{pmatrix}$, prove that $\mathbf{A}+(-\mathbf{A})=\mathbf{0}$, where $\mathbf{0}$ is the 2×2 zero matrix.

Answer: To prove the additive inverse property, we start with the expression $\mathbf{A}+(-\mathbf{A})$, substitute the definitions of the matrices, and apply the rules of matrix addition.The opposite matrix of $\mathbf{A}$ is $-\mathbf{A}=\begin{pmatrix}-x & -y\\ -z & -w\end{pmatrix}$.

$$\begin{aligned}\mathbf{A}+(-\mathbf{A}) &= \begin{pmatrix}x & y\\ z & w\end{pmatrix} + \begin{pmatrix}-x & -y\\ -z & -w\end{pmatrix} && \text{(By definition of opposite matrix)}\\ &= \begin{pmatrix}x+(-x) & y+(-y)\\ z+(-z) & w+(-w)\end{pmatrix} && \text{(By definition of matrix addition)}\\ &= \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} && \text{(By additive inverse property of real numbers)}\\ &= \mathbf{0}\end{aligned}$$

B.2 SCALAR MULTIPLICATION

B.2.1 CALCULATING SCALAR PRODUCTS

Ex 41: Calculate the scalar multiplication:

$$2\begin{pmatrix}\frac{1}{2} & 3\\ -1 & 0\end{pmatrix} = \begin{pmatrix}\boxed{1} & \boxed{6}\\ \boxed{-2} & \boxed{0}\end{pmatrix}$$

Answer: To multiply a matrix by a scalar, we multiply each entry of the matrix by that scalar.

$$\begin{aligned}2\begin{pmatrix}\frac{1}{2} & 3\\ -1 & 0\end{pmatrix} &= \begin{pmatrix}2\times\frac{1}{2} & 2\times 3\\ 2\times(-1) & 2\times 0\end{pmatrix}\\ &= \begin{pmatrix}1 & 6\\ -2 & 0\end{pmatrix}\end{aligned}$$

Ex 42: Calculate the scalar multiplication:

$$5\begin{pmatrix}2\\ 0\\ -3\\ 1\end{pmatrix} = \begin{pmatrix}\boxed{10}\\ \boxed{0}\\ \boxed{-15}\\ \boxed{5}\end{pmatrix}$$

Answer: To multiply a matrix by a scalar, we multiply each entry of the matrix by that scalar.

$$\begin{aligned}5\begin{pmatrix}2\\ 0\\ -3\\ 1\end{pmatrix} &= \begin{pmatrix}5\times 2\\ 5\times 0\\ 5\times(-3)\\ 5\times 1\end{pmatrix}\\ &= \begin{pmatrix}10\\ 0\\ -15\\ 5\end{pmatrix}\end{aligned}$$

Ex 43: Calculate the scalar multiplication:

$$\frac{1}{2}\begin{pmatrix}10 & -4 & 6\end{pmatrix} = \begin{pmatrix}\boxed{5} & \boxed{-2} & \boxed{3}\end{pmatrix}$$

Answer: To multiply a matrix by a scalar, we multiply each entry of the matrix by that scalar.

$$\begin{aligned}\frac{1}{2}\begin{pmatrix}10 & -4 & 6\end{pmatrix} &= \begin{pmatrix}\frac{1}{2}\times 10 & \frac{1}{2}\times(-4) & \frac{1}{2}\times 6\end{pmatrix}\\ &= \begin{pmatrix}5 & -2 & 3\end{pmatrix}\end{aligned}$$

Ex 44: Calculate the scalar multiplication:

$$-4\begin{pmatrix}1 & -3\\ 5 & 0\end{pmatrix}=\begin{pmatrix}\boxed{-4} & \boxed{12}\\ \boxed{-20} & \boxed{0}\end{pmatrix}$$

Answer: To multiply a matrix by a scalar, we multiply each entry of the matrix by that scalar.

$$\begin{aligned}-4\begin{pmatrix}1 & -3\\ 5 & 0\end{pmatrix}&=\begin{pmatrix}-4\times 1 & -4\times(-3)\\ -4\times 5 & -4\times 0\end{pmatrix}\\&=\begin{pmatrix}-4 & 12\\ -20 & 0\end{pmatrix}\end{aligned}$$

B.2.2 EVALUATING MATRIX EXPRESSIONS

Ex 45: For $\mathbf{A}=\begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix}$, find $2(\mathbf{A}+\mathbf{B})$.

Answer: We first calculate the sum inside the parentheses, $\mathbf{A}+\mathbf{B}$, and then multiply the resulting matrix by the scalar 2.

$$\begin{aligned}2(\mathbf{A}+\mathbf{B})&=2\left(\begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix}+\begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix}\right)\\&=2\begin{pmatrix}5-3 & -1+1\\ 2+4 & 8-2\end{pmatrix}\\&=2\begin{pmatrix}2 & 0\\ 6 & 6\end{pmatrix}\\&=\begin{pmatrix}2\times 2 & 2\times 0\\ 2\times 6 & 2\times 6\end{pmatrix}\\&=\begin{pmatrix}4 & 0\\ 12 & 12\end{pmatrix}\end{aligned}$$

Ex 46: For $\mathbf{A}=\begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -\frac{3}{2}\\ 2 & 0\end{pmatrix}$, find $2\mathbf{A}+3\mathbf{B}$.

Answer: We first perform the scalar multiplications for $2\mathbf{A}$ and $3\mathbf{B}$, and then add the resulting matrices.

$$\begin{aligned}2\mathbf{A}+3\mathbf{B}&=2\begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}+3\begin{pmatrix}1 & -\frac{3}{2}\\ 2 & 0\end{pmatrix}\\&=\begin{pmatrix}2\times\frac{1}{2} & 2\times 4\\ 2\times 0 & 2\times(-1)\end{pmatrix}+\begin{pmatrix}3\times 1 & 3\times(-\frac{3}{2})\\ 3\times 2 & 3\times 0\end{pmatrix}\\&=\begin{pmatrix}1 & 8\\ 0 & -2\end{pmatrix}+\begin{pmatrix}3 & -\frac{9}{2}\\ 6 & 0\end{pmatrix}\\&=\begin{pmatrix}1+3 & 8-\frac{9}{2}\\ 0+6 & -2+0\end{pmatrix}\\&=\begin{pmatrix}4 & \frac{16}{2}-\frac{9}{2}\\ 6 & -2\end{pmatrix}\\&=\begin{pmatrix}4 & \frac{7}{2}\\ 6 & -2\end{pmatrix}\end{aligned}$$

Ex 47: For $\mathbf{A}=\begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -\frac{3}{2}\\ 2 & 0\end{pmatrix}$, find $\frac{1}{2}(\mathbf{A}-\mathbf{B})$.

Answer: We first calculate the difference inside the parentheses, $\mathbf{A}-\mathbf{B}$, and then multiply the resulting matrix by the scalar $\frac{1}{2}$.

$$\begin{aligned}\frac{1}{2}(\mathbf{A}-\mathbf{B})&=\frac{1}{2}\left(\begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}-\begin{pmatrix}1 & -\frac{3}{2}\\ 2 & 0\end{pmatrix}\right)\\&=\frac{1}{2}\begin{pmatrix}\frac{1}{2}-1 & 4-(-\frac{3}{2})\\ 0-2 & -1-0\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}-\frac{1}{2} & \frac{8}{2}+\frac{3}{2}\\ -2 & -1\end{pmatrix}\\&=\frac{1}{2}\begin{pmatrix}-\frac{1}{2} & \frac{11}{2}\\ -2 & -1\end{pmatrix}\\&=\begin{pmatrix}\frac{1}{2}\times(-\frac{1}{2}) & \frac{1}{2}\times\frac{11}{2}\\ \frac{1}{2}\times(-2) & \frac{1}{2}\times(-1)\end{pmatrix}\\&=\begin{pmatrix}-\frac{1}{4} & \frac{11}{4}\\ -1 & -\frac{1}{2}\end{pmatrix}\end{aligned}$$

Ex 48: For $\mathbf{A}=\begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}$, find $2(3\mathbf{A})$.

