MATRICES

A STRUCTURE

A.1 DEFINITION

A.1.1 IDENTIFYING THE SIZE OF A MATRIX

Ex 1: What is the size of the following matrix?

$$\mathbf{A} = \begin{pmatrix} -2 & 0 & 7 \\ 1 & 9 & 4 \end{pmatrix}$$

Size: $\square \times \square$

Ex 2: What is the size of the following matrix?

$$\mathbf{B} = \begin{pmatrix} -5 \\ 0 \\ 3 \\ 1 \end{pmatrix}$$

Size: $\square \times \square$

Ex 3: What is the size of the following matrix?

$$\mathbf{C} = \begin{pmatrix} 10 & 20 & 30 & 40 & 50 \end{pmatrix}$$

Size: $\square \times \square$

Ex 4: What is the size of the following matrix?

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Size: $\square \times \square$

A.1.2 IDENTIFYING THE ENTRIES OF A MATRIX

Ex 5: Consider the matrix **A** defined as:

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 2 & 6 \end{pmatrix}$$

What is the value of the entry a_{13}?

$$a_{13} = \square$$

Ex 6: Consider the matrix **A** defined as:

$$\mathbf{A} = \begin{pmatrix} 8 & -1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & \frac{1}{2} \end{pmatrix}$$

What is the value of the entry a_{33}?

$$a_{33} = \square$$

Ex 7: Consider the matrix **B** defined as:

$$\mathbf{B} = \begin{pmatrix} 1 & \sqrt{2} & 0 & 9 \\ -5 & 3 & 11 & 8 \end{pmatrix}$$

What is the value of the entry b_{12}?

$$b_{12} = \square$$

Ex 8: Consider the matrix **C** defined as:

$$\mathbf{C} = \begin{pmatrix} \pi & 1 \\ -1 & 0 \\ 7 & \sqrt{3} \end{pmatrix}$$

What is the value of the entry c_{31}?

$$c_{31} = \square$$

A.2 SPECIAL MATRICES

A.2.1 IDENTIFYING TYPES OF MATRICES

MCQ 9: Which of the following matrices is a square matrix?

- ☐ $\mathbf{A} = \begin{pmatrix} 1 & 5 & 9 \\ 0 & 3 & 7 \end{pmatrix}$
- ☐ $\mathbf{B} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$
- ☐ $\mathbf{C} = \begin{pmatrix} 1 & 5 \\ 0 & 3 \end{pmatrix}$
- ☐ $\mathbf{D} = \begin{pmatrix} 1 & 5 & 9 \end{pmatrix}$

MCQ 10: Which of the following matrices is a column matrix?

- ☐ $\mathbf{A} = \begin{pmatrix} 2 & 0 & 9 \end{pmatrix}$
- ☐ $\mathbf{B} = \begin{pmatrix} 7 \\ -1 \\ 4 \end{pmatrix}$
- ☐ $\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
- ☐ $\mathbf{D} = \begin{pmatrix} 2 & 8 \\ 6 & 1 \end{pmatrix}$

MCQ 11: Which of the following is the identity matrix of order 3?

- ☐ $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- ☐ $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
- ☐ $\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
- ☐ $\mathbf{D} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

MCQ 12: Which of the following matrices is a row matrix?

- ☐ $\mathbf{A} = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}$

- ☐ $\mathbf{B} = \begin{pmatrix} 5 & 10 \\ 15 & 20 \end{pmatrix}$
- ☐ $\mathbf{C} = \begin{pmatrix} 5 & 10 & 15 \\ 20 & 25 & 30 \end{pmatrix}$
- ☐ $\mathbf{D} = \begin{pmatrix} 5 & 10 & 15 \end{pmatrix}$

MCQ 13: What type of special matrix is $\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$?

- ☐ A row matrix
- ☐ A column matrix
- ☐ A zero matrix
- ☐ An identity matrix

MCQ 14: What type of special matrix is $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$?

