A MACLAURIN SERIES

A.1 EXPANDING MACLAURIN SERIES FROM SIGMA NOTATION

Ex 1: Expand the series for $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ up to the term of order 3.

$$e^x = \boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}} + \sum_{k=4}^{\infty} \frac{x^k}{k!}$$

Answer: The notation $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ represents the infinite sum of terms where k starts at 0 and increases by 1 each time.
We can write this out term by term:

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

By definition, $x^0 = 1$ and $0! = 1$, so the first term is 1. The term of order 3 is the term where the power of x is 3.
Expanding up to order 3 gives us the first four terms ($k = 0, 1, 2, 3$):

$$\underbrace{\frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}}_{\text{Expanded terms}} + \underbrace{\frac{x^4}{4!} + \frac{x^5}{5!} + \dots}_{\text{Remainder}}$$

This simplifies to:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

The remainder of the series consists of all the terms from $k = 4$ onwards. We can represent this remaining infinite sum using sigma notation:

$$\sum_{k=4}^{\infty} \frac{x^k}{k!}$$

Therefore, the full expansion is:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \sum_{k=4}^{\infty} \frac{x^k}{k!}$$

Ex 2: Expand the series for $\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k$ up to the term of order 3.

$$\frac{1}{1+x} = \boxed{1 - x + x^2 - x^3} + \sum_{k=4}^{\infty} (-1)^k x^k$$

Answer: The notation $\sum_{k=0}^{\infty} (-1)^k x^k$ represents the infinite alternating sum where k starts at 0.
We can write this out term by term:

$$\sum_{k=0}^{\infty} (-1)^k x^k = (-1)^0 x^0 + (-1)^1 x^1 + (-1)^2 x^2 + (-1)^3 x^3 + (-1)^4 x^4 + \dots$$

The term of order 3 is the term where the power of x is 3.
Expanding up to order 3 gives us the first four terms ($k = 0, 1, 2, 3$):

$$\underbrace{x^0 - x^1 + x^2 - x^3}_{\text{Expanded terms}} + \underbrace{x^4 - x^5 + \dots}_{\text{Remainder}}$$

This simplifies to:

$$1 - x + x^2 - x^3$$

The remainder of the series consists of all the terms from $k = 4$ onwards, which can be written in sigma notation as:

$$\sum_{k=4}^{\infty} (-1)^k x^k$$

Therefore, the full expansion is:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \sum_{k=4}^{\infty} (-1)^k x^k$$

Ex 3: Expand the series for $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$ up to the term of order 4.

$$\ln(1+x) = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}} + \sum_{k=5}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

Answer: The notation $\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k}$ represents the infinite sum where k starts at 1.The term of order 4 is the term where the power of x is 4, which occurs when $k = 4$.So, we expand for $k = 1, 2, 3, 4$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} = \frac{(-1)^0 x^1}{1} + \frac{(-1)^1 x^2}{2} + \frac{(-1)^2 x^3}{3} + \frac{(-1)^3 x^4}{4} + \dots$$

The expanded terms up to order 4 are:

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

The remainder of the series consists of all the terms from $k = 5$ onwards:

$$\sum_{k=5}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

Therefore, the full expansion is:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \sum_{k=5}^{\infty} \frac{(-1)^{k-1} x^k}{k}$$

Ex 4: Expand the series for $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ up to the term of order 5.

$$\sin(x) = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!}} + \sum_{k=3}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Answer: The notation $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ represents the infinite sum of terms where k starts at 0.The term of order 5 is the term where the power of x is 5. This occurs when $2k + 1 = 5$, which means $k = 2$.So, we expand for $k = 0, 1, 2$:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = \frac{(-1)^0 x^1}{1!} + \frac{(-1)^1 x^3}{3!} + \frac{(-1)^2 x^5}{5!} + \frac{(-1)^3 x^7}{7!} + \dots$$

The expanded terms up to order 5 are:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

The remainder of the series consists of all the terms from $k = 3$ onwards:

$$\sum_{k=3}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Therefore, the full expansion is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \sum_{k=3}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

A.2 WRITING A SERIES USING SIGMA NOTATION

Ex 5: For the series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$, write the series using sigma notation.

$$\tfrac{1}{1-x} = \sum_{n=0}^{\infty} \boxed{x^n}$$

Answer: To find the general term, we analyze the pattern of the terms in the series:

$$\frac{1}{1-x} = x^0 + x^1 + x^2 + x^3 + \dots$$

We observe that for each term, the power of x simply matches the index of the term, starting from $n = 0$.The n-th term of the series (starting with $n = 0$) has the form:

$$x^n$$

The series is the sum of all these terms from $n = 0$ to infinity. We can therefore write the series using sigma notation as:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Ex 6: For the series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$, write the series using sigma notation.

$$e^x = \sum_{n=0}^{\infty} \boxed{\frac{x^n}{n!}}$$

Answer: To find the general term, we analyze the pattern of the terms in the series. Let's write out the first few terms, explicitly showing the powers and factorials:

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

The n-th term of the series (starting with $n = 0$) has the form:

$$\frac{x^n}{n!}$$

The series is the sum of all these terms from $n = 0$ to infinity. We can therefore write the series using sigma notation as:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Ex 7: For the series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$, write the series using sigma notation.

$$\ln(1+x) = \sum_{n=1}^{\infty} \boxed{\frac{(-1)^{n-1} x^n}{n}}$$

Answer: To find the general term, we analyze the pattern of the terms, noting that the summation starts at $n = 1$:

- The signs alternate, starting with positive for the first term ($n = 1$). This suggests a factor of $(-1)^{n-1}$ or $(-1)^{n+1}$.
- The power of x matches the index n, so x^n.
- The denominator matches the index n.

The n-th term of the series (starting with $n = 1$) has the form:

$$\frac{(-1)^{n-1} x^n}{n}$$

The series is the sum of these terms from $n = 1$ to infinity:

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

Ex 8: For the series $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$, write the series using sigma notation.

$$\cos(x) = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n x^{2n}}{(2n)!}}$$

Answer: To find the general term, we analyze the pattern of the terms:

- The signs alternate, starting with positive. This suggests a factor of $(-1)^n$.
- The powers of x are only even numbers $(x^0, x^2, x^4, \dots)$. This can be written as x^{2n}.
- The denominator is the factorial of the power of x, so $(2n)!$.

The n-th term of the series (starting with $n = 0$) has the form:

$$\frac{(-1)^n x^{2n}}{(2n)!}$$

The series is the sum of these terms from $n = 0$ to infinity:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

A.3 DERIVING STANDARD MACLAURIN SERIES FROM FIRST PRINCIPLES

Ex 9: For the function $f(x) = e^x$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, and $f^{(3)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, and $f^{(3)}(0)$.
3. Show that the Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Answer:

1. **Derivatives:** We find the first three derivatives.