Answer: We first calculate the product inside the parentheses, $3\mathbf{A}$, and then multiply the resulting matrix by the scalar 2.

$$\begin{aligned}2(3\mathbf{A})&=2\left(3\begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}\right)\\&=2\begin{pmatrix}3\times\frac{1}{2} & 3\times 4\\ 3\times 0 & 3\times(-1)\end{pmatrix}\\&=2\begin{pmatrix}\frac{3}{2} & 12\\ 0 & -3\end{pmatrix}\\&=\begin{pmatrix}2\times\frac{3}{2} & 2\times 12\\ 2\times 0 & 2\times(-3)\end{pmatrix}\\&=\begin{pmatrix}3 & 24\\ 0 & -6\end{pmatrix}\end{aligned}$$

Note that due to the associative property of scalar multiplication, this is the same as calculating $(2\times 3)\mathbf{A}=6\mathbf{A}$.

B.2.3 SIMPLIFYING MATRIX EXPRESSIONS

Ex 49: For any matrix $\mathbf{A}$, simplify the expression $2\mathbf{A}+2(4\mathbf{A})$.

Answer: To simplify the expression, we use the associative and distributive properties of scalar multiplication.

$$\begin{aligned}2\mathbf{A}+2(4\mathbf{A})&=2\mathbf{A}+(2\times 4)\mathbf{A} && \text{(Associativity: }\lambda(\mu\mathbf{A})=(\lambda\mu)\mathbf{A}\text{)}\\&=2\mathbf{A}+8\mathbf{A}\\&=(2+8)\mathbf{A} && \text{(Distributive property: }\lambda\mathbf{A}+\mu\mathbf{A}=(\lambda+\mu)\mathbf{A}\text{)}\\&=10\mathbf{A}\end{aligned}$$

Ex 50: For any two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, simplify the expression $(\mathbf{A}-\mathbf{B})+(\mathbf{A}+\mathbf{B})$.

Answer: To simplify the expression, we can rearrange the terms using the properties of matrix addition and then factor using the distributive property.

$$\begin{aligned}(\mathbf{A}-\mathbf{B})+(\mathbf{A}+\mathbf{B})&=\mathbf{A}-\mathbf{B}+\mathbf{A}+\mathbf{B} && \text{(Associativity of addition)}\\&=\mathbf{A}+\mathbf{A}+\mathbf{B}-\mathbf{B} && \text{(Commutativity of addition)}\\&=(1+1)\mathbf{A}+(1-1)\mathbf{B} && \text{(Distributive property)}\\&=2\mathbf{A}+0\mathbf{B}\\&=2\mathbf{A} && \text{(Property of the zero matrix)}\end{aligned}$$

Ex 51: For any two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, simplify the expression $3(\mathbf{A}+\mathbf{B})-3\mathbf{A}$.

Answer: To simplify the expression, we use the distributive property of scalar multiplication over matrix addition.

$$\begin{aligned}
3(\mathbf{A}+\mathbf{B})-3\mathbf{A} &= (3\mathbf{A}+3\mathbf{B})-3\mathbf{A} && \text{(Distributive property: } \lambda(\mathbf{A}+\mathbf{B})=\lambda\mathbf{A}+\lambda\mathbf{B}) \\
&= 3\mathbf{A}-3\mathbf{A}+3\mathbf{B} && \text{(Commutativity of addition)} \\
&= (3\mathbf{A}-3\mathbf{A})+3\mathbf{B} && \text{(Associativity of addition)} \\
&= \mathbf{0}+3\mathbf{B} && \text{(Additive inverse)} \\
&= 3\mathbf{B}
\end{aligned}$$

Ex 52: For any two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, simplify the expression $(\mathbf{A}+\mathbf{B})-(\mathbf{A}-\mathbf{B})$.

Answer: To simplify the expression, we use the fact that subtracting is equivalent to adding the opposite.

$$\begin{aligned}
(\mathbf{A}+\mathbf{B})-(\mathbf{A}-\mathbf{B}) &= \mathbf{A}+\mathbf{B}-\mathbf{A}-(-\mathbf{B}) && \text{(Definition of subtraction)} \\
&= \mathbf{A}+\mathbf{B}-\mathbf{A}+\mathbf{B} \\
&= \mathbf{A}-\mathbf{A}+\mathbf{B}+\mathbf{B} && \text{(Commutativity of addition)} \\
&= (1-1)\mathbf{A}+(1+1)\mathbf{B} && \text{(Distributive property)} \\
&= 0\mathbf{A}+2\mathbf{B} \\
&= 2\mathbf{B} && \text{(Property of the zero matrix)}
\end{aligned}$$

B.3 MATRIX MULTIPLICATION

B.3.1 VERIFYING THE CONDITION FOR MULTIPLICATION

MCQ 53: Which of the following matrix products is possible?

- ☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \times \begin{pmatrix}5 & 6\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} \times \begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$
- ☒ $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} \times \begin{pmatrix}7\\ 8\\ 9\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\end{pmatrix} \times \begin{pmatrix}4 & 5 & 6\end{pmatrix}$

Answer: The product of two matrices $\mathbf{A}\times\mathbf{B}$ is defined only if the number of columns in the first matrix ($\mathbf{A}$) is equal to the number of rows in the second matrix ($\mathbf{B}$).

- Option 1: A 2×2 matrix and a 1×2 matrix. The number of columns of the first (2) is not equal to the number of rows of the second (1). Not possible.
- Option 2: A 3×1 matrix and a 3×1 matrix. The number of columns of the first (1) is not equal to the number of rows of the second (3). Not possible.
- Option 3: A 2×3 matrix and a 3×1 matrix. The number of columns of the first (3) is equal to the number of rows of the second (3). The product is possible.
- Option 4: A 1×3 matrix and a 1×3 matrix. The number of columns of the first (3) is not equal to the number of rows of the second (1). Not possible.

MCQ 54: Which of the following matrix products is possible?

- ☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \times \begin{pmatrix}5 & 6\\ 7 & 8\\ 9 & 10\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\end{pmatrix} \times \begin{pmatrix}4 & 5\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} \times \begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$
- ☒ $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} \times \begin{pmatrix}0 & 1\\ 0 & 1\\ 0 & 1\end{pmatrix}$

Answer: The product of two matrices $\mathbf{A}\times\mathbf{B}$ is defined only if the number of columns in the first matrix ($\mathbf{A}$) is equal to the number of rows in the second matrix ($\mathbf{B}$).

- Option 1: A 2×2 matrix and a 3×2 matrix. The number of columns of the first (2) is not equal to the number of rows of the second (3). Not possible.
- Option 2: A 1×3 matrix and a 1×2 matrix. The number of columns of the first (3) is not equal to the number of rows of the second (1). Not possible.
- Option 3: A 3×1 matrix and a 3×1 matrix. The number of columns of the first (1) is not equal to the number of rows of the second (3). Not possible.
- Option 4: A 2×3 matrix and a 3×2 matrix. The number of columns of the first (3) is equal to the number of rows of the second (3). The product is possible.

MCQ 55: Which of the following matrix products is possible?

- ☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} \times \begin{pmatrix}5\end{pmatrix}$
- ☒ $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} \times \begin{pmatrix}5\\ 6\\ 7\\ 8\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2\end{pmatrix} \times \begin{pmatrix}3 & 4\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\ 2\end{pmatrix} \times \begin{pmatrix}3\\ 4\end{pmatrix}$

Answer: The product of two matrices $\mathbf{A}\times\mathbf{B}$ is defined only if the number of columns in the first matrix ($\mathbf{A}$) is equal to the number of rows in the second matrix ($\mathbf{B}$).

- Option 1: A 2×2 matrix and a 1×1 matrix. The number of columns of the first (2) is not equal to the number of rows of the second (1). Not possible.
- Option 2: A 1×4 matrix and a 4×1 matrix. The number of columns of the first (4) is equal to the number of rows of the second (4). The product is possible.
- Option 3: A 1×2 matrix and a 1×2 matrix. The number of columns of the first (2) is not equal to the number of rows of the second (1). Not possible.
- Option 4: A 2×1 matrix and a 2×1 matrix. The number of columns of the first (1) is not equal to the number of rows of the second (2). Not possible.

B.3.2 DETERMINING THE SIZE OF THE PRODUCT

Ex 56: Let $\mathbf{A}$ be a matrix of size 4×2 and $\mathbf{B}$ be a matrix of size 2×3. What is the size of the product $\mathbf{A} \times \mathbf{B}$?

$$\text{Size: } \boxed{4} \times \boxed{3}$$

Answer: The product $\mathbf{A} \times \mathbf{B}$ of a matrix $\mathbf{A}$ of size $n \times p$ and a matrix $\mathbf{B}$ of size $p \times q$ is defined because the number of columns of $\mathbf{A}$ (p) matches the number of rows of $\mathbf{B}$ (p). The resulting matrix has size $n \times q$.

- Matrix $\mathbf{A}$ has size 4×2.
- Matrix $\mathbf{B}$ has size 2×3.
- The number of columns of $\mathbf{A}$ (2) is equal to the number of rows of $\mathbf{B}$ (2), so the product is defined.
- The resulting matrix will have the number of rows of $\mathbf{A}$ and the number of columns of $\mathbf{B}$, which is 4×3.

Ex 57: Let $\mathbf{A}$ be a matrix of size 5×4 and $\mathbf{B}$ be a matrix of size 4×1. What is the size of the product $\mathbf{A} \times \mathbf{B}$?

$$\text{Size: } \boxed{5} \times \boxed{1}$$

Answer: The product $\mathbf{A} \times \mathbf{B}$ of a matrix $\mathbf{A}$ of size $n \times p$ and a matrix $\mathbf{B}$ of size $p \times q$ is defined because the number of columns of $\mathbf{A}$ (p) matches the number of rows of $\mathbf{B}$ (p). The resulting matrix has size $n \times q$.

- Matrix $\mathbf{A}$ has size 5×4.
- Matrix $\mathbf{B}$ has size 4×1.
- The number of columns of $\mathbf{A}$ (4) is equal to the number of rows of $\mathbf{B}$ (4), so the product is defined.
- The resulting matrix will have the number of rows of $\mathbf{A}$ (5) and the number of columns of $\mathbf{B}$ (1). Therefore, the size is 5×1.

Ex 58: Let $\mathbf{A}$ be a matrix of size 2×3 and $\mathbf{B}$ be a matrix of size 4×2. What is the size of the product $\mathbf{B} \times \mathbf{A}$?

$$\text{Size: } \boxed{4} \times \boxed{3}$$

Answer: The product $\mathbf{B} \times \mathbf{A}$ is defined if the number of columns of $\mathbf{B}$ equals the number of rows of $\mathbf{A}$. The size of the resulting matrix is (rows of $\mathbf{B}$) $\times$ (columns of $\mathbf{A}$).

- Matrix $\mathbf{B}$ has size 4×2.
- Matrix $\mathbf{A}$ has size 2×3.
- The number of columns of $\mathbf{B}$ (2) is equal to the number of rows of $\mathbf{A}$ (2), so the product is defined.
- The resulting matrix will have the number of rows of $\mathbf{B}$ (4) and the number of columns of $\mathbf{A}$ (3). Therefore, the size is 4×3.

B.3.3 CALCULATING MATRIX PRODUCTS

Ex 59: Calculate the multiplication of the following matrices:

$$\begin{pmatrix} 3 & 4 \end{pmatrix} \times \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \boxed{39} \end{pmatrix}$$

Answer: To multiply two matrices, we calculate the dot product of each row of the first matrix with each column of the second matrix. The product of a 1×2 matrix and a 2×1 matrix results in a 1×1 matrix.

$$\begin{array}{cc} & \begin{pmatrix} 5 \\ 6 \end{pmatrix} \\ \begin{pmatrix} 3 & 4 \end{pmatrix} & \begin{aligned} &= \begin{pmatrix} 3 \times 5 + 4 \times 6 \end{pmatrix} \\ &= \begin{pmatrix} 15 + 24 \end{pmatrix} \\ &= \begin{pmatrix} 39 \end{pmatrix} \end{aligned} \end{array}$$

Ex 60: Calculate the multiplication of the following matrices:

$$\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix} \times \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \boxed{17} \\ \boxed{-2} \end{pmatrix}$$

Answer: To multiply two matrices, we calculate the dot product of each row of the first matrix with each column of the second matrix. The product of a 2×2 matrix and a 2×1 matrix results in a 2×1 matrix.

$$\begin{array}{cc} & \begin{pmatrix} 5 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix} & \begin{aligned} &= \begin{pmatrix} 3 \times 5 + 1 \times 2 \\ (-2) \times 5 + 4 \times 2 \end{pmatrix} \\ &= \begin{pmatrix} 15 + 2 \\ -10 + 8 \end{pmatrix} \\ &= \begin{pmatrix} 17 \\ -2 \end{pmatrix} \end{aligned} \end{array}$$

Ex 61: Calculate the multiplication of the following matrices:

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \boxed{11} & \boxed{16} \\ \boxed{1} & \boxed{2} \end{pmatrix}$$

Answer: To multiply two matrices, we calculate the dot product of each row of the first matrix with each column of the second matrix. The product of a 2×2 matrix and a 2×2 matrix results in a 2×2 matrix.

$$\begin{array}{cc} & \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} & \begin{aligned} &= \begin{pmatrix} 2 \times 1 + 3 \times 3 & 2 \times 2 + 3 \times 4 \\ 1 \times 1 + 0 \times 3 & 1 \times 2 + 0 \times 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 16 \\ 1 & 2 \end{pmatrix} \end{aligned} \end{array}$$

Ex 62: Calculate the multiplication of the following matrices:

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 5 \\ -1 & 0 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} \boxed{0} & \boxed{11} \\ \boxed{9} & \boxed{3} \end{pmatrix}$$

Answer: To multiply two matrices, we calculate the dot product of each row of the first matrix with each column of the second matrix. The product of a 2×3 matrix and a 3×2 matrix results in a 2×2 matrix.

$$\begin{pmatrix} 2 & 5 \\ -1 & 0 \\ 1 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot 2 + 0 \cdot (-1) + (-2) \cdot 1 & 1 \cdot 5 + 0 \cdot 0 + (-2) \cdot (-3) \\ 3 \cdot 2 + 1 \cdot (-1) + 4 \cdot 1 & 3 \cdot 5 + 1 \cdot 0 + 4 \cdot (-3) \end{pmatrix}$$

$$= \begin{pmatrix} 2 + 0 - 2 & 5 + 0 + 6 \\ 6 - 1 + 4 & 15 + 0 - 12 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 11 \\ 9 & 3 \end{pmatrix}$$

Ex 63: Calculate the multiplication of the following matrices:

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \end{pmatrix} = \begin{pmatrix} \boxed{15} & \boxed{10} \\ \boxed{-3} & \boxed{-2} \end{pmatrix}$$

Answer: To multiply two matrices, we calculate the dot product of each row of the first matrix with each column of the second matrix. The product of a 2×1 matrix and a 1×2 matrix results in a 2×2 matrix.

$$\begin{pmatrix} 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \times 3 & 5 \times 2 \\ (-1) \times 3 & (-1) \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 15 & 10 \\ -3 & -2 \end{pmatrix}$$

Ex 64: Calculate the multiplication of the following matrices:

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} \boxed{4} & \boxed{16} \\ \boxed{10} & \boxed{38} \end{pmatrix}$$

Answer: To multiply two matrices, we calculate the dot product of each row of the first matrix with each column of the second matrix. The product of a 2×2 matrix and a 2×2 matrix results in another 2×2 matrix.

$$\begin{pmatrix} 1 & 5 \\ 2 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 1 \times 2 & 2 \times 5 + 1 \times 6 \\ 4 \times 1 + 3 \times 2 & 4 \times 5 + 3 \times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 2 + 2 & 10 + 6 \\ 4 + 6 & 20 + 18 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 16 \\ 10 & 38 \end{pmatrix}$$

B.3.4 INVESTIGATING COMMUTATIVITY

Ex 65: Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}$.

1. Calculate the product $\mathbf{AB}$.
2. Calculate the product $\mathbf{BA}$.
3. Hence, conclude whether $\mathbf{AB} = \mathbf{BA}$.

Answer:

1. **Calculate AB**:

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} (1)(3) + (2)(1) & (1)(0) + (2)(-1) \\ (0)(3) + (1)(1) & (0)(0) + (1)(-1) \end{pmatrix} \\ &= \begin{pmatrix} 3 + 2 & 0 - 2 \\ 0 + 1 & 0 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -2 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

2. **Calculate BA**:

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (3)(1) + (0)(0) & (3)(2) + (0)(1) \\ (1)(1) + (-1)(0) & (1)(2) + (-1)(1) \end{pmatrix} \\ &= \begin{pmatrix} 3 + 0 & 6 + 0 \\ 1 - 0 & 2 - 1 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 6 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

3. **Conclusion**: Since $\begin{pmatrix} 5 & -2 \\ 1 & -1 \end{pmatrix} \neq \begin{pmatrix} 3 & 6 \\ 1 & 1 \end{pmatrix}$, we conclude that $\mathbf{AB} \neq \mathbf{BA}$. Matrix multiplication is not commutative.

Ex 66: Let $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix}$.

1. Calculate the product $\mathbf{AB}$.
2. Calculate the product $\mathbf{BA}$.
3. Hence, conclude whether $\mathbf{AB} = \mathbf{BA}$.

Answer:

1. **Calculate AB**:

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (0)(2) + (1)(3) & (0)(5) + (1)(1) \\ (1)(2) + (0)(3) & (1)(5) + (0)(1) \end{pmatrix} \\ &= \begin{pmatrix} 0 + 3 & 0 + 1 \\ 2 + 0 & 5 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \end{aligned}$$

2. **Calculate BA**:

$$\begin{aligned} \mathbf{BA} &= \begin{pmatrix} 2 & 5 \\ 3 & 1 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (2)(0) + (5)(1) & (2)(1) + (5)(0) \\ (3)(0) + (1)(1) & (3)(1) + (1)(0) \end{pmatrix} \\ &= \begin{pmatrix} 0 + 5 & 2 + 0 \\ 0 + 1 & 3 + 0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix} \end{aligned}$$

3. **Conclusion**: Since $\begin{pmatrix} 3 & 1 \\ 2 & 5 \end{pmatrix} \neq \begin{pmatrix} 5 & 2 \\ 1 & 3 \end{pmatrix}$, we conclude that $\mathbf{AB} \neq \mathbf{BA}$.

Ex 67: Let $\mathbf{A} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Calculate the product $\mathbf{AI}_2$.
2. Calculate the product $\mathbf{I}_2\mathbf{A}$.
3. Hence, conclude whether $\mathbf{AI}_2 = \mathbf{I}_2\mathbf{A}$.

Answer:

1. **Calculate $\mathbf{AI}_2$:**

$$\begin{aligned} \mathbf{AI}_2 &= \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} (3)(1)+(5)(0) & (3)(0)+(5)(1) \\ (1)(1)+(2)(0) & (1)(0)+(2)(1) \end{pmatrix} \\ &= \begin{pmatrix} 3+0 & 0+5 \\ 1+0 & 0+2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

2. **Calculate $\mathbf{I}_2\mathbf{A}$:**

$$\begin{aligned} \mathbf{I}_2\mathbf{A} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (1)(3)+(0)(1) & (1)(5)+(0)(2) \\ (0)(3)+(1)(1) & (0)(5)+(1)(2) \end{pmatrix} \\ &= \begin{pmatrix} 3+0 & 5+0 \\ 0+1 & 0+2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

3. **Conclusion**: We observe that both products are equal to the original matrix $\mathbf{A}$.

$$\begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$$

Therefore, we conclude that $\mathbf{AI}_2 = \mathbf{I}_2\mathbf{A}$. This illustrates that the identity matrix is the neutral element for matrix multiplication and commutes with any square matrix of the same order.

B.3.5 EXPANDING MATRIX EXPRESSIONS

Ex 68: For any square matrix $\mathbf{A}$, expand and simplify the expression $\mathbf{A}(\mathbf{A}+\mathbf{I})$, where $\mathbf{I}$ is the identity matrix of the same order as $\mathbf{A}$.

Answer: To simplify the expression, we use the left distributive property of matrix multiplication and the property of the identity matrix.

$$\begin{aligned} \mathbf{A}(\mathbf{A}+\mathbf{I}) &= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{I} && \text{(Left distributivity: } \mathbf{A(B+C)=AB+AC}\text{)} \\ &= \mathbf{A}^2 + \mathbf{A} && \text{(Property of identity matrix: } \mathbf{AI=A}\text{)} \end{aligned}$$

Ex 69: For any square matrix $\mathbf{A}$, expand and simplify the expression $(\mathbf{A}+\mathbf{I})^2$, where $\mathbf{I}$ is the identity matrix of the same order as $\mathbf{A}$.

Answer: To expand and simplify the expression, we first write it as a product and then apply the distributive property of matrix multiplication.

$$\begin{aligned} (\mathbf{A}+\mathbf{I})^2 &= (\mathbf{A}+\mathbf{I})(\mathbf{A}+\mathbf{I}) && \text{(Definition of square)} \\ &= \mathbf{A}(\mathbf{A}+\mathbf{I}) + \mathbf{I}(\mathbf{A}+\mathbf{I}) && \text{(Left distributivity)} \\ &= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{I} + \mathbf{I}\cdot\mathbf{A} + \mathbf{I}\cdot\mathbf{I} && \text{(Right distributivity)} \\ &= \mathbf{A}^2 + \mathbf{A} + \mathbf{A} + \mathbf{I} && \text{(Property of identity matrix: } \mathbf{AI=A,I}\text{)} \\ &= \mathbf{A}^2 + 2\mathbf{A} + \mathbf{I} \end{aligned}$$

Ex 70: For any two square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, expand and simplify the expression $(\mathbf{A}+\mathbf{B})^2$.