- ☐ A square matrix
- ☐ A column matrix
- ☐ A zero matrix
- ☐ An identity matrix

A.2.2 CONSTRUCTING SPECIAL MATRICES

Ex 15: Write the identity matrix of order 2, denoted $\mathbf{I}_2$.

$$\mathbf{I}_2 = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 16: Find the opposite matrix of $\mathbf{A} = \begin{pmatrix} -1 & 7 \\ 0 & -2 \end{pmatrix}$.

$$-\mathbf{A} = \begin{pmatrix} \square & \square \\ \square & \square \end{pmatrix}$$

Ex 17: Find the opposite matrix of $\mathbf{A} = \begin{pmatrix} 9 & -2 & 0 & -11 \end{pmatrix}$.

$$-\mathbf{A} = \begin{pmatrix} \square & \square & \square & \square \end{pmatrix}$$

A.3 EQUALITY

A.3.1 IDENTIFYING EQUAL MATRICES

MCQ 18: Which of the following matrices is equal to matrix $\mathbf{A} = \begin{pmatrix} 4 & 0 \\ 9 & 1 \end{pmatrix}$?

- ☐ $\mathbf{B} = \begin{pmatrix} 4 & 9 \\ 0 & 1 \end{pmatrix}$
- ☐ $\mathbf{C} = \begin{pmatrix} 2^2 & 0 \\ 3^2 & 1^2 \end{pmatrix}$
- ☐ $\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 9 & 1 & 0 \end{pmatrix}$
- ☐ $\mathbf{E} = \begin{pmatrix} 4 \\ 9 \\ 0 \\ 1 \end{pmatrix}$

MCQ 19: Which of the following matrices is equal to the row matrix $\mathbf{A} = \begin{pmatrix} \sqrt{9} & 5 & 2^3 \end{pmatrix}$?

- ☐ $\mathbf{B} = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$
- ☐ $\mathbf{C} = \begin{pmatrix} 8 & 5 & 3 \end{pmatrix}$
- ☐ $\mathbf{D} = \begin{pmatrix} 3 & 5 & 8 \end{pmatrix}$
- ☐ $\mathbf{E} = \begin{pmatrix} 3 & 5 & 8 \\ 3 & 5 & 8 \end{pmatrix}$

MCQ 20: Let $\mathbf{A} = \begin{pmatrix} \frac{10}{2} & 0 \\ 1 & -3 \end{pmatrix}$. Which of the following statements is true?

- ☐ $\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 0 & -3 \end{pmatrix}$
- ☐ $\mathbf{A} = \begin{pmatrix} 5 & 0 \end{pmatrix}$
- ☐ $\mathbf{A} = \begin{pmatrix} 5 \\ 1 \\ 0 \\ -3 \end{pmatrix}$
- ☐ $\mathbf{A} = \begin{pmatrix} 5 & 0 \\ \sin(\frac{\pi}{2}) & -3 \end{pmatrix}$

A.3.2 SOLVING FOR UNKNOWNS USING MATRIX EQUALITY

Ex 21: Find the values of x and y that make the two matrices equal:

$$\begin{pmatrix} x & 7 \\ -2 & y+1 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ -2 & 3 \end{pmatrix}$$

$x = \square$ and $y = \square$

Ex 22: Find the values of a and b such that the following matrices are equal:

$$\begin{pmatrix} a+b & 5 \\ 1 & a-b \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ 1 & 4 \end{pmatrix}$$

$a = \square$ and $b = \square$

Ex 23: Find the values of x and y for which the following matrix equality holds:

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} y & -x \\ -x & y \end{pmatrix}$$

$x = \square$ and $y = \square$

B MATRIX OPERATIONS

B.1 MATRIX ADDITION

B.1.1 VERIFYING THE CONDITION FOR ADDITION

MCQ 24: Which of the following matrix sums is possible?

☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} + \begin{pmatrix}5 & 6\end{pmatrix}$

☐ $\begin{pmatrix}1\\ 2\end{pmatrix} + \begin{pmatrix}3\\ 4\\ 5\end{pmatrix}$

☐ $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} + \begin{pmatrix}7 & 8\\ 9 & 10\end{pmatrix}$

☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} + \begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$

MCQ 25: Which of the following matrix sums is possible?

☐ $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} + \begin{pmatrix}4 & 5 & 6\end{pmatrix}$

☐ $\begin{pmatrix}1 & 2 & 3\end{pmatrix} + \begin{pmatrix}4 & 5\end{pmatrix}$

☐ $\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} + \begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$

☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} + \begin{pmatrix}5\\ 6\end{pmatrix}$

MCQ 26: Which of the following matrix sums is possible?

☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} + \begin{pmatrix}5 & 6\\ 7 & 8\end{pmatrix}$

☐ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} + \begin{pmatrix}5 & 6\end{pmatrix}$

☐ $\begin{pmatrix}1\\ 2\end{pmatrix} + \begin{pmatrix}3\\ 4\\ 5\end{pmatrix}$

☐ $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix} + \begin{pmatrix}7 & 8\\ 9 & 10\end{pmatrix}$

B.1.2 CALCULATING MATRIX SUMS

Ex 27: Calculate the sum of the following matrices:

$$\begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix} + \begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix} = \begin{pmatrix}\square & \square\\ \square & \square\end{pmatrix}$$

Ex 28: Calculate the sum of the following column matrices:

$$\begin{pmatrix}4\\ -2\\ 7\end{pmatrix} + \begin{pmatrix}-1\\ 5\\ 3\end{pmatrix} = \begin{pmatrix}\square\\ \square\\ \square\end{pmatrix}$$

Ex 29: Calculate the sum of the following row matrices:

$$\begin{pmatrix}10 & 0 & -5\end{pmatrix} + \begin{pmatrix}2 & 4 & 5\end{pmatrix} = \begin{pmatrix}\square & \square & \square\end{pmatrix}$$

Ex 30: Calculate the sum of the following matrices:

$$\begin{pmatrix}\frac{1}{2} & 3\\ -1 & 0\end{pmatrix} + \begin{pmatrix}\frac{1}{2} & -2\\ 1 & 5\end{pmatrix} = \begin{pmatrix}\square & \square\\ \square & \square\end{pmatrix}$$

B.1.3 CALCULATING MATRIX DIFFERENCES

Ex 31: Calculate the difference of the following matrices:

$$\begin{pmatrix}10 & 8\\ 5 & 6\end{pmatrix} - \begin{pmatrix}2 & -1\\ 5 & 7\end{pmatrix} = \begin{pmatrix}\square & \square\\ \square & \square\end{pmatrix}$$

Ex 32: Calculate the difference of the following column matrices:

$$\begin{pmatrix}9\\ 1\\ -4\end{pmatrix} - \begin{pmatrix}5\\ -2\\ 3\end{pmatrix} = \begin{pmatrix}\square\\ \square\\ \square\end{pmatrix}$$

Ex 33: Calculate the difference of the following row matrices:

$$\begin{pmatrix}1 & -2 & 3\end{pmatrix} - \begin{pmatrix}1 & 2 & -3\end{pmatrix} = \begin{pmatrix}\square & \square & \square\end{pmatrix}$$

B.1.4 EVALUATING MATRIX EXPRESSIONS

Ex 34: For $\mathbf{A} = \begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix}$, $\mathbf{B} = \begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix}4 & 2\\ -2 & 3\end{pmatrix}$, find $\mathbf{A} - (\mathbf{B} + \mathbf{C})$.

Ex 35: For $\mathbf{A} = \begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix}$, find $-(\mathbf{A} - \mathbf{B})$.

Ex 36: For $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$, find $\mathbf{A} + (\mathbf{B} - \mathbf{C})$.

Ex 37: For $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 8 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -3 & 1 \\ 4 & -2 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$, find $\mathbf{A} - (\mathbf{B} - \mathbf{C})$.

B.1.5 PROVING THE PROPERTIES OF ADDITION

Ex 38: For two square matrices of order 2, $\mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}$, prove that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

Ex 39: For three square matrices of order 2, $\mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} x' & y' \\ z' & w' \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} x'' & y'' \\ z'' & w'' \end{pmatrix}$, prove that $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$.

Ex 40: For a square matrix of order 2, $\mathbf{A} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$, prove that $\mathbf{A} + (-\mathbf{A}) = \mathbf{0}$, where $\mathbf{0}$ is the 2×2 zero matrix.

B.2 SCALAR MULTIPLICATION

B.2.1 CALCULATING SCALAR PRODUCTS

Ex 41: Calculate the scalar multiplication:

$$2\begin{pmatrix}\frac{1}{2} & 3\\ -1 & 0\end{pmatrix} = \begin{pmatrix}\square & \square\\ \square & \square\end{pmatrix}$$

Ex 42: Calculate the scalar multiplication:

$$5\begin{pmatrix}2\\ 0\\ -3\\ 1\end{pmatrix} = \begin{pmatrix}\square\\ \square\\ \square\\ \square\end{pmatrix}$$