- $f^{(1)}(x) = (e^x)' = e^x$.
- $f^{(2)}(x) = (e^x)' = e^x$.
- $f^{(3)}(x) = (e^x)' = e^x$.

We can see that for all $k \geq 0$, $f^{(k)}(x) = e^x$.

2. **Evaluation at 0:** We evaluate the function and its derivatives at $x = 0$.

- $f(0) = e^0 = 1$.
- $f^{(1)}(0) = e^0 = 1$.
- $f^{(2)}(0) = e^0 = 1$.
- $f^{(3)}(0) = e^0 = 1$.

We have $f^{(k)}(0) = 1$ for all k.

3. **Series Construction:** Using the Maclaurin formula, we have:

$$\begin{aligned}
e^x &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots \\
&= 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \\
&= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \\
&= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!}
\end{aligned}$$

Ex 10: For the function $f(x) = \cos(x)$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, $f^{(3)}(x)$ and $f^{(4)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, $f^{(3)}(0)$ and $f^{(4)}(0)$.
3. Show that the Maclaurin series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

Answer:

1. **Derivatives:** We find the first four derivatives. We can observe a repeating pattern of four.

- $f^{(1)}(x) = (\cos x)' = -\sin x$.
- $f^{(2)}(x) = (-\sin x)' = -\cos x$.
- $f^{(3)}(x) = (-\cos x)' = \sin x$.
- $f^{(4)}(x) = (\sin x)' = \cos x$.

2. **Evaluation at 0:** We evaluate the function and its derivatives at $x = 0$.

- $f(0) = \cos(0) = 1$.
- $f^{(1)}(0) = -\sin(0) = 0$.
- $f^{(2)}(0) = -\cos(0) = -1$.
- $f^{(3)}(0) = \sin(0) = 0$.
- $f^{(4)}(0) = \cos(0) = 1$.

3. **Series Construction:** Using the Maclaurin formula, we have:

$$\begin{aligned}
\cos x &= f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \\
&= 1 + 0 \cdot x + \frac{-1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \dots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \\
&= \frac{(-1)^0}{(2 \cdot 0)!}x^{2\cdot 0} + \frac{(-1)^1}{(2 \cdot 1)!}x^{2\cdot 1} + \frac{(-1)^2}{(2 \cdot 2)!}x^{2\cdot 2} + \dots \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}
\end{aligned}$$

Ex 11: For the function $f(x) = \frac{1}{1-x}$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, and $f^{(3)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, and $f^{(3)}(0)$.
3. Show that the Maclaurin series for $\frac{1}{1-x}$ is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{k=0}^{\infty} x^k$$

Answer:

1. **Derivatives:** We write $f(x) = (1-x)^{-1}$ and find the first three derivatives.

- $f^{(1)}(x) = \left((1-x)^{-1}\right)' = -1(1-x)^{-2}(-1) = (1-x)^{-2}$.
- $f^{(2)}(x) = \left((1-x)^{-2}\right)' = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$.
- $f^{(3)}(x) = \left(2(1-x)^{-3}\right)' = 2(-3)(1-x)^{-4}(-1) = 6(1-x)^{-4}$.

2. **Evaluation at 0:** We evaluate the function and its derivatives at $x = 0$.

- $f(0) = (1-0)^{-1} = 1$.
- $f^{(1)}(0) = (1-0)^{-2} = 1$.
- $f^{(2)}(0) = 2(1-0)^{-3} = 2$.
- $f^{(3)}(0) = 6(1-0)^{-4} = 6$.

We can observe a pattern: $f^{(k)}(0) = k!$.

3. **Series Construction:** Using the Maclaurin formula, we have:

$$\begin{aligned}
\frac{1}{1-x} &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots \\
&= 1 + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{6}{3!}x^3 + \dots \\
&= 1 + x + x^2 + x^3 + \dots \\
&= \sum_{k=0}^{\infty} x^k
\end{aligned}$$

Ex 12: For the function $f(x) = \ln(1-x)$,

1. Find $f^{(1)}(x)$, $f^{(2)}(x)$, and $f^{(3)}(x)$.
2. Find $f(0)$, $f'(0)$, $f^{(2)}(0)$, and $f^{(3)}(0)$.

3. Show that the Maclaurin series for $\ln(1-x)$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

Answer:

1. **Derivatives:** We find the first three derivatives.
   - $f^{(1)}(x) = (\ln(1-x))' = \frac{-1}{1-x} = -(1-x)^{-1}$.
   - $f^{(2)}(x) = (-(1-x)^{-1})' = -(-1)(1-x)^{-2}(-1) = -(1-x)^{-2}$.
   - $f^{(3)}(x) = (-(1-x)^{-2})' = -(-2)(1-x)^{-3}(-1) = -2(1-x)^{-3}$.
2. **Evaluation at 0:** We evaluate the function and its derivatives at $x = 0$.
   - $f(0) = \ln(1-0) = \ln(1) = 0$.
   - $f^{(1)}(0) = -(1-0)^{-1} = -1$.
   - $f^{(2)}(0) = -(1-0)^{-2} = -1$.
   - $f^{(3)}(0) = -2(1-0)^{-3} = -2$.

   We can observe a pattern: $f^{(k)}(0) = -(k-1)!$ for $k \geq 1$.
3. **Series Construction:** Using the Maclaurin formula, we have:

$$\begin{aligned}\ln(1-x) &= f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots \\ &= 0 + \frac{-1}{1!}x + \frac{-1}{2!}x^2 + \frac{-2}{3!}x^3 + \dots \\ &= -x - \frac{x^2}{2} - \frac{2x^3}{6} - \dots \\ &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \\ &= -\sum_{k=1}^{\infty} \frac{x^k}{k}\end{aligned}$$

A.4 FINDING SERIES WITH THE BINOMIAL FORMULA

Ex 13: The general Maclaurin series for the function $f(x) = (1+x)^p$, known as the binomial series, is given by:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

Use this formula to find the first four non-zero terms of the Maclaurin series for $f(x) = \sqrt{1+x}$.

$$\sqrt{1+x} = \boxed{1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}} + \dots$$

Answer: First, we write the function in the form $(1+x)^p$:

$$f(x) = \sqrt{1+x} = (1+x)^{1/2}$$

From this, we can identify that $p = \frac{1}{2}$.
Now, we substitute this value of p into the general binomial series formula to find the coefficients for each term.

