LOGARITHMS

In the previous chapter, we explored exponentiation, which answers the question of what we get when we multiply a number by itself a certain number of times (for example, $10^3 = 10 \times 10 \times 10 = 1000$).
Now we will ask the inverse question: **To what exponent must 10 be raised to get 1000?**
This question leads us to the concept of the **logarithm**. A logarithm is the inverse operation of exponentiation: it is the tool we use to find an unknown exponent.
Before the invention of calculators, logarithms were a revolutionary tool for scientists, turning complex multiplications into simpler additions. Today, they remain essential for solving exponential equations and are used to describe phenomena in science, such as the pH scale in chemistry or the Richter scale for earthquakes.
In this chapter, we will:

- define logarithms (in base 10),
- explore their main properties (the laws of logarithms),
- and use them to solve exponential equations and real-world problems.

Throughout this chapter, unless stated otherwise, we use $\log x$ to mean the logarithm *base 10*: $\log x = \log_{10} x$.

A LOGARITHMS IN BASE 10

Definition Logarithm

The **(base 10) logarithm** of a number y (where $y > 0$) is the exponent to which 10 must be raised to obtain y. It is denoted as:

$$\log y = x \quad \Leftrightarrow \quad 10^x = y$$

In other words, $\log y$ is the exponent x such that $10^x = y$.

Proposition Inverse Properties of Logarithms

For real numbers:

- $10^{\log x} = x$ for any $x > 0$,
- $\log(10^x) = x$ for any real x.

These properties reflect that the exponential function 10^x and the logarithm function $\log x$ (base 10) are inverses of each other.

Ex: Evaluate $\log 100$.

Answer: $\log(100) = \log\left(10^2\right)$
$= 2$
Since $10^2 = 100$, the logarithm of 100 (base 10) is 2.

B LOGARITHMS IN BASE a

In the previous section we defined the logarithm in base 10 of a number as the exponent to which 10 must be raised in order to obtain that number.
We can use the same principle to define logarithms in other bases:

Definition Logarithm in Base a

For a positive number $a \neq 1$, the **logarithm base a** of a number y (where $y > 0$) is the exponent to which a must be raised to obtain y. It is denoted as:

$$\log_a y = x \quad \Leftrightarrow \quad a^x = y.$$

Proposition Inverse Properties of Logarithms in Base a

Let $a > 0$ with $a \neq 1$.

- $a^{\log_a x} = x$ for any $x > 0$,
- $\log_a(a^x) = x$ for any real x.

These properties show that the exponential function a^x and the logarithm function $\log_a x$ are inverses of each other.

Ex: Evaluate $\log_2 8$.

Answer: $\log_2 8 = \log_2(2^3)$
$= 3$

Proposition Laws of Logarithms in Base a

For $x > 0$, $y > 0$, and $a > 0$, $a \neq 1$:

- **Product Rule**: $\log_a(xy) = \log_a x + \log_a y$
- **Quotient Rule**: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- **Power Rule**: $\log_a(x^k) = k \log_a x$

Proof

$$\begin{aligned} \log_a(xy) &= \log_a\left(a^{\log_a x} \times a^{\log_a y}\right) \\ &= \log_a\left(a^{\log_a x + \log_a y}\right) \\ &= \log_a x + \log_a y \end{aligned}$$

$$\begin{aligned} \log_a\left(\frac{x}{y}\right) &= \log_a\left(\frac{a^{\log_a x}}{a^{\log_a y}}\right) \\ &= \log_a\left(a^{\log_a x - \log_a y}\right) \\ &= \log_a x - \log_a y \end{aligned}$$

$$\begin{aligned} \log_a(x^k) &= \log_a\left(\left(a^{\log_a x}\right)^k\right) \\ &= \log_a\left(a^{k \log_a x}\right) \\ &= k \log_a x \end{aligned}$$

Method Solving Exponential Equations with Base a

To solve $a^x = b$ (where $a > 0$, $a \neq 1$, $b > 0$):

1. Take the logarithm base a of both sides: $\log_a(a^x) = \log_a b$
2. Simplify: $x = \log_a b$

Ex: Solve $3^x = 81$.

Answer:

$$\begin{aligned} 3^x &= 81 \\ \log_3(3^x) &= \log_3(81) \\ x &= \log_3(3^4) \\ x &= 4 \end{aligned}$$

C NATURAL LOGARITHM

The natural logarithm, often denoted $\ln x$, is the logarithm with base e (where $e \approx 2.71828$ is the base of the natural exponential function). It is the inverse of the exponential function e^x and plays a central role in calculus, growth models, and many scientific applications.