Ex 43: Calculate the scalar multiplication:

$$\frac{1}{2}\begin{pmatrix}10 & -4 & 6\end{pmatrix} = \begin{pmatrix}\square & \square & \square\end{pmatrix}$$

Ex 44: Calculate the scalar multiplication:

$$-4\begin{pmatrix}1 & -3\\ 5 & 0\end{pmatrix} = \begin{pmatrix}\square & \square\\ \square & \square\end{pmatrix}$$

B.2.2 EVALUATING MATRIX EXPRESSIONS

Ex 45: For $\mathbf{A} = \begin{pmatrix}5 & -1\\ 2 & 8\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}-3 & 1\\ 4 & -2\end{pmatrix}$, find $2(\mathbf{A}+\mathbf{B})$.

Ex 46: For $\mathbf{A} = \begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}1 & -\frac{3}{2}\\ 2 & 0\end{pmatrix}$, find $2\mathbf{A}+3\mathbf{B}$.

Ex 47: For $\mathbf{A} = \begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}1 & -\frac{3}{2}\\ 2 & 0\end{pmatrix}$, find $\frac{1}{2}(\mathbf{A}-\mathbf{B})$.

Ex 48: For $\mathbf{A} = \begin{pmatrix}\frac{1}{2} & 4\\ 0 & -1\end{pmatrix}$, find $2(3\mathbf{A})$.

B.2.3 SIMPLIFYING MATRIX EXPRESSIONS

Ex 49: For any matrix $\mathbf{A}$, simplify the expression $2\mathbf{A}+2(4\mathbf{A})$.

Ex 50: For any two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, simplify the expression $(\mathbf{A}-\mathbf{B})+(\mathbf{A}+\mathbf{B})$.

Ex 51: For any two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, simplify the expression $3(\mathbf{A}+\mathbf{B})-3\mathbf{A}$.

Ex 52: For any two matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, simplify the expression $(\mathbf{A}+\mathbf{B})-(\mathbf{A}-\mathbf{B})$.

B.3 MATRIX MULTIPLICATION

B.3.1 VERIFYING THE CONDITION FOR MULTIPLICATION

MCQ 53: Which of the following matrix products is possible?

- ☐ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\times\begin{pmatrix}5 & 6\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}4\\5\\6\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\times\begin{pmatrix}7\\8\\9\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\times\begin{pmatrix}4 & 5 & 6\end{pmatrix}$

MCQ 54: Which of the following matrix products is possible?

- ☐ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\times\begin{pmatrix}5 & 6\\7 & 8\\9 & 10\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\end{pmatrix}\times\begin{pmatrix}4 & 5\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}4\\5\\6\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3\\4 & 5 & 6\end{pmatrix}\times\begin{pmatrix}0 & 1\\0 & 1\\0 & 1\end{pmatrix}$

MCQ 55: Which of the following matrix products is possible?

- ☐ $\begin{pmatrix}1 & 2\\3 & 4\end{pmatrix}\times\begin{pmatrix}5\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}\times\begin{pmatrix}5\\6\\7\\8\end{pmatrix}$
- ☐ $\begin{pmatrix}1 & 2\end{pmatrix}\times\begin{pmatrix}3 & 4\end{pmatrix}$
- ☐ $\begin{pmatrix}1\\2\end{pmatrix}\times\begin{pmatrix}3\\4\end{pmatrix}$

B.3.2 DETERMINING THE SIZE OF THE PRODUCT

Ex 56: Let $\mathbf{A}$ be a matrix of size 4×2 and $\mathbf{B}$ be a matrix of size 2×3. What is the size of the product $\mathbf{A}\times\mathbf{B}$?

Size: ☐ × ☐

Ex 57: Let $\mathbf{A}$ be a matrix of size 5×4 and $\mathbf{B}$ be a matrix of size 4×1. What is the size of the product $\mathbf{A}\times\mathbf{B}$?

Size: ☐ × ☐

Ex 58: Let $\mathbf{A}$ be a matrix of size 2×3 and $\mathbf{B}$ be a matrix of size 4×2. What is the size of the product $\mathbf{B}\times\mathbf{A}$?