- **Constant term (x^0):** The first term is always 1.
- **Term in x:** The coefficient is $p = \frac{1}{2}$. The term is $\frac{1}{2}x$.
- **Term in x^2:** The coefficient is $\frac{p(p-1)}{2!} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} = \frac{\frac{1}{2}(-\frac{1}{2})}{2} = \frac{-1/4}{2} = -\frac{1}{8}$. The term is $-\frac{1}{8}x^2$.
- **Term in x^3:** The coefficient is $\frac{p(p-1)(p-2)}{3!} = \frac{\frac{1}{2}(-\frac{1}{2})(\frac{1}{2}-2)}{6} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6} = \frac{3/8}{6} = \frac{3}{48} = \frac{1}{16}$. The term is $\frac{1}{16}x^3$.

Combining these terms gives the Maclaurin series expansion for $\sqrt{1+x}$:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$$

Ex 14: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

to find the first four non-zero terms of the Maclaurin series for $f(x) = \frac{1}{(1+x)^2}$.

$$\frac{1}{(1+x)^2} = \boxed{1 - 2x + 3x^2 - 4x^3} + \dots$$

Answer: First, we write the function in the form $(1+x)^p$:

$$f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2}$$

From this, we can identify that $p = -2$.
Now, we substitute $p = -2$ into the general binomial series formula to find the coefficients for each term.

- **Constant term (x^0):** The first term is always 1.
- **Term in x:** The coefficient is $p = -2$. The term is $-2x$.
- **Term in x^2:** The coefficient is $\frac{p(p-1)}{2!} = \frac{(-2)(-2-1)}{2} = \frac{(-2)(-3)}{2} = \frac{6}{2} = 3$. The term is $3x^2$.
- **Term in x^3:** The coefficient is $\frac{p(p-1)(p-2)}{3!} = \frac{(-2)(-3)(-2-2)}{6} = \frac{(-2)(-3)(-4)}{6} = \frac{-24}{6} = -4$. The term is $-4x^3$.

Combining these terms gives the Maclaurin series expansion for $\frac{1}{(1+x)^2}$:

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots$$

Ex 15: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

to find the Maclaurin series for $f(x) = (1+x)^3$. Explain why the series terminates.

$$(1+x)^3 = \boxed{1 + 3x + 3x^2 + x^3}$$

Answer: For the function $f(x) = (1+x)^3$, we can identify that $p = 3$.We substitute this value of p into the general binomial series formula.

- **Constant term (x^0):** The first term is always 1.
- **Term in x:** The coefficient is $p = 3$. The term is $3x$.
- **Term in x^2:** The coefficient is $\frac{p(p-1)}{2!} = \frac{(3)(3-1)}{2} = \frac{(3)(2)}{2} = 3$. The term is $3x^2$.

- **Term in x^3:** The coefficient is $\frac{p(p-1)(p-2)}{3!} = \frac{(3)(2)(3-2)}{6} = \frac{(3)(2)(1)}{6} = 1$. The term is x^3.
- **Term in x^4:** The coefficient is $\frac{p(p-1)(p-2)(p-3)}{4!} = \frac{(3)(2)(1)(3-3)}{24} = \frac{(3)(2)(1)(0)}{24} = 0$. The term is 0.

Because the coefficient for the x^4 term is zero, all subsequent coefficients will also be zero, as they will all contain the $(p-3)$ factor. This is a general feature: when p is a non-negative integer, the binomial series terminates and becomes a finite polynomial. Combining the non-zero terms gives the exact expansion:

$$(1+x)^3 = 1 + 3x + 3x^2 + x^3$$

This result is identical to the expansion found using the standard Binomial Theorem.

B MACLAURIN POLYNOMIALS FOR APPROXIMATION

B.1 FINDING MACLAURIN POLYNOMIALS FROM A GIVEN SERIES

Ex 16: Given the Maclaurin series for $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$, write down the Maclaurin polynomial of degree 3, $P_3(x)$.

$$P_3(x) = \boxed{-x - \frac{x^2}{2} - \frac{x^3}{3}}$$

Answer: A Maclaurin polynomial of degree 3, $P_3(x)$, includes all terms from the Maclaurin series with a power of x less than or equal to 3.The series for $\ln(1-x)$ is:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

We select all terms up to degree 3:

- Term of degree 1: $-x$
- Term of degree 2: $-\frac{x^2}{2}$
- Term of degree 3: $-\frac{x^3}{3}$

So, the Maclaurin polynomial of degree 3 is $P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$.

Ex 17: Given the Maclaurin series $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, write down the Maclaurin polynomial of degree 4, $P_4(x)$, in expanded form.

$$P_4(x) = \boxed{1 - x + x^2 - x^3 + x^4}$$

Answer: A Maclaurin polynomial of degree 4, $P_4(x)$, is the sum of the terms of the series from $k=0$ up to $k=4$.
We evaluate the first five terms:

- For $k=0$: $(-1)^0 x^0 = 1$
- For $k=1$: $(-1)^1 x^1 = -x$
- For $k=2$: $(-1)^2 x^2 = x^2$
- For $k=3$: $(-1)^3 x^3 = -x^3$
- For $k=4$: $(-1)^4 x^4 = x^4$

Summing these terms gives the polynomial:

$$P_4(x) = 1 - x + x^2 - x^3 + x^4$$

Ex 18: Given the Maclaurin series for $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$, write down the Maclaurin polynomial of degree 4, $P_4(x)$.

$$P_4(x) = \boxed{x - \frac{x^3}{6}}$$

Answer: A Maclaurin polynomial of degree 4, $P_4(x)$, includes all terms from the Maclaurin series with a power of x less than or equal to 4.
The series for $\sin(x)$ is:

$$\sin(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

We select all terms up to degree 4:

- Term of degree 1: x
- Term of degree 2: $0x^2$ (it is not present in the series)
- Term of degree 3: $-\frac{x^3}{3!} = -\frac{x^3}{6}$
- Term of degree 4: $0x^4$ (it is not present in the series)

So, the Maclaurin polynomial of degree 4 is $P_4(x) = x - \frac{x^3}{6}$.

B.2 FINDING MACLAURIN POLYNOMIALS FROM FIRST PRINCIPLES

Ex 19: Find the Maclaurin polynomial of degree 3 for the function $f(x) = e^{2x}$.

$$P_3(x) = \boxed{1 + 2x + 2x^2 + \frac{4}{3}x^3}$$

Answer: We find the Maclaurin polynomial of degree 3, $P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$, by calculating the first three derivatives of $f(x) = e^{2x}$ and evaluating them at $x=0$.

- $f(x) = e^{2x} \implies f(0) = e^0 = 1$
- $f'(x) = 2e^{2x} \implies f'(0) = 2e^0 = 2$
- $f''(x) = 4e^{2x} \implies f''(0) = 4e^0 = 4$
- $f'''(x) = 8e^{2x} \implies f'''(0) = 8e^0 = 8$

Now, we substitute these values into the formula:

$$\begin{aligned} P_3(x) &= 1 + \frac{2}{1!}x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 \\ &= 1 + 2x + \frac{4}{2}x^2 + \frac{8}{6}x^3 \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 \end{aligned}$$

Ex 20: Find the Maclaurin polynomial of degree 3 for the function $f(x) = \ln(1+x)$.

$$P_3(x) = \boxed{x - \frac{x^2}{2} + \frac{x^3}{3}}$$

Answer: We find the first three derivatives of $f(x) = \ln(1+x)$ and evaluate them at $x=0$.