Answer: To expand and simplify the expression, we first write it as a product and then carefully apply the distributive property of matrix multiplication.

$$\begin{aligned} (\mathbf{A}+\mathbf{B})^2 &= (\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) && \text{(Definition of square)} \\ &= \mathbf{A}(\mathbf{A}+\mathbf{B}) + \mathbf{B}(\mathbf{A}+\mathbf{B}) && \text{(Left distributivity)} \\ &= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} && \text{(Right distributivity)} \\ &= \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 \end{aligned}$$

Note: Since matrix multiplication is not commutative in general ($\mathbf{AB} \neq \mathbf{BA}$), the expression $\mathbf{AB}+\mathbf{BA}$ cannot be simplified to $2\mathbf{AB}$. Therefore, the expression is left in the form $\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2$.

Ex 71: For any square matrix $\mathbf{A}$, expand and simplify the expression $(\mathbf{A}+3\mathbf{I})^2$, where $\mathbf{I}$ is the identity matrix of the same order as $\mathbf{A}$.

Answer: To expand and simplify the expression, we write it as a product and then apply the distributive property of matrix multiplication.

$$\begin{aligned} &(\mathbf{A}+3\mathbf{I})^2 \\ &= (\mathbf{A}+3\mathbf{I})(\mathbf{A}+3\mathbf{I}) && \text{(Definition of square)} \\ &= \mathbf{A}(\mathbf{A}+3\mathbf{I}) + 3\mathbf{I}(\mathbf{A}+3\mathbf{I}) && \text{(Left distributivity)} \\ &= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot(3\mathbf{I}) + (3\mathbf{I})\cdot\mathbf{A} + (3\mathbf{I})\cdot(3\mathbf{I}) && \text{(Right distributivity)} \\ &= \mathbf{A}^2 + 3(\mathbf{AI}) + 3(\mathbf{IA}) + (3\cdot 3)(\mathbf{I}\cdot\mathbf{I}) && \text{(Associativity of scalar multiplication)} \\ &= \mathbf{A}^2 + 3\mathbf{A} + 3\mathbf{A} + 9\mathbf{I}^2 && \text{(Property of identity matrix: } \mathbf{AI=A,IA=A}\text{)} \\ &= \mathbf{A}^2 + 6\mathbf{A} + 9\mathbf{I} && \text{(Since } \mathbf{I}^2=\mathbf{I}\text{)} \end{aligned}$$

B.3.6 SIMPLIFYING POWERS OF A MATRIX

Ex 72: Given that a square matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2 = \mathbf{A}+\mathbf{I}$, find the expressions for $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars and $\mathbf{I}$ is the identity matrix.

Answer: We are given the relation $\mathbf{A}^2 = \mathbf{A}+\mathbf{I}$. We will use this relation to find expressions for higher powers of $\mathbf{A}$.

1. **Finding $\mathbf{A}^3$:**
We start by writing $\mathbf{A}^3$ as $\mathbf{A}\cdot\mathbf{A}^2$.

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A}\cdot\mathbf{A}^2 \\ &= \mathbf{A}(\mathbf{A}+\mathbf{I}) && \text{(Substitute the given relation for } \mathbf{A}^2\text{)} \\ &= \mathbf{A}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{I} && \text{(Distributive property)} \\ &= \mathbf{A}^2 + \mathbf{A} && \text{(Property of the identity matrix, } \mathbf{AI=A}\text{)} \\ &= (\mathbf{A}+\mathbf{I}) + \mathbf{A} && \text{(Substitute the relation for } \mathbf{A}^2 \text{ again)} \\ &= 2\mathbf{A} + \mathbf{I} \end{aligned}$$

So, $\mathbf{A}^3 = 2\mathbf{A}+\mathbf{I}$, which is in the form $k\mathbf{A}+l\mathbf{I}$ with $k=2$ and $l=1$.

2. **Finding $\mathbf{A}^4$**:
We can write $\mathbf{A}^4$ as $\mathbf{A}\cdot\mathbf{A}^3$.

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{A}\cdot\mathbf{A}^3\\ &= \mathbf{A}(2\mathbf{A}+\mathbf{I}) && \text{(Using the result for } \mathbf{A}^3\text{)}\\ &= 2(\mathbf{A}\cdot\mathbf{A})+\mathbf{A}\cdot\mathbf{I} && \text{(Distributive property)}\\ &= 2\mathbf{A}^2+\mathbf{A} && \text{(Property of the identity matrix, } \mathbf{AI}=\mathbf{A}\text{)}\\ &= 2(\mathbf{A}+\mathbf{I})+\mathbf{A} && \text{(Substitute the relation for } \mathbf{A}^2\text{)}\\ &= 2\mathbf{A}+2\mathbf{I}+\mathbf{A}\\ &= 3\mathbf{A}+2\mathbf{I}\end{aligned}$$

So, $\mathbf{A}^4 = 3\mathbf{A}+2\mathbf{I}$, which is in the form $k\mathbf{A}+l\mathbf{I}$ with $k=3$ and $l=2$.

Ex 73: Given that a square matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2 = \mathbf{A}-\mathbf{I}$, find the expressions for $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars and $\mathbf{I}$ is the identity matrix.

Answer: We are given the relation $\mathbf{A}^2=\mathbf{A}-\mathbf{I}$. We will use this relation to find expressions for higher powers of $\mathbf{A}$.

1. **Finding $\mathbf{A}^3$**:
We start by writing $\mathbf{A}^3$ as $\mathbf{A}\cdot\mathbf{A}^2$.

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{A}\cdot\mathbf{A}^2\\ &= \mathbf{A}(\mathbf{A}-\mathbf{I}) && \text{(Substitute the given relation for } \mathbf{A}^2\text{)}\\ &= \mathbf{A}\cdot\mathbf{A}-\mathbf{A}\cdot\mathbf{I} && \text{(Distributive property)}\\ &= \mathbf{A}^2-\mathbf{A} && \text{(Property of the identity matrix, } \mathbf{AI}=\mathbf{A}\text{)}\\ &= (\mathbf{A}-\mathbf{I})-\mathbf{A} && \text{(Substitute the relation for } \mathbf{A}^2 \text{ again)}\\ &= \mathbf{A}-\mathbf{A}-\mathbf{I}\\ &= -\mathbf{I}\end{aligned}$$

So, $\mathbf{A}^3 = 0\mathbf{A}-\mathbf{I}$, which is in the form $k\mathbf{A}+l\mathbf{I}$ with $k=0$ and $l=-1$.

2. **Finding $\mathbf{A}^4$**:
We can write $\mathbf{A}^4$ as $\mathbf{A}\cdot\mathbf{A}^3$.

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{A}\cdot\mathbf{A}^3\\ &= \mathbf{A}(-\mathbf{I}) && \text{(Using the result for } \mathbf{A}^3\text{)}\\ &= -(\mathbf{A}\cdot\mathbf{I})\\ &= -\mathbf{A} && \text{(Property of the identity matrix, } \mathbf{AI}=\mathbf{A}\text{)}\end{aligned}$$

So, $\mathbf{A}^4 = -\mathbf{A}+0\mathbf{I}$, which is in the form $k\mathbf{A}+l\mathbf{I}$ with $k=-1$ and $l=0$.

Ex 74: Given that a square matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2 = 2\mathbf{A}+3\mathbf{I}$, find the expressions for $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars and $\mathbf{I}$ is the identity matrix.

Answer: We are given the relation $\mathbf{A}^2=2\mathbf{A}+3\mathbf{I}$. We will use this relation to find expressions for higher powers of $\mathbf{A}$.

