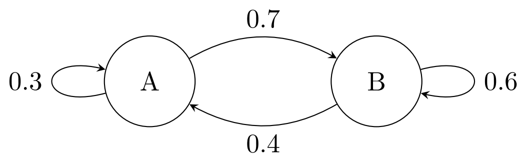
MARKOV CHAINS

A MARKOV CHAIN

A.1 READING TRANSITION DIAGRAMS

Ex 1: A laboratory rat is placed in a maze with two rooms, Room A and Room B. The transition diagram below shows the probability of the rat moving between rooms or staying in the same room every minute.

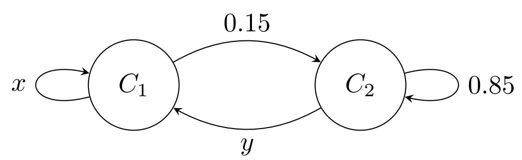


1. If the rat is in Room A, what is the probability it moves to Room B in the next minute?
2. If the rat is in Room B, what is the probability it stays in Room B in the next minute?
3. Verify that the sum of probabilities leaving each state is equal to 1.

Answer:

1. The arrow from A to B is labeled 0.7. The probability is 0.7.
2. The loop on B is labeled 0.6. The probability is 0.6.
3. For state A: 0.3(stay) + 0.7(move) = 1. For state B: 0.6(stay) + 0.4(move) = 1.

Ex 2: Two companies, C_1 and C_2, compete for customers. The transition diagram below shows the weekly change in customers. Find the missing values x and y.



Answer: The sum of the probabilities on all arrows leaving a single state must equal 1.

- For C_1: $x + 0.15 = 1 \implies x = 1 - 0.15 = 0.85$.
- For C_2: $0.85 + y = 1 \implies y = 1 - 0.85 = 0.15$.

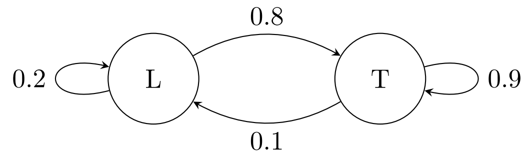
Ex 3: A student can be either "Late" (L) or "On Time" (T) for class.

- If the student is Late one day, there is a 20% probability they will be Late again the next day.
- If the student is On Time one day, there is a 10% probability they will be Late the next day.

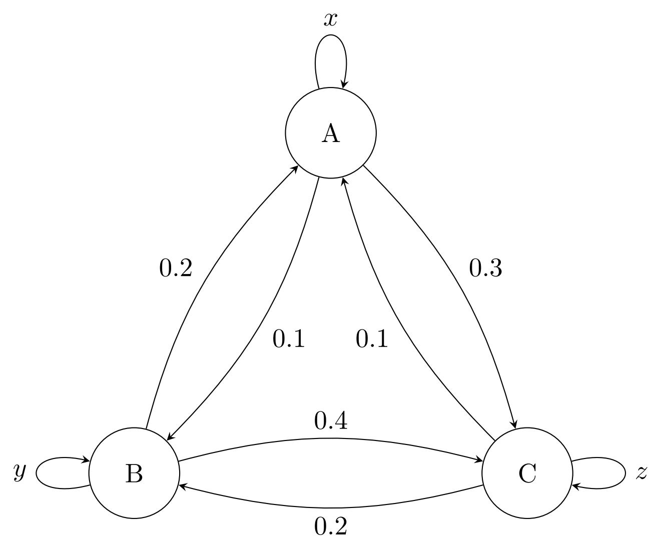
Draw the transition diagram representing this situation.

Answer:

- From Late (L): $P(L \to L) = 0.2$. Therefore $P(L \to T) = 1 - 0.2 = 0.8$.
- From On Time (T): $P(T \to L) = 0.1$. Therefore $P(T \to T) = 1 - 0.1 = 0.9$.