- $f(x) = \ln(1+x) \implies f(0) = \ln(1) = 0$
- $f'(x) = \frac{1}{1+x} = (1+x)^{-1} \implies f'(0) = 1$
- $f''(x) = -(1+x)^{-2} \implies f''(0) = -1$
- $f'''(x) = 2(1+x)^{-3} \implies f'''(0) = 2$

Now, we substitute these values into the formula for $P_3(x)$:

$$\begin{aligned} P_3(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &= 0 + \frac{1}{1!}x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 \\ &= x - \frac{x^2}{2} + \frac{2x^3}{6} \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} \end{aligned}$$

Ex 21: Find the Maclaurin polynomial of degree 2 for the function $f(x) = e^{x^2}$.

$$P_2(x) = \boxed{1 + x^2}$$

Answer: We find the Maclaurin polynomial of degree 2, $P_2(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2$, by calculating the first two derivatives of $f(x) = e^{x^2}$ and evaluating them at $x = 0$.

- $f(x) = e^{x^2} \implies f(0) = e^0 = 1$
- $f'(x) = e^{x^2} \cdot (2x) = 2xe^{x^2} \implies f'(0) = 2(0)e^0 = 0$
- $f''(x) = (2e^{x^2}) + (2x)(2xe^{x^2})$ (using the product rule) $f''(x) = 2e^{x^2} + 4x^2e^{x^2} \implies f''(0) = 2e^0 + 4(0)^2e^0 = 2$

Now, we substitute these values into the formula:

$$\begin{aligned} P_2(x) &= 1 + \frac{0}{1!}x + \frac{2}{2!}x^2 \\ &= 1 + 0x + \frac{2}{2}x^2 \\ &= 1 + x^2 \end{aligned}$$

B.3 APPROXIMATING FUNCTION VALUES USING MACLAURIN POLYNOMIALS

Ex 22: The Maclaurin polynomial of degree 3 for the function $f(x) = \ln(1+x)$ is

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

Use this polynomial to find an approximation for $\ln(1.4)$ (round to 3 decimal places).

$$\ln(1.4) \approx \boxed{0.341}$$

Answer: To approximate $\ln(1.4)$, we must first identify the value of x to use in the polynomial for $\ln(1+x)$.We set $1+x = 1.4$, which gives $x = 0.4$.Now we substitute $x = 0.4$ into the polynomial $P_3(x)$:

$$\begin{aligned} \ln(1.4) \approx P_3(0.4) &= (0.4) - \frac{(0.4)^2}{2} + \frac{(0.4)^3}{3} \\ &= 0.4 - \frac{0.16}{2} + \frac{0.064}{3} \\ &= 0.4 - 0.08 + 0.021333... \\ &= 0.341333... \end{aligned}$$

Rounding to 3 decimal places, the approximation is 0.341.

Ex 23: The Maclaurin polynomial of degree 3 for the function $f(x) = e^x$ is

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Use this polynomial to find an approximation for e (round to 3 decimal places).

$$e \approx \boxed{2.667}$$

Answer: To approximate $e = e^1$, we substitute $x = 1$ into the polynomial $P_3(x)$:

$$\begin{aligned} e \approx P_3(1) &= 1 + 1 + \frac{1^2}{2!} + \frac{1^3}{3!} \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{6} \\ &= 2 + 0.5 + 0.1666... \\ &= 2.6666... \end{aligned}$$

Rounding to 3 decimal places, the approximation is 2.667.

Ex 24: The Maclaurin polynomial of degree 5 for the function $f(x) = \arctan(x)$ is

$$P_5(x) = x - \frac{x^3}{3} + \frac{x^5}{5}$$

Use this polynomial and the fact that $\arctan(1) = \frac{\pi}{4}$ to find an approximation for π (round to 3 decimal places).

$$\pi \approx \boxed{3.467}$$

Answer: First, we use the polynomial to approximate $\arctan(1)$ by substituting $x = 1$:

$$\begin{aligned} \arctan(1) \approx P_5(1) &= 1 - \frac{1^3}{3} + \frac{1^5}{5} \\ &= 1 - \frac{1}{3} + \frac{1}{5} \\ &= \frac{15 - 5 + 3}{15} \\ &= \frac{13}{15} \approx 0.8666... \end{aligned}$$

Since $\arctan(1) = \frac{\pi}{4}$, we have the approximation:

$$\frac{\pi}{4} \approx \frac{13}{15}$$

To find an approximation for π, we multiply by 4:

$$\pi \approx 4 \times \frac{13}{15} = \frac{52}{15} = 3.4666...$$

Rounding to 3 decimal places, the approximation is 3.467.
Note:This approximation is not very accurate because the series for $\arctan(x)$ converges very slowly when $x = 1$. Modern calculators use similar but much more efficient series-based algorithms to compute constants like π to a very high degree of precision.

B.4 ESTIMATING THE ERROR OF MACLAURIN APPROXIMATIONS

Ex 25: Consider the function $f(x) = \cos(x)$.

1. Find the Maclaurin polynomial of degree 4, $P_4(x)$, for $f(x) = \cos(x)$.
2. Use this polynomial to approximate the value of $\cos(0.5)$.
3. The Lagrange form of the remainder term, $R_n(x)$, gives the exact error of a Maclaurin approximation of degree n, and is defined as:
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$
for some value c between 0 and x. Write down the Lagrange form of the remainder $R_4(x)$ for the approximation in part (b).
4. By finding the maximum possible absolute value of $R_4(0.5)$, determine the upper bound for the error in your approximation of $\cos(0.5)$.