Size: ☐ × ☐

B.3.3 CALCULATING MATRIX PRODUCTS

Ex 59: Calculate the multiplication of the following matrices:

$$\begin{pmatrix}3 & 4\end{pmatrix} \times \begin{pmatrix}5\\6\end{pmatrix} = \begin{pmatrix}\square\end{pmatrix}$$

Ex 60: Calculate the multiplication of the following matrices:

$$\begin{pmatrix}3 & 1\\-2 & 4\end{pmatrix} \times \begin{pmatrix}5\\2\end{pmatrix} = \begin{pmatrix}\square\\\square\end{pmatrix}$$

Ex 61: Calculate the multiplication of the following matrices:

$$\begin{pmatrix}2 & 3\\1 & 0\end{pmatrix} \times \begin{pmatrix}1 & 2\\3 & 4\end{pmatrix} = \begin{pmatrix}\square & \square\\\square & \square\end{pmatrix}$$

Ex 62: Calculate the multiplication of the following matrices:

$$\begin{pmatrix}1 & 0 & -2\\3 & 1 & 4\end{pmatrix} \times \begin{pmatrix}2 & 5\\-1 & 0\\1 & -3\end{pmatrix} = \begin{pmatrix}\square & \square\\\square & \square\end{pmatrix}$$

Ex 63: Calculate the multiplication of the following matrices:

$$\begin{pmatrix}5\\-1\end{pmatrix} \times \begin{pmatrix}3 & 2\end{pmatrix} = \begin{pmatrix}\square & \square\\\square & \square\end{pmatrix}$$

Ex 64: Calculate the multiplication of the following matrices:

$$\begin{pmatrix}2 & 1\\4 & 3\end{pmatrix} \times \begin{pmatrix}1 & 5\\2 & 6\end{pmatrix} = \begin{pmatrix}\square & \square\\\square & \square\end{pmatrix}$$

B.3.4 INVESTIGATING COMMUTATIVITY

Ex 65: Let $\mathbf{A} = \begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}3 & 0\\1 & -1\end{pmatrix}$.

1. Calculate the product $\mathbf{AB}$.
2. Calculate the product $\mathbf{BA}$.
3. Hence, conclude whether $\mathbf{AB} = \mathbf{BA}$.

Ex 66: Let $\mathbf{A} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}2 & 5\\3 & 1\end{pmatrix}$.

1. Calculate the product $\mathbf{AB}$.
2. Calculate the product $\mathbf{BA}$.
3. Hence, conclude whether $\mathbf{AB} = \mathbf{BA}$.

Ex 67: Let $\mathbf{A} = \begin{pmatrix}3 & 5\\1 & 2\end{pmatrix}$ and $\mathbf{I}_2 = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.

1. Calculate the product $\mathbf{AI}_2$.
2. Calculate the product $\mathbf{I}_2\mathbf{A}$.
3. Hence, conclude whether $\mathbf{AI}_2 = \mathbf{I}_2\mathbf{A}$.

B.3.5 EXPANDING MATRIX EXPRESSIONS

Ex 68: For any square matrix $\mathbf{A}$, expand and simplify the expression $\mathbf{A}(\mathbf{A}+\mathbf{I})$, where $\mathbf{I}$ is the identity matrix of the same order as $\mathbf{A}$.

Ex 69: For any square matrix $\mathbf{A}$, expand and simplify the expression $(\mathbf{A}+\mathbf{I})^2$, where $\mathbf{I}$ is the identity matrix of the same order as $\mathbf{A}$.

Ex 70: For any two square matrices $\mathbf{A}$ and $\mathbf{B}$ of the same order, expand and simplify the expression $(\mathbf{A}+\mathbf{B})^2$.

Ex 71: For any square matrix $\mathbf{A}$, expand and simplify the expression $(\mathbf{A}+3\mathbf{I})^2$, where $\mathbf{I}$ is the identity matrix of the same order as $\mathbf{A}$.

B.3.6 SIMPLIFYING POWERS OF A MATRIX

Ex 72: Given that a square matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2=\mathbf{A}+\mathbf{I}$, find the expressions for $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars and $\mathbf{I}$ is the identity matrix.

Ex 73: Given that a square matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2=\mathbf{A}-\mathbf{I}$, find the expressions for $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars and $\mathbf{I}$ is the identity matrix.

Ex 74: Given that a square matrix $\mathbf{A}$ satisfies the relation $\mathbf{A}^2=2\mathbf{A}+3\mathbf{I}$, find the expressions for $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$, where k and l are scalars and $\mathbf{I}$ is the identity matrix.

1. Calculate the product **AB**.
2. Calculate the product **BA**.
3. What can you conclude about the relationship between matrices **A** and **B**?