Ex 4: Three brands of cereal, A, B, and C, dominate the market. The transition diagram below shows the probability of a customer switching brands or staying with the same brand next week.
Find the missing probabilities x, y and z.



Answer: The sum of all probabilities leaving a node must equal 1.

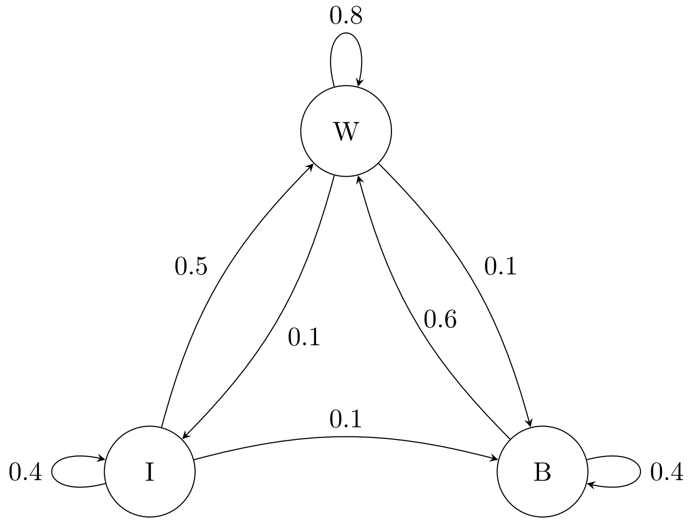
- **State A:** $x + 0.1 + 0.3 = 1 \implies x = 1 - 0.4 = 0.6$.
- **State B:** $0.2 + y + 0.4 = 1 \implies y = 1 - 0.6 = 0.4$.
- **State C:** $0.1 + 0.2 + z = 1 \implies z = 1 - 0.3 = 0.7$.

Ex 5: A machine in a factory can be in one of three states: Working (W), Idle (I), or Broken (B).
The transition probabilities per hour are:

- If Working: 80% chance to stay Working, 10% to become Idle, 10% to become Broken.
- If Idle: 50% chance to become Working, 40% to stay Idle, 10% to Break.
- If Broken: 0% chance to become Idle, 60% chance to be Repaired (Working), 40% to stay Broken.

Draw the transition diagram for this system.

Answer:



B MATRIX REPRESENTATION

B.1 DEFINING AND VALIDATING STATE VECTORS

Ex 6: A system has two states, A and B. Currently, the system is certainly in state A. Write down the initial state matrix $\mathbf{s}_0$.

Answer:

$$\mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Ex 7: A population is divided between City (C) and Suburbs (S). Initially, 70% of people live in the City and 30% in the Suburbs. Write the initial state matrix $\mathbf{s}_0$.

Answer:

$$\mathbf{s}_0 = \begin{pmatrix} 0.7 \\ 0.3 \end{pmatrix}$$

Ex 8: Which of the following cannot be a valid state matrix? Explain why.

1. $\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$
2. $\begin{pmatrix} 0.2 \\ 0.9 \end{pmatrix}$
3. $\begin{pmatrix} -0.1 \\ 1.1 \end{pmatrix}$

Answer:

- Valid ($0.5 + 0.5 = 1$).
- Invalid. The sum is $0.2 + 0.9 = 1.1 \neq 1$.
- Invalid. Probabilities cannot be negative (-0.1).

Ex 9: A market study tracks the preference between two smartphone brands, Brand X and Brand Y. The state of the market after 2 months is given by the state matrix:

$$\mathbf{s}_2 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$$

1. Interpret the meaning of the value 0.4 in the context of the problem.
2. Interpret the meaning of the value 0.6 in the context of the problem.

Answer:

1. The value 0.4 represents the probability (or proportion) that a customer chooses **Brand X** after 2 months.
2. The value 0.6 represents the probability (or proportion) that a customer chooses **Brand Y** after 2 months.