Definition Natural Logarithm

The **natural logarithm**, denoted $\ln x$, is the logarithm base e, defined for $x > 0$:

$$\ln x = \log_e(x).$$

Proposition Inverse Properties of Natural Logarithms

- $e^{\ln x} = x$ for $x > 0$,
- $\ln(e^x) = x$ for any real x.

D LAWS OF LOGARITHMS

Proposition Laws of Logarithms

For $x > 0$, $y > 0$ and $a > 0$ with $a \neq 1$:

- **Product Rule**: $\log_a(xy) = \log_a x + \log_a y$
- **Quotient Rule**: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
- **Power Rule**: $\log_a(x^k) = k\log_a x$

Proof

$$\begin{aligned}\log_a(xy) &= \log_a\left(a^{\log_a x} \times a^{\log_a y}\right)\\ &= \log_a\left(a^{\log_a x + \log_a y}\right)\\ &= \log_a x + \log_a y\end{aligned}$$

$$\begin{aligned}\log_a\left(\frac{x}{y}\right) &= \log_a\left(\frac{a^{\log_a x}}{a^{\log_a y}}\right)\\ &= \log_a\left(a^{\log_a x - \log_a y}\right)\\ &= \log_a x - \log_a y\end{aligned}$$

$$\begin{aligned}\log_a(x^k) &= \log_a\left(\left(a^{\log_a x}\right)^k\right)\\ &= \log_a\left(a^{k\log_a x}\right)\\ &= k\log_a x\end{aligned}$$

Ex: Write as a single logarithm: $\log(5) + \log(3)$.

Answer: $\begin{aligned}\log(5) + \log(3) &= \log(5 \times 3)\\ &= \log 15\end{aligned}$

In particular, for the natural logarithm, for $x > 0$ and $y > 0$:

- $\ln(xy) = \ln x + \ln y$
- $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
- $\ln(x^k) = k\ln x$

Ex: Evaluate $\ln\left(e^4\right)$.

Answer: $\begin{aligned}\ln\left(e^4\right) &= 4\ln e\\ &= 4 \times 1\\ &= 4\end{aligned}$

E CHANGE OF BASE RULE

Proposition Change of Base Rule

For any $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$, and $x > 0$:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

Proof

Let $y = \log_a x$.

$$\begin{aligned} y &= \log_a x \\ a^y &= a^{\log_a x} \\ a^y &= x \\ \log_b a^y &= \log_b x \\ y \log_b a &= \log_b x \quad \text{(power rule)} \\ y &= \frac{\log_b x}{\log_b a} \\ \log_a x &= \frac{\log_b x}{\log_b a} \quad (\text{as } y = \log_a x). \end{aligned}$$

This rule allows us to compute logarithms in any base using a calculator that computes base 10 logarithms or natural logarithms.

Ex: Compute $\log_2 10$ using natural logarithms.

Answer:

$$\begin{aligned} \log_2 10 &= \frac{\ln 10}{\ln 2} \\ &\approx \frac{2.302585}{0.693147} \\ &\approx 3.321928 \end{aligned}$$

Method Using Change of Base

To compute $\log_a b$ using base 10 logarithms:

$$\log_a b = \frac{\log b}{\log a}.$$

Using natural logarithms:

$$\log_a b = \frac{\ln b}{\ln a}.$$

F APPLICATIONS OF LOGARITHMS

Logarithms are used to describe quantities that vary over many orders of magnitude. Some important examples in science include:

- **pH scale** in chemistry: $\text{pH} = -\log_{10}[H^+]$
  (where $[H^+]$ is the hydrogen ion concentration in moles per litre)
- **Richter scale** for earthquakes: Magnitude $= \log_{10}\left(\frac{I}{I_0}\right)$
  (where I is the intensity of the earthquake, I_0 is a reference intensity)
- **Decibel (dB) scale** for sound: $L = 10\log_{10}\left(\frac{P}{P_0}\right)$
  (where P is the power/intensity measured, P_0 is a reference level)

On calculators, the log key usually means $\log_{10}$, and the ln key means the natural logarithm (base e).

Ex: The pH of a solution is 3.2. Find the hydrogen ion concentration $[H^+]$.

Answer: We know:

$$\begin{aligned} \text{pH} &= -\log_{10}[H^+] \\ 3.2 &= -\log_{10}[H^+] && \text{(substituting the value)} \\ -3.2 &= \log_{10}[H^+] && \text{(multiplying both sides by } -1) \\ 10^{-3.2} &= 10^{\log_{10}[H^+]} && \text{(exponentiating both sides, base 10)} \\ [H^+] &= 10^{-3.2} \\ [H^+] &\approx 6.31 \times 10^{-4} \text{ mol/L} && \text{(using calculator)} \end{aligned}$$