Answer:

1. **Maclaurin Polynomial:** The Maclaurin series for $\cos(x)$ is $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$. The polynomial of degree 4 is:
$$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
2. **Approximation:** We substitute $x = 0.5$ into the polynomial.
$$\cos(0.5) \approx P_4(0.5) = 1 - \frac{(0.5)^2}{2} + \frac{(0.5)^4}{24} = 1 - \frac{0.25}{2} + \frac{0.0625}{24}$$
$$\approx 1 - 0.125 + 0.002604... = 0.877604...$$
3. **Lagrange Remainder Term:** We need the $(n+1)$-th derivative, which is the 5th derivative of $f(x) = \cos(x)$.
   - $f'(x) = -\sin(x)$
   - $f''(x) = -\cos(x)$
   - $f'''(x) = \sin(x)$
   - $f^{(4)}(x) = \cos(x)$
   - $f^{(5)}(x) = -\sin(x)$

   The Lagrange form of the remainder for $n = 4$ and $x = 0.5$ is:
$$R_4(0.5) = \frac{f^{(5)}(c)}{5!}(0.5)^5 = \frac{-\sin(c)}{120}(0.5)^5$$
   for some value $c \in (0, 0.5)$.
4. **Error Bound:** The error is the absolute value of the remainder term: $|R_4(0.5)| = \left|\frac{-\sin(c)}{120}(0.5)^5\right| = \frac{|\sin(c)|}{120}(0.5)^5$. To find the maximum possible error, we need to find the maximum value of $|\sin(c)|$ on the interval $c \in [0, 0.5]$. Since $\sin(c)$ is a positive and increasing function on this interval, its maximum value occurs at $c = 0.5$. However, we know that for any real number c, the absolute value of $\sin(c)$ never exceeds 1. Therefore, we can establish an upper bound:
$$|\sin(c)| \leq 1 \quad \text{for all } c$$
   Now we can bound the error:
$$|R_4(0.5)| \leq \frac{1}{120}(0.5)^5 = \frac{0.03125}{120} \approx 0.00026$$
   This means our approximation in part (b) is accurate to at least 3 decimal places, as the error is less than 0.0005.

Ex 26: Consider the function $f(x) = \ln(1 + x)$.

1. Find the Maclaurin polynomial of degree 3, $P_3(x)$, for $f(x) = \ln(1 + x)$.
2. Use this polynomial to approximate the value of $\ln(1.2)$.
3. The Lagrange form of the remainder term, $R_n(x)$, is given by:
$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$
for some value c between 0 and x. Write down the Lagrange form of the remainder $R_3(x)$ for the approximation in part (b).
4. By finding the maximum possible absolute value of $R_3(0.2)$, determine the upper bound for the error in your approximation of $\ln(1.2)$.

Answer:

1. **Maclaurin Polynomial:** The Maclaurin series for $\ln(1+x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$. The polynomial of degree 3 is:
$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$
2. **Approximation:** To find $\ln(1.2)$, we substitute $x = 0.2$ into the polynomial.
$$\ln(1.2) \approx P_3(0.2) = 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} = 0.2 - \frac{0.04}{2} + \frac{0.008}{3}$$
$$\approx 0.2 - 0.02 + 0.002666... = 0.182666...$$
3. **Lagrange Remainder Term:** We need the $(n+1)$-th derivative, which is the 4th derivative of $f(x) = \ln(1 + x)$.
   - $f'(x) = (1 + x)^{-1}$
   - $f''(x) = -(1 + x)^{-2}$
   - $f'''(x) = 2(1 + x)^{-3}$
   - $f^{(4)}(x) = -6(1 + x)^{-4} = \frac{-6}{(1+x)^4}$

   The Lagrange form of the remainder for $n = 3$ and $x = 0.2$ is:
$$R_3(0.2) = \frac{f^{(4)}(c)}{4!}(0.2)^4 = \frac{-6/(1 + c)^4}{24}(0.2)^4 = \frac{-1}{4(1 + c)^4}(0.2)^4$$
   for some value $c \in (0, 0.2)$.
4. **Error Bound:** The error is the absolute value of the remainder term:
$$|R_3(0.2)| = \left|\frac{-1}{4(1 + c)^4}(0.2)^4\right| = \frac{(0.2)^4}{4(1 + c)^4}$$
   To find the maximum possible error, we need to find the maximum value of this expression for $c \in [0, 0.2]$. The expression is maximized when its denominator, $4(1 + c)^4$, is minimized. Since $g(c) = 4(1 + c)^4$ is an increasing function, its minimum value on the interval $[0, 0.2]$ occurs at the left endpoint, $c = 0$. Therefore, we can bound the error by substituting $c = 0$ into the expression:
$$|R_3(0.2)| \leq \frac{(0.2)^4}{4(1 + 0)^4} = \frac{0.0016}{4} = 0.0004$$
   This means our approximation in part (b) is guaranteed to be within 0.0004 of the true value of $\ln(1.2)$.

C SUBSTITUTION AND DIFFERENTIATION/INTEGRATION TERM-BY-TERM

C.1 FINDING NEW SERIES BY SUBSTITUTION

Ex 27: Starting with the geometric series $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$, find the Maclaurin series for $f(x) = \frac{1}{1-2x}$.

Answer:

$$\begin{aligned} \frac{1}{1-u} &= 1 + u + u^2 + u^3 + \dots \\ \frac{1}{1-(2x)} &= 1 + (2x) + (2x)^2 + (2x)^3 + \dots \quad \text{(letting } u = 2x\text{)} \\ \frac{1}{1-2x} &= 1 + 2x + 4x^2 + 8x^3 + \dots \end{aligned}$$

Ex 28: Starting with the Maclaurin series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$, find the Maclaurin series for $f(x) = e^{x^2}$.

Answer:

$$\begin{aligned} e^u &= 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots \\ e^{(x^2)} &= 1 + (x^2) + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \dots \quad \text{(letting } u = x^2\text{)} \\ e^{x^2} &= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \dots \end{aligned}$$

Ex 29: Starting with the Maclaurin series $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots$, find the Maclaurin series for $f(x) = \cos(2x)$.

Answer:

$$\begin{aligned} \cos(u) &= 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \dots \\ \cos(2x) &= 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots \quad \text{(letting } u = 2x\text{)} \\ \cos(2x) &= 1 - \frac{4x^2}{2} + \frac{16x^4}{24} - \frac{64x^6}{720} + \dots \\ &= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots \end{aligned}$$

Ex 30: Starting with the Maclaurin series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots$, use a substitution to find the first four non-zero terms of the series for $f(x) = e^{ix}$.

Answer: We use the standard series for e^u and make the substitution $u = ix$.

$$\begin{aligned} e^u &= 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots \\ e^{(ix)} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots \quad \text{(letting } u = ix\text{)} \\ e^{ix} &= 1 + ix + \frac{i^2x^2}{2} + \frac{i^3x^3}{6} + \dots \\ &= 1 + ix - \frac{x^2}{2} - \frac{ix^3}{6} + \dots \end{aligned}$$

C.2 FINDING NEW SERIES WITH THE BINOMIAL FORMULA BY SUBSTITUTION

Ex 31: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

to find the first four non-zero terms of the Maclaurin series for $f(x) = \frac{1}{1-2x}$.

$$\frac{1}{1-2x} = \boxed{1 + 2x + 4x^2 + 8x^3} + \dots$$

Answer: First, we write the function in the form $(1+x)^p$. This requires a slight manipulation:

$$f(x) = \frac{1}{1-2x} = (1 + (-2x))^{-1}$$

From this, we can identify that $p = -1$ and we are substituting $-2x$ for x in the binomial series.
Now, we substitute $p = -1$ and replace x with $(-2x)$ in the general binomial series formula to find the coefficients for each term.