B.2 BUILDING AND INTERPRETING TRANSITION MATRICES

Ex 10: A transition matrix is given by $\mathbf{T} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$. What is the probability of moving from state 2 (column 2) to state 1 (row 1)?

Answer: The value is $t_{12} = 0.3$. The probability is 0.3.

Ex 11: Find the missing value x in the following transition matrix:

$$\mathbf{T} = \begin{pmatrix} 0.2 & 0.5 \\ x & 0.5 \end{pmatrix}$$

Answer: The sum of the first column must be 1.

$$0.2 + x = 1 \implies x = 0.8$$

Ex 12: A computer can be either "On" or "Sleep".

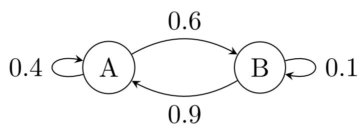
- If it is On, there is a 90% chance it stays On.
- If it is Sleep, there is a 20% chance it turns On.

Construct the transition matrix $\mathbf{T}$ (using the order On, then Sleep).

Answer: From "On" (Column 1): 0.9 to "On", so $1 - 0.9 = 0.1$ to "Sleep".
From "Sleep" (Column 2): 0.2 to "On", so $1 - 0.2 = 0.8$ to "Sleep".

$$\mathbf{T} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$$

Ex 13: Consider the transition diagram below. Write the corresponding transition matrix $\mathbf{T}$.



Answer:

$$\mathbf{T} = \begin{pmatrix} 0.4 & 0.9 \\ 0.6 & 0.1 \end{pmatrix}$$

Ex 14: A marketing model tracks customers buying Brand A, Brand B, or Brand C. The transition matrix is:

$$\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}$$

If a customer buys Brand B this week, what is the probability they will buy Brand C next week?

Answer: We look at Column 2 (From B) and Row 3 (To C). The value is 0.2 (20%).

Ex 15: Construct the transition matrix for a 3-state system (X, Y, Z) given:

- X always stays X.
- Y goes to X half the time, and Z half the time.
- Z goes to Y with probability 1.

Answer:

$$\mathbf{T} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 1 \\ 0 & 0.5 & 0 \end{pmatrix}$$

(Rows are X, Y, Z; Columns are X, Y, Z).

C STATE VECTORS AND FUTURE PROBABILITIES

C.1 CALCULATING AND PREDICTING STATES AFTER ONE STEP

Ex 16: A rental car company has cars at Location X and Location Y. The transition matrix for weekly movement is $\mathbf{T} = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$.
Initially, 50% of the cars are at X and 50% are at Y.
Find the distribution of cars after 1 week.

Answer: The initial state is $\mathbf{s}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$.

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{T}\mathbf{s}_0 \\ &= \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0.8 \cdot 0.5 + 0.4 \cdot 0.5 \\ 0.2 \cdot 0.5 + 0.6 \cdot 0.5 \end{pmatrix} \\ &= \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \end{aligned}$$

After 1 week, 60% of cars are at X and 40% are at Y.

Ex 17: In an election campaign, voters support either Party A or Party B. The transition matrix representing the shift in support each month is $\mathbf{T} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$.
Currently, 40% of voters support Party A and 60% support Party B.
Find the distribution of voter support after 1 month.

Answer: The initial state is $\mathbf{s}_0 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$.

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{T}\mathbf{s}_0 \\ &= \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} \\ &= \begin{pmatrix} 0.9 \cdot 0.4 + 0.2 \cdot 0.6 \\ 0.1 \cdot 0.4 + 0.8 \cdot 0.6 \end{pmatrix} \\ &= \begin{pmatrix} 0.36 + 0.12 \\ 0.04 + 0.48 \end{pmatrix} \\ &= \begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix} \end{aligned}$$

After 1 month, 48% of voters support Party A and 52% support Party B.