1. **Finding $\mathbf{A}^3$**:
We start by writing $\mathbf{A}^3$ as $\mathbf{A}\cdot\mathbf{A}^2$.

$$\begin{aligned}\mathbf{A}^3 &= \mathbf{A}\cdot\mathbf{A}^2\\ &= \mathbf{A}(2\mathbf{A}+3\mathbf{I}) && \text{(Substitute the given relation for } \mathbf{A}^2\text{)}\\ &= 2(\mathbf{A}\cdot\mathbf{A})+3(\mathbf{A}\cdot\mathbf{I}) && \text{(Distributive property)}\\ &= 2\mathbf{A}^2+3\mathbf{A} && \text{(Property of the identity matrix, } \mathbf{AI}=\mathbf{A}\text{)}\\ &= 2(2\mathbf{A}+3\mathbf{I})+3\mathbf{A} && \text{(Substitute the relation for } \mathbf{A}^2 \text{ again)}\\ &= 4\mathbf{A}+6\mathbf{I}+3\mathbf{A}\\ &= 7\mathbf{A}+6\mathbf{I}\end{aligned}$$

So, $\mathbf{A}^3 = 7\mathbf{A}+6\mathbf{I}$, which is in the form $k\mathbf{A}+l\mathbf{I}$ with $k=7$ and $l=6$.

2. **Finding $\mathbf{A}^4$**:
We can write $\mathbf{A}^4$ as $\mathbf{A}\cdot\mathbf{A}^3$.

$$\begin{aligned}\mathbf{A}^4 &= \mathbf{A}\cdot\mathbf{A}^3\\ &= \mathbf{A}(7\mathbf{A}+6\mathbf{I}) && \text{(Using the result for } \mathbf{A}^3\text{)}\\ &= 7(\mathbf{A}\cdot\mathbf{A})+6(\mathbf{A}\cdot\mathbf{I}) && \text{(Distributive property)}\\ &= 7\mathbf{A}^2+6\mathbf{A} && \text{(Property of the identity matrix, } \mathbf{AI}=\mathbf{A}\text{)}\\ &= 7(2\mathbf{A}+3\mathbf{I})+6\mathbf{A} && \text{(Substitute the relation for } \mathbf{A}^2\text{)}\\ &= 14\mathbf{A}+21\mathbf{I}+6\mathbf{A}\\ &= 20\mathbf{A}+21\mathbf{I}\end{aligned}$$

So, $\mathbf{A}^4 = 20\mathbf{A}+21\mathbf{I}$, which is in the form $k\mathbf{A}+l\mathbf{I}$ with $k=20$ and $l=21$.

C INVERTIBLE MATRICES

C.1 DEFINITION

C.1.1 VERIFYING AN INVERSE BY DEFINITION

Ex 75: Let $\mathbf{A}=\begin{pmatrix}2&5\\1&3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3&-5\\-1&2\end{pmatrix}$.

1. Calculate the product $\mathbf{AB}$.
2. Calculate the product $\mathbf{BA}$.
3. What can you conclude about the relationship between matrices $\mathbf{A}$ and $\mathbf{B}$?

Answer:

1. **Calculate $\mathbf{AB}$**:

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix}2&5\\1&3\end{pmatrix}\times\begin{pmatrix}3&-5\\-1&2\end{pmatrix}\\ &= \begin{pmatrix}(2)(3)+(5)(-1) & (2)(-5)+(5)(2)\\ (1)(3)+(3)(-1) & (1)(-5)+(3)(2)\end{pmatrix}\\ &= \begin{pmatrix}6-5 & -10+10\\ 3-3 & -5+6\end{pmatrix}\\ &= \begin{pmatrix}1&0\\0&1\end{pmatrix} = \mathbf{I}_2\end{aligned}$$

2. **Calculate $\mathbf{BA}$**:

$$\begin{aligned}\mathbf{BA} &= \begin{pmatrix}3&-5\\-1&2\end{pmatrix}\times\begin{pmatrix}2&5\\1&3\end{pmatrix}\\ &= \begin{pmatrix}(3)(2)+(-5)(1) & (3)(5)+(-5)(3)\\ (-1)(2)+(2)(1) & (-1)(5)+(2)(3)\end{pmatrix}\\ &= \begin{pmatrix}6-5 & 15-15\\ -2+2 & -5+6\end{pmatrix}\\ &= \begin{pmatrix}1&0\\0&1\end{pmatrix} = \mathbf{I}_2\end{aligned}$$

3. **Conclusion**: Since we have shown that $\mathbf{AB}=\mathbf{I}_2$ and $\mathbf{BA}=\mathbf{I}_2$, by the definition of an inverse matrix, we can conclude that matrix $\mathbf{B}$ is the inverse of matrix $\mathbf{A}$. We can write $\mathbf{B}=\mathbf{A}^{-1}$. This also means that $\mathbf{A}$ is the inverse of $\mathbf{B}$.

Ex 76: Let $\mathbf{A}=\begin{pmatrix}7&2\\3&1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1&-2\\-3&7\end{pmatrix}$.

1. Calculate the product **AB**.
2. Calculate the product **BA**.
3. What can you conclude about the relationship between matrices **A** and **B**?

Answer:

1. **Calculate AB**:

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix}7 & 2\\3 & 1\end{pmatrix} \times \begin{pmatrix}1 & -2\\-3 & 7\end{pmatrix}\\ &= \begin{pmatrix}(7)(1)+(2)(-3) & (7)(-2)+(2)(7)\\(3)(1)+(1)(-3) & (3)(-2)+(1)(7)\end{pmatrix}\\ &= \begin{pmatrix}7-6 & -14+14\\3-3 & -6+7\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I}_2\end{aligned}$$

2. **Calculate BA**:

$$\begin{aligned}\mathbf{BA} &= \begin{pmatrix}1 & -2\\-3 & 7\end{pmatrix} \times \begin{pmatrix}7 & 2\\3 & 1\end{pmatrix}\\ &= \begin{pmatrix}(1)(7)+(-2)(3) & (1)(2)+(-2)(1)\\(-3)(7)+(7)(3) & (-3)(2)+(7)(1)\end{pmatrix}\\ &= \begin{pmatrix}7-6 & 2-2\\-21+21 & -6+7\end{pmatrix}\\ &= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbf{I}_2\end{aligned}$$

3. **Conclusion**: Since we have shown that $\mathbf{AB} = \mathbf{I}_2$ and $\mathbf{BA} = \mathbf{I}_2$, by the definition of an inverse matrix, we can conclude that matrix **B** is the inverse of matrix **A**. We can write $\mathbf{B} = \mathbf{A}^{-1}$.

Ex 77: Let $\mathbf{A} = \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}2 & 0\\1 & 3\end{pmatrix}$.

1. Calculate the product **AB**.
2. Based on this result, can you conclude whether **B** is the inverse of **A**?

Answer:

1. **Calculate AB**:

$$\begin{aligned}\mathbf{AB} &= \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} \times \begin{pmatrix}2 & 0\\1 & 3\end{pmatrix}\\ &= \begin{pmatrix}(1)(2)+(2)(1) & (1)(0)+(2)(3)\\(3)(2)+(4)(1) & (3)(0)+(4)(3)\end{pmatrix}\\ &= \begin{pmatrix}2+2 & 0+6\\6+4 & 0+12\end{pmatrix}\\ &= \begin{pmatrix}4 & 6\\10 & 12\end{pmatrix}\end{aligned}$$

2. **Conclusion**: The definition of an inverse states that for **B** to be the inverse of **A**, the product **AB** must be equal to the identity matrix $\mathbf{I}_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Since we found that

$$\mathbf{AB} = \begin{pmatrix}4 & 6\\10 & 12\end{pmatrix} \neq \mathbf{I}_2$$

the condition is not met. Therefore, we can conclude that **B is not the inverse of A**. It is not necessary to calculate **BA**.

C.1.2 PROVING PROPERTIES OF THE INVERSE

Ex 78: Prove that the identity matrix, **I**, is invertible and that its inverse is itself (i.e., $\mathbf{I}^{-1} = \mathbf{I}$).