C INVERTIBLE MATRICES

C.1 DEFINITION

C.1.1 VERIFYING AN INVERSE BY DEFINITION

Ex 75: Let $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

1. Calculate the product **AB**.
2. Calculate the product **BA**.
3. What can you conclude about the relationship between matrices **A** and **B**?

Ex 76: Let $\mathbf{A} = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -2 \\ -3 & 7 \end{pmatrix}$.

Ex 77: Let $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$.

1. Calculate the product **AB**.
2. Based on this result, can you conclude whether **B** is the inverse of **A**?

C.1.2 PROVING PROPERTIES OF THE INVERSE

Ex 78: Prove that the identity matrix, $\mathbf{I}$, is invertible and that its inverse is itself (i.e., $\mathbf{I}^{-1} = \mathbf{I}$).

Ex 79: Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be square matrices of the same order. Suppose that $\mathbf{A}$ is an invertible matrix. Prove that if $\mathbf{AB} = \mathbf{AC}$, then $\mathbf{B} = \mathbf{C}$.

Ex 80: Let $\mathbf{A}$ be an invertible square matrix, and let $\mathbf{X}$ and $\mathbf{B}$ be matrices of compatible sizes. Prove that if $\mathbf{AX} = \mathbf{B}$, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.

Ex 81: Let $\mathbf{A}$ be an invertible matrix. Suppose there are two matrices, $\mathbf{B}$ and $\mathbf{C}$, such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$ and $\mathbf{AC} = \mathbf{CA} = \mathbf{I}$. Prove that $\mathbf{B} = \mathbf{C}$. (This shows the inverse is unique).

C.2 FINDING THE INVERSE OF A 2X2 MATRIX

C.2.1 CALCULATING THE DETERMINANT

Ex 82: Calculate the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 4 \end{pmatrix}$.

$$\det(\mathbf{A}) = \square$$

Ex 83: Calculate the determinant of the matrix $\mathbf{B} = \begin{pmatrix} -1 & 0 \\ 7 & -5 \end{pmatrix}$.

$$\det(\mathbf{B}) = \square$$

Ex 84: Calculate the determinant of the matrix $\mathbf{C} = \begin{pmatrix} 6 & 3 \\ 8 & 4 \end{pmatrix}$.

$$\det(\mathbf{C}) = \square$$

C.2.2 FINDING THE INVERSE OF A 2X2 MATRIX

Ex 85: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ exists, and if so, find it.

Ex 86: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ exists, and if so, find it.

Ex 87: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ exists, and if so, find it.

Ex 88: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$ exists, and if so, find it.

Ex 89: Determine if the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ exists, and if so, find it.

C.2.3 FINDING THE CONDITION FOR INVERTIBILITY

Ex 90: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & k \end{pmatrix}$ is invertible.

Ex 91: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} 2k & 3 \\ k & 1 \end{pmatrix}$ is invertible.

Ex 92: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} k & 1 \\ 0 & k+1 \end{pmatrix}$ is invertible.

Ex 93: Find the value(s) of k for which the matrix $\mathbf{A} = \begin{pmatrix} 1 & k-1 \\ k & 2 \end{pmatrix}$ is invertible.

D APPLICATIONS

D.1 SOLVING SYSTEMS OF LINEAR EQUATIONS

D.1.1 WRITING A SYSTEM IN MATRIX FORM

Ex 94: Write the system $\begin{cases} 2x+5y &= 2 \\ x+3y &= 5 \end{cases}$ in matrix form.

Ex 95: Write the system $\begin{cases} x-2y &= 7 \\ 3x+y &= 0 \end{cases}$ in matrix form.

Ex 96: Write the system $\begin{cases} x+y-z &= 9 \\ 2y+4z &= -2 \\ 5x-6z &= 0 \end{cases}$ in matrix form.

D.1.2 SOLVING SYSTEMS WITH THE INVERSE METHOD

Ex 97: Use the matrix method to solve the following system of linear equations:

$$\begin{cases} 2x+5y &= 2 \\ x+3y &= 5 \end{cases}$$

Ex 98: Use the matrix method to solve the following system of linear equations:

$$\begin{cases} 3x+y &= 8 \\ x+2y &= 9 \end{cases}$$

Ex 99: Use the matrix method to solve the following system of linear equations:

$$\begin{cases} 5x-2y &= 1 \\ 4x-y &= 4 \end{cases}$$