Ex 18: Two internet service providers, FastNet and SpeedWeb, compete for subscribers. The transition matrix representing annual customer switching is $\mathbf{T} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix}$.
Currently, FastNet has 30% of the market and SpeedWeb has 70%.
Find the market share of each provider after 1 year.

Answer: The initial state is $\mathbf{s}_0 = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$.

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{T}\mathbf{s}_0 \\ &= \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} \\ &= \begin{pmatrix} 0.8 \cdot 0.3 + 0.1 \cdot 0.7 \\ 0.2 \cdot 0.3 + 0.9 \cdot 0.7 \end{pmatrix} \\ &= \begin{pmatrix} 0.24 + 0.07 \\ 0.06 + 0.63 \end{pmatrix} \\ &= \begin{pmatrix} 0.31 \\ 0.69 \end{pmatrix} \end{aligned}$$

After 1 year, FastNet has 31% of the market and SpeedWeb has 69%.

C.2 CALCULATING AND PREDICTING STATES AFTER TWO STEPS

Ex 19: A rental car company has cars at Location X and Location Y. The transition matrix for weekly movement is $\mathbf{T} = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$.
Initially, 50% of the cars are at X and 50% are at Y.
Find the distribution of cars after **2 weeks**.

Answer: The initial state is $\mathbf{s}_0 = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$.

- **Method 1: Using Matrix Powers ($\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0$)**

$$\mathbf{T}^2 = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} = \begin{pmatrix} 0.72 & 0.56 \\ 0.28 & 0.44 \end{pmatrix}$$

$$\mathbf{s}_2 = \begin{pmatrix} 0.72 & 0.56 \\ 0.28 & 0.44 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.64 \\ 0.36 \end{pmatrix}$$

- **Method 2: Step-by-Step ($\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0$ then $\mathbf{s}_2 = \mathbf{T}\mathbf{s}_1$)**

$$\mathbf{s}_1 = \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$$

$$\begin{aligned} \mathbf{s}_2 &= \begin{pmatrix} 0.8 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \\ &= \begin{pmatrix} 0.64 \\ 0.36 \end{pmatrix} \end{aligned}$$

After 2 weeks, 64% of cars are at X and 36% are at Y.

Ex 20: In an election campaign, voters support either Party A or Party B. The transition matrix representing the shift in support each month is $\mathbf{T} = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$.
Currently, 40% of voters support Party A and 60% support Party B.
Find the distribution of voter support after **2 months**.

Answer: The initial state is $\mathbf{s}_0 = \begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix}$.

- **Method 1: Using Matrix Powers ($\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0$)**

$$\mathbf{T}^2 = \begin{pmatrix}0.9 & 0.2\\0.1 & 0.8\end{pmatrix}\begin{pmatrix}0.9 & 0.2\\0.1 & 0.8\end{pmatrix} = \begin{pmatrix}0.83 & 0.34\\0.17 & 0.66\end{pmatrix}$$

$$\mathbf{s}_2 = \begin{pmatrix}0.83 & 0.34\\0.17 & 0.66\end{pmatrix}\begin{pmatrix}0.4\\0.6\end{pmatrix} = \begin{pmatrix}0.536\\0.464\end{pmatrix}$$

- **Method 2: Step-by-Step ($\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0$ then $\mathbf{s}_2 = \mathbf{T}\mathbf{s}_1$)**

$$\mathbf{s}_1 = \begin{pmatrix}0.9 & 0.2\\0.1 & 0.8\end{pmatrix}\begin{pmatrix}0.4\\0.6\end{pmatrix} = \begin{pmatrix}0.48\\0.52\end{pmatrix}$$

$$\begin{aligned}\mathbf{s}_2 &= \begin{pmatrix}0.9 & 0.2\\0.1 & 0.8\end{pmatrix}\begin{pmatrix}0.48\\0.52\end{pmatrix}\\ &= \begin{pmatrix}0.536\\0.464\end{pmatrix}\end{aligned}$$

After 2 months, 53.6% of voters support Party A and 46.4% support Party B.