Answer: To prove that **I** is its own inverse, we must verify that $\mathbf{I}\cdot\mathbf{I} = \mathbf{I}$, according to the definition of an inverse ($\mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{I}$ and $\mathbf{A}^{-1}\cdot\mathbf{A} = \mathbf{I}$).

$$\mathbf{I}\cdot\mathbf{I} = \mathbf{I} \quad \text{(By the property of the identity matrix as the neutral element)}$$

Since the condition is met, **I** is invertible and its inverse is **I**.

Ex 79: Let **A**, **B**, and **C** be square matrices of the same order. Suppose that **A** is an invertible matrix. Prove that if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

Answer: To prove this cancellation property, we start with the given equation and use the properties of the inverse matrix.

$$\begin{aligned}\mathbf{AB} &= \mathbf{AC} && \text{(Given)}\\ \mathbf{A}^{-1}(\mathbf{AB}) &= \mathbf{A}^{-1}(\mathbf{AC}) && \text{(Left-multiply by } \mathbf{A}^{-1}\text{, which exists since } \mathbf{A} \text{ is invertible)}\\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{B} &= (\mathbf{A}^{-1}\mathbf{A})\mathbf{C} && \text{(Associativity of matrix multiplication)}\\ \mathbf{I}\cdot\mathbf{B} &= \mathbf{I}\cdot\mathbf{C} && \text{(Definition of inverse: } \mathbf{A}^{-1}\mathbf{A}=\mathbf{I})\\ \mathbf{B} &= \mathbf{C} && \text{(Property of the identity matrix)}\end{aligned}$$

This completes the proof.

Ex 80: Let **A** be an invertible square matrix, and let **X** and **B** be matrices of compatible sizes. Prove that if $\mathbf{AX} = \mathbf{B}$, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Answer: To prove this, we start with the equation $\mathbf{AX} = \mathbf{B}$ and isolate **X** by using the properties of the inverse matrix.

$$\begin{aligned}\mathbf{AX} &= \mathbf{B} && \text{(Given)}\\ \mathbf{A}^{-1}(\mathbf{AX}) &= \mathbf{A}^{-1}\mathbf{B} && \text{(Left-multiply by } \mathbf{A}^{-1}\text{, which exists since } \mathbf{A} \text{ is invertible)}\\ (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} && \text{(Associativity of matrix multiplication)}\\ \mathbf{I}\cdot\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} && \text{(Definition of inverse: } \mathbf{A}^{-1}\mathbf{A}=\mathbf{I})\\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} && \text{(Property of the identity matrix)}\end{aligned}$$

This shows that if $\mathbf{AX} = \mathbf{B}$ and **A** is invertible, then **X** must be equal to $\mathbf{A}^{-1}\mathbf{B}$.

Ex 81: Let **A** be an invertible matrix. Suppose there are two matrices, **B** and **C**, such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$. Prove that $\mathbf{B} = \mathbf{C}$. (This shows the inverse is unique).

Answer: To prove that the inverse is unique, we start with the assumption that a matrix **A** has two inverses, **B** and **C**. We will show they must be the same.

$$\begin{aligned}\mathbf{B} &= \mathbf{B}\cdot\mathbf{I} && \text{(Property of the identity matrix)}\\ &= \mathbf{B}(\mathbf{AC}) && \text{(Given that } \mathbf{C} \text{ is an inverse of } \mathbf{A}\text{, so } \mathbf{AC}=\mathbf{I})\\ &= (\mathbf{BA})\mathbf{C} && \text{(Associativity of matrix multiplication)}\\ &= \mathbf{I}\cdot\mathbf{C} && \text{(Given that } \mathbf{B} \text{ is an inverse of } \mathbf{A}\text{, so } \mathbf{BA}=\mathbf{I})\\ &= \mathbf{C} && \text{(Property of the identity matrix)}\end{aligned}$$

Since we started with **B** and ended with **C**, we have shown that $\mathbf{B} = \mathbf{C}$. Therefore, the inverse of a matrix, if it exists, is unique.

C.2 FINDING THE INVERSE OF A 2X2 MATRIX

C.2.1 CALCULATING THE DETERMINANT

Ex 82: Calculate the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\det(\mathbf{A}) = \boxed{14}$$

Answer: The determinant of a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by the formula $\det(\mathbf{A}) = ad - bc$.For the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$, we have $a = 5$, $b = 2$, $c = 3$, and $d = 4$.

$$\begin{aligned} \det(\mathbf{A}) &= (5)(4) - (2)(3) \\ &= 20 - 6 \\ &= 14 \end{aligned}$$

Ex 83: Calculate the determinant of the matrix $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 7 & -5 \end{pmatrix}$.

$$\det(\mathbf{B}) = \boxed{5}$$

Answer: For the matrix $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 7 & -5 \end{pmatrix}$, we use the formula $\det(\mathbf{B}) = ad - bc$ with $a = -1$, $b = 0$, $c = 7$, and $d = -5$.

$$\begin{aligned} \det(\mathbf{B}) &= (-1)(-5) - (0)(7) \\ &= 5 - 0 \\ &= 5 \end{aligned}$$

Ex 84: Calculate the determinant of the matrix $\mathbf{C} = \begin{pmatrix} 6 & 3 \\ 8 & 4 \end{pmatrix}$.

$$\det(\mathbf{C}) = \boxed{0}$$

Answer: For the matrix $\mathbf{C} = \begin{pmatrix} 6 & 3 \\ 8 & 4 \end{pmatrix}$, we use the formula $\det(\mathbf{C}) = ad - bc$ with $a = 6$, $b = 3$, $c = 8$, and $d = 4$.

$$\begin{aligned} \det(\mathbf{C}) &= (6)(4) - (3)(8) \\ &= 24 - 24 \\ &= 0 \end{aligned}$$

Since the determinant is 0, this matrix is singular (not invertible).

C.2.2 FINDING THE INVERSE OF A 2X2 MATRIX

Ex 85: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ exists, and if so, find it.

Answer: To find the inverse of a 2x2 matrix, we must first calculate its determinant to determine if it is invertible.

1. **Calculate the determinant**:
   For $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, the determinant is:
   $$\det(\mathbf{A}) = (1)(2) - (0)(0) = 2$$
   Since $\det(\mathbf{A}) = 2 \neq 0$, the matrix is invertible, and its inverse exists.
2. **Calculate the inverse**:
   We use the formula $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
   $$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{2} \begin{pmatrix} 2 & -0 \\ -0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}(2) & \frac{1}{2}(0) \\ \frac{1}{2}(0) & \frac{1}{2}(1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \end{aligned}$$

Ex 86: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ exists, and if so, find it.

Answer: To determine if the inverse of a 2x2 matrix exists, we must first calculate its determinant.

1. **Calculate the determinant**:
   For $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, the determinant is:
   $$\det(\mathbf{A}) = (1)(1) - (1)(1) = 1 - 1 = 0$$
2. **Conclusion**:
   A matrix is invertible if and only if its determinant is non-zero. Since $\det(\mathbf{A}) = 0$, the matrix $\mathbf{A}$ is singular and therefore **not invertible**. Its inverse does not exist.

Ex 87: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ exists, and if so, find it.

Answer: To find the inverse of a 2x2 matrix, we first calculate its determinant to check for invertibility.

1. **Calculate the determinant**:
   For $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, the determinant is:
   $$\det(\mathbf{A}) = (2)(3) - (5)(1) = 6 - 5 = 1$$
   Since $\det(\mathbf{A}) = 1 \neq 0$, the matrix is invertible.
2. **Calculate the inverse**:
   We use the formula $\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
   $$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{1} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \end{aligned}$$

Ex 88: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$ exists, and if so, find it.