- **Constant term (x^0):** The first term is always 1.
- **Term in x:** The coefficient is $p = -1$. The term is $p \times$ (substitution for x) $= (-1)(-2x) = 2x$.
- **Term in x^2:** The coefficient is $\frac{p(p-1)}{2!} = \frac{(-1)(-1-1)}{2} = \frac{(-1)(-2)}{2} = 1$. The term is $\frac{p(p-1)}{2!} \times$ (substitution for x)$^2 = (1)(-2x)^2 = 4x^2$.
- **Term in x^3:** The coefficient is $\frac{p(p-1)(p-2)}{3!} = \frac{(-1)(-2)(-1-2)}{6} = \frac{(-1)(-2)(-3)}{6} = \frac{-6}{6} = -1$. The term is $\frac{p(p-1)(p-2)}{3!} \times$ (substitution for x)$^3 = (-1)(-2x)^3 = (-1)(-8x^3) = 8x^3$.

Combining these terms gives the Maclaurin series expansion for $\frac{1}{1-2x}$:

$$\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + \dots$$

This is consistent with the geometric series formula $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \dots$ with $u = 2x$.

Ex 32: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \dots$$

to find the first four non-zero terms of the Maclaurin series for $f(x) = \sqrt{1+x^2}$.

$$\sqrt{1+x^2} = \boxed{1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16}} + \dots$$

Answer: First, we write the function in the form $(1+x)^p$.

$$f(x) = \sqrt{1+x^2} = (1 + (x^2))^{1/2}$$

From this, we can identify that $p = \frac{1}{2}$ and we are substituting x^2 for x in the binomial series.
Now, we substitute $p = 1/2$ and replace x with (x^2) in the general binomial series formula to find the terms.

- **First term (x^0):** The first term is always 1.

- **Second non-zero term:** The coefficient is $p = \frac{1}{2}$. The term is $p \times \text{(substitution)} = \frac{1}{2}(x^2) = \frac{x^2}{2}$.

- **Third non-zero term:** The coefficient is $\frac{p(p-1)}{2!} = \frac{\frac{1}{2}(\frac{1}{2}-1)}{2} = \frac{-1/4}{2} = -\frac{1}{8}$. The term is $\frac{p(p-1)}{2!} \times \text{(substitution)}^2 = -\frac{1}{8}(x^2)^2 = -\frac{x^4}{8}$.

- **Fourth non-zero term:** The coefficient is $\frac{p(p-1)(p-2)}{3!} = \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{6} = \frac{3/8}{6} = \frac{1}{16}$. The term is $\frac{p(p-1)(p-2)}{3!} \times \text{(substitution)}^3 = \frac{1}{16}(x^2)^3 = \frac{x^6}{16}$.

Combining these terms gives the Maclaurin series expansion for $\sqrt{1+x^2}$:

$$\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} + \dots$$

The series is valid for $|x^2| < 1$, which simplifies to $|x| < 1$.

Ex 33: Use the general binomial series formula:

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \dots$$

to find the first three non-zero terms of the Maclaurin series for $f(x) = \sqrt{2+x}$.

$$\sqrt{2+x} = \boxed{\sqrt{2} + (\sqrt{2}/4) \times x - (\sqrt{2}/32) \times x^2} + \dots$$

Answer: First, we must manipulate the function to fit the form $C(1+u)^p$. We do this by factoring out the constant term inside the square root.

$$f(x) = \sqrt{2+x} = \sqrt{2\left(1+\frac{x}{2}\right)} = \sqrt{2} \cdot \left(1+\frac{x}{2}\right)^{1/2}$$

From this, we see that we will find the series for $\left(1+\frac{x}{2}\right)^{1/2}$ and then multiply the entire result by $\sqrt{2}$. We identify that $p = \frac{1}{2}$ and we are substituting $\frac{x}{2}$ for x in the binomial series.

Let's find the series for $\left(1+\frac{x}{2}\right)^{1/2}$ term by term:

- **First non-zero term:** The first term is 1.

- **Second non-zero term:** The coefficient is $p = \frac{1}{2}$. The term is $p \times \text{(substitution)} = \frac{1}{2}\left(\frac{x}{2}\right) = \frac{x}{4}$.

- **This non-zero term:** The coefficient is $\frac{p(p-1)}{2!} = \frac{\frac{1}{2}(-\frac{1}{2})}{2} = -\frac{1}{8}$. The term is $-\frac{1}{8}\left(\frac{x}{2}\right)^2 = -\frac{1}{8}\left(\frac{x^2}{4}\right) = -\frac{x^2}{32}$.

So, the series for $\left(1+\frac{x}{2}\right)^{1/2}$ is $1+\frac{x}{4}-\frac{x^2}{32}+\dots$. Finally, we multiply this entire series by the $\sqrt{2}$ factor we extracted at the beginning:

$$\sqrt{2+x} = \sqrt{2}\left(1 + \frac{x}{4} - \frac{x^2}{32} + \dots\right)$$

$$\sqrt{2+x} = \sqrt{2} + \frac{\sqrt{2}}{4}x - \frac{\sqrt{2}}{32}x^2 + \dots$$

The series is valid when the substitution is within the interval of convergence: $|\frac{x}{2}| < 1$, which means $|x| < 2$.

C.3 DIFFERENTIATING MACLAURIN POLYNOMIALS

Ex 34: Find the derivative of the Maclaurin polynomial of degree 4 for $\frac{1}{1-x}$: $P_4(x) = 1 + x + x^2 + x^3 + x^4$.

$$P_4'(x) = \boxed{1 + 2x + 3x^2 + 4x^3}$$

Answer: We differentiate the polynomial term-by-term using the power rule, $\frac{d}{dx}(x^n) = nx^{n-1}$.

$$\begin{aligned} P_4'(x) &= \frac{d}{dx}(1 + x + x^2 + x^3 + x^4) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) + \frac{d}{dx}(x^4) \\ &= 0 + 1 + 2x + 3x^2 + 4x^3 \\ &= 1 + 2x + 3x^2 + 4x^3 \end{aligned}$$

Notice that this is the Maclaurin polynomial of degree 3 for the function $g(x) = \frac{1}{(1-x)^2}$, which is consistent with the fact that $\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$.

Ex 35: Find the derivative of the Maclaurin polynomial of degree 5 for e^x:

$$P_5(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$P_5'(x) = \boxed{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}$$

Answer: We differentiate the polynomial term-by-term.

$$\begin{aligned} P_5'(x) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}\right) \\ &= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \frac{5x^4}{120} \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \end{aligned}$$

Notice that the derivative of the degree 5 polynomial for e^x is the degree 4 polynomial for e^x. This is consistent with the fact that $\frac{d}{dx}(e^x) = e^x$.