Ex 21: A student is studying. If they study today, there is a 60% chance they study tomorrow. If they don't study today, there is a 30% chance they study tomorrow.
Assume the student studies today. Calculate the probability they will **not** study in 2 days.

Answer: Let S be "Studying" and N be "Not Studying". The initial state is $\mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}$ (Studies today).

The transition matrix is $\mathbf{T} = \begin{pmatrix}0.6 & 0.3\\0.4 & 0.7\end{pmatrix}$ (where the first column is from S and the second from N).

- **Method 1: Using Matrix Powers ($\mathbf{s}_2 = \mathbf{T}^2\mathbf{s}_0$)**

$$\mathbf{T}^2 = \begin{pmatrix}0.6 & 0.3\\0.4 & 0.7\end{pmatrix}\begin{pmatrix}0.6 & 0.3\\0.4 & 0.7\end{pmatrix} = \begin{pmatrix}0.48 & 0.39\\0.52 & 0.61\end{pmatrix}$$

$$\mathbf{s}_2 = \begin{pmatrix}0.48 & 0.39\\0.52 & 0.61\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0.48\\0.52\end{pmatrix}$$

- **Method 2: Step-by-Step ($\mathbf{s}_1 = \mathbf{T}\mathbf{s}_0$ then $\mathbf{s}_2 = \mathbf{T}\mathbf{s}_1$)**

$$\mathbf{s}_1 = \begin{pmatrix}0.6 & 0.3\\0.4 & 0.7\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0.6\\0.4\end{pmatrix}$$

$$\begin{aligned}\mathbf{s}_2 &= \begin{pmatrix}0.6 & 0.3\\0.4 & 0.7\end{pmatrix}\begin{pmatrix}0.6\\0.4\end{pmatrix}\\ &= \begin{pmatrix}0.6(0.6) + 0.3(0.4)\\0.4(0.6) + 0.7(0.4)\end{pmatrix}\\ &= \begin{pmatrix}0.36 + 0.12\\0.24 + 0.28\end{pmatrix}\\ &= \begin{pmatrix}0.48\\0.52\end{pmatrix}\end{aligned}$$

The probability they will **not** study is the second component: 0.52 or 52%.

D STEADY STATE

D.1 FINDING STEADY STATE

Ex 22: Two supermarket chains, A and B, compete for customers. The transition matrix representing the weekly change in customer preference is given by $\mathbf{T} = \begin{pmatrix}0.7 & 0.2\\0.3 & 0.8\end{pmatrix}$.

1. Find the steady state vector $\mathbf{s} = \begin{pmatrix}x\\y\end{pmatrix}$ algebraically by solving the system $\mathbf{T}\mathbf{s} = \mathbf{s}$.
2. Interpret the result in terms of market share in the long run.

Answer:

1. **Find an eigenvector for $\lambda = 1$:**

$$\begin{aligned}\mathbf{T}\mathbf{s} &= \mathbf{s}\\ \begin{pmatrix}0.7 & 0.2\\0.3 & 0.8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}x\\y\end{pmatrix}\\ \begin{pmatrix}0.7x + 0.2y\\0.3x + 0.8y\end{pmatrix} &= \begin{pmatrix}x\\y\end{pmatrix}\end{aligned}$$

   This gives $0.7x + 0.2y = x$ and $0.3x + 0.8y = y$.
   Simplifying, we get $-0.3x + 0.2y = 0$ and $0.3x - 0.2y = 0$.
   Thus $0.3x = 0.2y$, which implies $3x = 2y$.
   Letting $y = 3t$, we have $x = 2t$, so:

$$\mathbf{s} = \begin{pmatrix}2t\\3t\end{pmatrix}$$

   Since $\mathbf{s}$ is a state vector, the sum of its components must be 1:

$$2t + 3t = 1 \implies 5t = 1 \implies t = 0.2$$

$$\mathbf{s} = \begin{pmatrix}0.2\cdot 2\\0.2\cdot 3\end{pmatrix} = \begin{pmatrix}0.4\\0.6\end{pmatrix}$$

2. In the long run, Supermarket A will have 40% of the market share and Supermarket B will have 60%.

Ex 23: In a region, people move between the City (C) and the Suburbs (S). The transition matrix representing the annual migration is given by $\mathbf{T} = \begin{pmatrix}0.9 & 0.2\\0.1 & 0.8\end{pmatrix}$.