Answer: To find the inverse of a 2x2 matrix, we first calculate its determinant to check for invertibility.

1. **Calculate the determinant**:
   For $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$, the determinant is:
   $$\det(\mathbf{A}) = (5)(4) - (6)(3) = 20 - 18 = 2$$
   Since $\det(\mathbf{A}) = 2 \neq 0$, the matrix is invertible.

2. **Calculate the inverse**:
We use the formula $\mathbf{A}^{-1} = \dfrac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{2}\begin{pmatrix} 4 & -6 \\ -3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} \frac{4}{2} & \frac{-6}{2} \\ \frac{-3}{2} & \frac{5}{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}\end{aligned}$$

Ex 89: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ exists, and if so, find it.

Answer: To determine if the inverse of a 2x2 matrix exists, we must first calculate its determinant.

1. **Calculate the determinant**:
For $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$, the determinant is:

$$\det(\mathbf{A}) = (2)(2) - (4)(1) = 4 - 4 = 0$$

2. **Conclusion**:
Since $\det(\mathbf{A}) = 0$, the matrix $\mathbf{A}$ is singular and therefore **not invertible**. Its inverse does not exist.

C.2.3 FINDING THE CONDITION FOR INVERTIBILITY

Ex 90: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & k \end{pmatrix}$ is invertible.

Answer:

$$\begin{aligned}\mathbf{A} \text{ is invertible } &\Leftrightarrow \det(\mathbf{A}) \neq 0 \\ &\Leftrightarrow (2)(k) - (3)(1) \neq 0 \\ &\Leftrightarrow 2k - 3 \neq 0 \\ &\Leftrightarrow k \neq \frac{3}{2}\end{aligned}$$

Ex 91: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} 2k & 3 \\ k & 1 \end{pmatrix}$ is invertible.

Answer:

$$\begin{aligned}\mathbf{A} \text{ is invertible } &\Leftrightarrow \det(\mathbf{A}) \neq 0 \\ &\Leftrightarrow (2k)(1) - (3)(k) \neq 0 \\ &\Leftrightarrow 2k - 3k \neq 0 \\ &\Leftrightarrow -k \neq 0 \\ &\Leftrightarrow k \neq 0\end{aligned}$$

Ex 92: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} k & 1 \\ 0 & k+1 \end{pmatrix}$ is invertible.

Answer:

$$\begin{aligned}\mathbf{A} \text{ is invertible } &\Leftrightarrow \det(\mathbf{A}) \neq 0 \\ &\Leftrightarrow (k)(k+1) - (1)(0) \neq 0 \\ &\Leftrightarrow k(k+1) \neq 0 \\ &\Leftrightarrow k \neq 0 \text{ and } k \neq -1\end{aligned}$$

Ex 93: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} 1 & k-1 \\ k & 2 \end{pmatrix}$ is invertible.

Answer:

$$\begin{aligned}\mathbf{A} \text{ is invertible } &\Leftrightarrow \det(\mathbf{A}) \neq 0 \\ &\Leftrightarrow (1)(2) - (k-1)(k) \neq 0 \\ &\Leftrightarrow 2 - (k^2 - k) \neq 0 \\ &\Leftrightarrow -k^2 + k + 2 \neq 0 \\ &\Leftrightarrow -(k-2)(k+1) \neq 0 \\ &\Leftrightarrow k \neq 2 \text{ and } k \neq -1\end{aligned}$$

D APPLICATIONS

D.1 SOLVING SYSTEMS OF LINEAR EQUATIONS

D.1.1 WRITING A SYSTEM IN MATRIX FORM

Ex 94: Write the system $\begin{cases} 2x + 5y &= 2 \\ x + 3y &= 5 \end{cases}$ in matrix form.

Answer: In matrix form, the system is

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Ex 95: Write the system $\begin{cases} x - 2y &= 7 \\ 3x + y &= 0 \end{cases}$ in matrix form.

Answer: In matrix form, the system is

$$\begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Ex 96: Write the system $\begin{cases} x + y - z &= 9 \\ 2y + 4z &= -2 \\ 5x - 6z &= 0 \end{cases}$ in matrix form.

Answer: In matrix form, the system is

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \\ 5 & 0 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -2 \\ 0 \end{pmatrix}$$

D.1.2 SOLVING SYSTEMS WITH THE INVERSE METHOD

Ex 97: Use the matrix method to solve the following system of linear equations:

$$\begin{cases} 2x + 5y &= 2 \\ x + 3y &= 5 \end{cases}$$

Answer: To solve the system using matrices, we follow three main steps: represent the system in matrix form, find the inverse of the coefficient matrix, and then multiply to find the solution.

1. **Write the system in matrix form $\mathbf{AX} = \mathbf{B}$**:

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Here, $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

2. **Find the inverse of the coefficient matrix $\mathbf{A}$**: First, we calculate the determinant of $\mathbf{A}$:

$$\det(\mathbf{A}) = (2)(3) - (5)(1) = 6 - 5 = 1$$

Since $\det(\mathbf{A}) \neq 0$, the matrix is invertible. We use the formula for the inverse:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

3. **Calculate the solution using $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$**:

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} (3)(2) + (-5)(5) \\ (-1)(2) + (2)(5) \end{pmatrix} \\ &= \begin{pmatrix} 6 - 25 \\ -2 + 10 \end{pmatrix} \\ &= \begin{pmatrix} -19 \\ 8 \end{pmatrix}\end{aligned}$$

4. **Conclusion**: By matrix equality, the solution is $x = -19$ and $y = 8$.

Ex 98: Use the matrix method to solve the following system of linear equations:

$$\begin{cases} 3x + y &= 8 \\ x + 2y &= 9 \end{cases}$$

Answer: We follow the three steps of the matrix inverse method.

1. **Write the system in matrix form $\mathbf{AX} = \mathbf{B}$**:

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

Here, $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$.

2. **Find the inverse of the coefficient matrix $\mathbf{A}$**: First, the determinant: $\det(\mathbf{A}) = (3)(2) - (1)(1) = 6 - 1 = 5$. Since $\det(\mathbf{A}) \neq 0$, the inverse exists.

$$\mathbf{A}^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

3. **Calculate the solution using $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$**:

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 8 \\ 9 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} (2)(8) + (-1)(9) \\ (-1)(8) + (3)(9) \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 16 - 9 \\ -8 + 27 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ 19 \end{pmatrix} \\ &= \begin{pmatrix} 7/5 \\ 19/5 \end{pmatrix}\end{aligned}$$

4. **Conclusion**: The solution is $x = 7/5$ and $y = 19/5$.

Ex 99: Use the matrix method to solve the following system of linear equations:

$$\begin{cases} 5x - 2y &= 1 \\ 4x - y &= 4 \end{cases}$$

Answer: We follow the three steps of the matrix inverse method.

1. **Write the system in matrix form $\mathbf{AX} = \mathbf{B}$**:

$$\begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

2. **Find the inverse of the coefficient matrix $\mathbf{A}$**: First, the determinant: $\det(\mathbf{A}) = (5)(-1) - (-2)(4) = -5 - (-8) = 3$. Since $\det(\mathbf{A}) \neq 0$, the inverse exists.

$$\mathbf{A}^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}$$

3. **Calculate the solution using $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$**:

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{3}\begin{pmatrix} -1 & 2 \\ -4 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} (-1)(1) + (2)(4) \\ (-4)(1) + (5)(4) \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -1 + 8 \\ -4 + 20 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 7 \\ 16 \end{pmatrix} \\ &= \begin{pmatrix} 7/3 \\ 16/3 \end{pmatrix}\end{aligned}$$

4. **Conclusion**: The solution is $x = 7/3$ and $y = 16/3$.