Ex 36: Find the derivative of the Maclaurin polynomial of degree 5 for $\ln(1+x)$:

$$P_5(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$P_5'(x) = \boxed{1 - x + x^2 - x^3 + x^4}$$

Answer: We differentiate the polynomial term-by-term.

$$\begin{aligned} P_5'(x) &= \frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}\right) \\ &= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{4x^3}{4} + \frac{5x^4}{5} \\ &= 1 - x + x^2 - x^3 + x^4 \end{aligned}$$

Notice that the derivative of the degree 5 polynomial for $\ln(1+x)$ is the degree 4 polynomial for $\frac{1}{1+x}$. This is consistent with the fact that $\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$.

Ex 37: Find the derivative of the Maclaurin polynomial of degree 4 for $\cos(x)$:

$$P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$P_4'(x) = \boxed{-x + \frac{x^3}{6}}$$

Answer: We differentiate the polynomial term-by-term.

$$\begin{aligned} P_4'(x) &= \frac{d}{dx}\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) \\ &= 0 - \frac{2x}{2} + \frac{4x^3}{24} \\ &= -x + \frac{x^3}{6} \\ &= -x + \frac{x^3}{3!} \end{aligned}$$

Notice that this is the negative of the Maclaurin polynomial of degree 3 for $\sin(x)$. This is consistent with the fact that $\frac{d}{dx}(\cos(x)) = -\sin(x)$.

C.4 INTEGRATING MACLAURIN POLYNOMIALS

Ex 38: Find the indefinite integral of the Maclaurin polynomial of degree 3 for $\frac{1}{1-x}$: $P_3(x) = 1 + x + x^2 + x^3$.

$$\int P_3(x)dx = \boxed{C + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}}$$

Answer: We integrate the polynomial term-by-term using the power rule, $\int x^n dx = \frac{x^{n+1}}{n+1} + C$.

$$\begin{aligned} \int P_3(x)dx &= \int (1 + x + x^2 + x^3)dx \\ &= \int 1dx + \int xdx + \int x^2dx + \int x^3dx \\ &= x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + C \end{aligned}$$

Notice that this is the Maclaurin polynomial of degree 4 for the function $g(x) = -\ln(1-x)$, which is consistent with the fact that $\int \frac{1}{1-x}dx = -\ln(1-x) + C$.

Ex 39: Find the indefinite integral of the Maclaurin polynomial of degree 4 for e^x:

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\int P_4(x)dx = \boxed{C + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120}}$$

Answer: We integrate the polynomial term-by-term.

$$\begin{aligned} \int P_4(x)dx &= \int \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)dx \\ &= \int 1dx + \int xdx + \int \frac{x^2}{2!}dx + \int \frac{x^3}{3!}dx + \int \frac{x^4}{4!}dx \\ &= C + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2!} + \frac{x^4}{4\cdot 3!} + \frac{x^5}{5\cdot 4!} \\ &= C + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \end{aligned}$$

Notice that the integral of the degree 4 polynomial for e^x (plus a constant) is the degree 5 polynomial for e^x. This is consistent with the fact that $\int e^x dx = e^x + C$.

Ex 40: Find the indefinite integral of the Maclaurin polynomial of degree 3 for $\cos(x)$:

$$P_3(x) = 1 - \frac{x^2}{2!}$$

$$\int P_3(x)dx = \boxed{C + x - \frac{x^3}{6}}$$

Answer: We integrate the polynomial term-by-term.

$$\begin{aligned} \int P_3(x)dx &= \int \left(1 - \frac{x^2}{2!}\right)dx \\ &= \int 1dx - \int \frac{x^2}{2!}dx \\ &= C + x - \frac{x^3}{3\cdot 2!} \\ &= C + x - \frac{x^3}{3!} \end{aligned}$$

Notice that this result (with C=0) is the Maclaurin polynomial of degree 4 for $\sin(x)$. This is consistent with the fact that $\int \cos(x)dx = \sin(x) + C$.

C.5 FINDING NEW SERIES BY DIFFERENTIATION

Ex 41: Consider the Maclaurin series:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

By differentiating both sides of this equation, find the Maclaurin series for the function $\frac{1}{(1-x)^2}$.

Answer: We start with the given equality and differentiate both sides with respect to x.

- **Differentiating the Left-Hand Side (LHS):**

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{1-x}\right) &= \frac{d}{dx}\left((1-x)^{-1}\right) \\ &= -1(1-x)^{-2}(-1) \\ &= \frac{1}{(1-x)^2} \end{aligned}$$

- **Differentiating the Right-Hand Side (RHS) term-by-term:**

$$\begin{aligned} \frac{d}{dx}\left(1 + x + x^2 + x^3 + \dots\right) &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) + \dots \\ &= 0 + 1 + 2x + 3x^2 + \dots \end{aligned}$$

By equating the derivative of the LHS with the derivative of the RHS, we find the new series:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Ex 42: Consider the Maclaurin series for $\ln(1+x)$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

By differentiating both sides of this equation, find the Maclaurin series for the function $f(x) = \frac{1}{1+x}$.

Answer: We start with the given equality and differentiate both sides with respect to x.

- **Differentiating the Left-Hand Side (LHS):**
$$\frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x}$$
- **Differentiating the Right-Hand Side (RHS) term-by-term:**
$$\begin{aligned}\frac{d}{dx}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) &= \frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\frac{x^3}{3}\right) - \dots \\ &= 1 - \frac{2x}{2} + \frac{3x^2}{3} - \frac{4x^3}{4} + \dots \\ &= 1 - x + x^2 - x^3 + \dots\end{aligned}$$

By equating the derivative of the LHS with the derivative of the RHS, we find the new series:
$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Ex 43: Consider the Maclaurin series for $\sin(x)$:
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$
By differentiating both sides of this equation, find the Maclaurin series for the function $f(x) = \cos(x)$.

Answer: We start with the given equality and differentiate both sides with respect to x.

- **Differentiating the Left-Hand Side (LHS):**
$$\frac{d}{dx}(\sin(x)) = \cos(x)$$
- **Differentiating the Right-Hand Side (RHS) term-by-term:**
$$\begin{aligned}\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) &= \frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{x^3}{6}\right) + \frac{d}{dx}\left(\frac{x^5}{120}\right) - \dots \\ &= 1 - \frac{3x^2}{6} + \frac{5x^4}{120} - \frac{7x^6}{5040} + \dots \\ &= 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\end{aligned}$$

By equating the derivative of the LHS with the derivative of the RHS, we find the series for $\cos(x)$:
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Ex 44: Consider the Maclaurin series for e^x:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
Show that differentiating the series term-by-term reproduces the original series.