1. Find the steady state vector $\mathbf{s} = \begin{pmatrix}x\\y\end{pmatrix}$ algebraically by solving the system $\mathbf{T}\mathbf{s} = \mathbf{s}$.
2. Interpret the result in terms of the long-term population distribution.

Answer:

1. **Find an eigenvector for $\lambda = 1$:**

$$\begin{aligned}\mathbf{T}\mathbf{s} &= \mathbf{s}\\ \begin{pmatrix}0.9 & 0.2\\0.1 & 0.8\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} &= \begin{pmatrix}x\\y\end{pmatrix}\\ \begin{pmatrix}0.9x + 0.2y\\0.1x + 0.8y\end{pmatrix} &= \begin{pmatrix}x\\y\end{pmatrix}\end{aligned}$$

   This gives $0.9x + 0.2y = x$ and $0.1x + 0.8y = y$.
   Simplifying, we get $-0.1x + 0.2y = 0$ and $0.1x - 0.2y = 0$.
   Thus $0.1x = 0.2y$, which implies $x = 2y$.
   Letting $y = t$, we have $x = 2t$, so:

$$\mathbf{s} = \begin{pmatrix}2t\\t\end{pmatrix}$$

   Since $\mathbf{s}$ is a state vector, the sum of its components must be 1:

$$2t + t = 1 \implies 3t = 1 \implies t = \frac{1}{3}$$

$$\mathbf{s} = \begin{pmatrix}2\cdot\frac{1}{3}\\\frac{1}{3}\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\end{pmatrix} \approx \begin{pmatrix}0.67\\0.33\end{pmatrix}$$

2. In the long run, about 67% of the population will live in the City and 33% in the Suburbs.

Ex 24: Two streaming services, Netstream and MoviePlus, compete for subscribers. The transition matrix representing the monthly change is $\mathbf{T} = \begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix}$.

1. Find the steady state vector $\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix}$ algebraically by solving the system $\mathbf{Ts} = \mathbf{s}$.
2. Interpret the result in terms of market share in the long run.

Answer:

1. **Find an eigenvector for $\lambda = 1$:**

$$\mathbf{Ts} = \mathbf{s}$$
$$\begin{pmatrix} 0.6 & 0.3 \\ 0.4 & 0.7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} 0.6x + 0.3y \\ 0.4x + 0.7y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This gives $0.6x + 0.3y = x$ and $0.4x + 0.7y = y$.
Simplifying, we get $-0.4x + 0.3y = 0$ and $0.4x - 0.3y = 0$.
Thus $0.4x = 0.3y$, which implies $4x = 3y$.
Letting $x = 3t$, we have $y = 4t$, so:

$$\mathbf{s} = \begin{pmatrix} 3t \\ 4t \end{pmatrix}$$

Since $\mathbf{s}$ is a state vector, the sum of its components must be 1:

$$3t + 4t = 1 \implies 7t = 1 \implies t = \frac{1}{7}$$
$$\mathbf{s} = \begin{pmatrix} 3 \cdot \frac{1}{7} \\ 4 \cdot \frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{3}{7} \\ \frac{4}{7} \end{pmatrix} \approx \begin{pmatrix} 0.43 \\ 0.57 \end{pmatrix}$$

2. In the long run, Netstream will have approximately 43% of the market share and MoviePlus will have 57%.