Answer: We start with the series for e^x and differentiate it term-by-term.
$$\begin{aligned}\frac{d}{dx}(e^x) &= \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \\ &= \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\frac{x^3}{6}\right) + \dots \\ &= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x^3}{24} + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\end{aligned}$$

The resulting series is identical to the original series for e^x. This is consistent with the fact that $\frac{d}{dx}(e^x) = e^x$.

C.6 FINDING NEW SERIES BY INTEGRATION

Ex 45: Consider the Maclaurin series:
$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \dots$$
By integrating both sides of this equation from 0 to x, find the Maclaurin series for the function $f(x) = \ln(1+x)$.

Answer: We start with the given equality and integrate both sides from $t = 0$ to $t = x$.

- **Integrating the Left-Hand Side (LHS):**
$$\begin{aligned}\int_0^x \frac{1}{1+t}\,dt &= [\ln|1+t|]_0^x \\ &= \ln|1+x| - \ln|1+0| \\ &= \ln(1+x) \quad (\text{for } |x| < 1)\end{aligned}$$
- **Integrating the Right-Hand Side (RHS) term-by-term:**
$$\begin{aligned}\int_0^x \left(1 - t + t^2 - t^3 + \dots\right) dt &= \left[t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots\right]_0^x \\ &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) - (0) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\end{aligned}$$

By equating the integral of the LHS with the integral of the RHS, we find the new series:
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Ex 46: Consider the Maclaurin series for $\cos(t)$:
$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$$
By integrating both sides of this equation from 0 to x, find the Maclaurin series for the function $f(x) = \sin(x)$.

Answer: We start with the given equality and integrate both sides from $t = 0$ to $t = x$.

- **Integrating the Left-Hand Side (LHS):**
$$\int_0^x \cos(t)\,dt = [\sin(t)]_0^x = \sin(x) - \sin(0) = \sin(x)$$
- **Integrating the Right-Hand Side (RHS) term-by-term:**
$$\begin{aligned}\int_0^x \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots\right) dt &= \left[t - \frac{t^3}{3\cdot 2!} + \frac{t^5}{5\cdot 4!} - \dots\right]_0^x \\ &= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) - (0) \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\end{aligned}$$

By equating the integral of the LHS with the integral of the RHS, we find the series for $\sin(x)$:
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Ex 47:

1. Starting with the geometric series $\frac{1}{1-u} = 1 + u + u^2 + \dots$, use a substitution to find the Maclaurin series for $\frac{1}{1+t^2}$.
2. By integrating the resulting series from 0 to x, find the Maclaurin series for $f(x) = \arctan(x)$.

Answer:

- **Find the series for** $\frac{1}{1+t^2}$: We start with the geometric series and substitute $u = -t^2$.

$$\begin{aligned}\frac{1}{1-u} &= 1 + u + u^2 + u^3 + \dots \\ \frac{1}{1-(-t^2)} &= 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \dots \\ \frac{1}{1+t^2} &= 1 - t^2 + t^4 - t^6 + \dots\end{aligned}$$

- **Integrate to find the series for** $\arctan(x)$: We know that $\int_0^x \frac{1}{1+t^2} dt = [\arctan(t)]_0^x = \arctan(x) - \arctan(0) = \arctan(x)$. We now integrate the series term-by-term:

$$\begin{aligned}\arctan(x) &= \int_0^x \left(1 - t^2 + t^4 - t^6 + \dots\right) dt \\ &= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots\right]_0^x \\ &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \dots\right) - (0) \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\end{aligned}$$

D LINEARITY OF MACLAURIN SERIES

D.1 COMBINING SERIES TO FIND NEW SERIES

Ex 48: This exercise guides you to find the Maclaurin series for the hyperbolic cosine function, $\cosh(x)$.

1. Start with the Maclaurin series for e^u. By substituting $u = -x$, find the Maclaurin series for $f(x) = e^{-x}$.
2. The hyperbolic cosine is defined as $\cosh(x) = \frac{e^x + e^{-x}}{2}$. Use your series for e^x and e^{-x} to find the Maclaurin series for $\cosh(x)$.

Answer:

1. We substitute $u = -x$ into the series $e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$

$$\begin{aligned}e^{-x} &= 1 + (-x) + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \frac{(-x)^4}{4!} + \dots \\ &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\end{aligned}$$

2. We add the series for e^x and e^{-x} term by term.

$$\begin{array}{rl} e^x & = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ + e^{-x} & = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \\ \hline e^x + e^{-x} & = 2 + 0 + 2\frac{x^2}{2!} + 0 + 2\frac{x^4}{4!} + \dots \end{array}$$

Notice that all the odd-powered terms cancel out.

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

Now, we divide by 2 to find the series for $\cosh(x)$:

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

This is remarkably similar to the series for $\cos(x)$, but without the alternating signs.

Ex 49: This exercise guides you to find the Maclaurin series for the hyperbolic sine function, $\sinh(x)$.

1. You have already found the series for e^x and e^{-x}. Recall them here.
2. The hyperbolic sine is defined as $\sinh(x) = \frac{e^x - e^{-x}}{2}$. Use your series for e^x and e^{-x} to find the Maclaurin series for $\sinh(x)$.

Answer:

1. The required Maclaurin series are:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots$$

2. We subtract the series for e^{-x} from the series for e^x term by term.

$$\begin{array}{rl} e^x & = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ -\,(e^{-x}) & = (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots) \\ \hline e^x - e^{-x} & = 0 + 2x + 0 + 2\frac{x^3}{3!} + 0 + \dots \end{array}$$

Notice that all the even-powered terms cancel out.

$$e^x - e^{-x} = 2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

Now, we divide by 2 to find the series for $\sinh(x)$:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

This is remarkably similar to the series for $\sin(x)$, but without the alternating signs.

Ex 50: Consider the Maclaurin series for the real exponential function:

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \dots$$

By formally substituting $u = ix$ and rearranging the terms, show how this series relates to the series for $\cos(x)$ and $\sin(x)$.

Answer: We substitute $u = ix$ into the series for e^u:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

Next, we simplify the powers of i, remembering that $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i$, and so on.

$$e^{ix} = 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \frac{i^5x^5}{5!} + \dots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \dots$$

Now, we group the terms into real and imaginary parts:

$$e^{ix} = \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)}_{\text{Real Part}} + i \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}_{\text{Imaginary Part}}$$

We recognize these two series:

- The real part is the Maclaurin series for $\cos(x)$.
- The imaginary part is the Maclaurin series for $\sin(x)$.

This demonstrates Euler's Formula: $e^{ix} = \cos(x) + i\sin(x)$.