LOGARITHMS

A LOGARITHMS IN BASE 10

A.1 EVALUATING LOGARITHMS

Ex 1: Evaluate:

$$\log 100 = \boxed{2}$$

Answer:

$$\begin{aligned}\log(100) &= \log\left(10^2\right)\\ &= 2\end{aligned}$$

Ex 2: Evaluate:

$$\log 0.1 = \boxed{-1}$$

Answer:

$$\begin{aligned}\log(0.1) &= \log\left(10^{-1}\right)\\ &= -1\end{aligned}$$

Ex 3: Evaluate:

$$\log\left(\frac{1}{100}\right) = \boxed{-2}$$

Answer:

$$\begin{aligned}\log\left(\frac{1}{100}\right) &= \log\left(\frac{1}{10^2}\right)\\ &= \log\left(10^{-2}\right)\\ &= -2\end{aligned}$$

Ex 4: Evaluate:

$$\log\sqrt{10} = \boxed{0.5}$$

Answer:

$$\begin{aligned}\log(\sqrt{10}) &= \log(10^{0.5})\\ &= 0.5\end{aligned}$$

Ex 5: Evaluate:

$$\log 1 = \boxed{0}$$

Answer:

$$\begin{aligned}\log(1) &= \log\left(10^0\right)\\ &= 0\end{aligned}$$

A.2 EVALUATING USING A CALCULATOR

Ex 6: Evaluate (round to 2 decimal places).

$$\log(2) \approx \boxed{0.30}$$

Answer: By entering log(2) and pressing the equal button, the calculator displays: 0.30103.
So, $\log(2) \approx 0.30$ (rounded to two decimal places).

Ex 7: Evaluate (round to 2 decimal places).

$$\log(0.2) \approx \boxed{-0.70}$$

Answer: By entering log(0.2) and pressing the equal button, the calculator displays: -0.69897.
So, $\log(0.2) \approx -0.70$ (rounded to two decimal places).

Ex 8: Evaluate (round to 2 decimal places).

$$\log(2\times 10^9) \approx \boxed{9.30}$$

Answer: By entering $\log(2\times 10^9)$ and pressing the equal button, the calculator displays: 9.30103.
So, $\log(2\times 10^9) \approx 9.30$ (rounded to two decimal places).

A.3 SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

Ex 9: Find x such that $8 = 10^x$.

$$x \approx \boxed{0.903} \text{ (rounded to 3 decimal places)}$$

Answer: To solve $8 = 10^x$, take the logarithm (base 10) of both sides:

$$\begin{aligned}8 &= 10^x\\ \log(8) &= \log(10^x)\\ \log(8) &= x\\ x &\approx 0.903\end{aligned}$$

So, $x \approx 0.903$ (rounded to 3 decimal places).

Ex 10: Find x such that $0.4 = 10^x$.

$$x \approx \boxed{-0.398} \text{ (rounded to 3 decimal places)}$$

Answer: To solve $0.4 = 10^x$, take the logarithm (base 10) of both sides:

$$\begin{aligned}0.4 &= 10^x\\ \log(0.4) &= \log(10^x)\\ \log(0.4) &= x\\ x &\approx -0.398\end{aligned}$$

So, $x \approx -0.398$ (rounded to 3 decimal places).

Ex 11: Find x such that $250 = 10^x$.

$$x \approx \boxed{2.398} \text{ (rounded to 3 decimal places)}$$

Answer: To solve $250 = 10^x$, take the logarithm (base 10) of both sides:

$$\begin{aligned}250 &= 10^x\\ \log(250) &= \log(10^x)\\ \log(250) &= x\\ x &\approx 2.398\end{aligned}$$

So, $x \approx 2.398$ (rounded to 3 decimal places).

A.4 SOLVING FOR x WHEN $\log(x)$ IS GIVEN

Ex 12: Find x such that $\log(x) = 3$.

$$x = \boxed{1000}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned} \log(x) &= 3 \\ 10^{\log(x)} &= 10^3 \\ x &= 10^3 \\ x &= 1000 \end{aligned}$$

Ex 13: Find x such that $\log(x) = -1$.

$$x = \boxed{0.1}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned} \log(x) &= -1 \\ 10^{\log(x)} &= 10^{-1} \\ x &= 10^{-1} \\ x &= 0.1 \end{aligned}$$

Ex 14: Find x such that $\log(x) = 0$.

$$x = \boxed{1}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned} \log(x) &= 0 \\ 10^{\log(x)} &= 10^0 \\ x &= 10^0 \\ x &= 1 \end{aligned}$$

Ex 15: Find x such that $\log(x) = 7$.

$$x = \boxed{10000000}$$

Answer: Take $10^{\cdot}$ on both sides:

$$\begin{aligned} \log(x) &= 7 \\ 10^{\log(x)} &= 10^7 \\ x &= 10^7 \\ x &= 10\,000\,000 \end{aligned}$$

B LOGARITHMS IN BASE a

B.1 EVALUATING LOGARITHMS

Ex 16: Evaluate:

$$\log_2 8 = \boxed{3}$$

Answer:

$$\begin{aligned} \log_2(8) &= \log_2\left(2^3\right) \\ &= 3 \end{aligned}$$

Ex 17: Evaluate:

$$\log_3 1/9 = \boxed{-2}$$

Answer:

$$\begin{aligned} \log_3(1/9) &= \log_3\left(3^{-2}\right) \\ &= -2 \end{aligned}$$

Ex 18: Evaluate:

$$\log_5\left(\frac{1}{25}\right) = \boxed{-2}$$

Answer:

$$\begin{aligned} \log_5\left(\frac{1}{25}\right) &= \log_5\left(\frac{1}{5^2}\right) \\ &= \log_5\left(5^{-2}\right) \\ &= -2 \end{aligned}$$

Ex 19: Evaluate:

$$\log_4 \sqrt{4} = \boxed{0.5}$$

Answer:

$$\begin{aligned} \log_4(\sqrt{4}) &= \log_4(4^{0.5}) \\ &= 0.5 \end{aligned}$$

Ex 20: Evaluate:

$$\log_7 1 = \boxed{0}$$

Answer:

$$\begin{aligned} \log_7(1) &= \log_7\left(7^0\right) \\ &= 0 \end{aligned}$$

B.2 SOLVING FOR x WHEN $\log_a(x)$ IS GIVEN

Ex 21: Find x such that $\log_2(x) = 3$.

$$x = \boxed{8}$$

Answer: Take $2^{\cdot}$ on both sides:

$$\begin{aligned} \log_2(x) &= 3 \\ 2^{\log_2(x)} &= 2^3 \\ x &= 2^3 \\ x &= 8 \end{aligned}$$

Ex 22: Find x such that $\log_2(x) = -1$.

$$x = \boxed{0.5}$$

Answer: Take $2^{\cdot}$ on both sides:

$$\begin{aligned} \log_2(x) &= -1 \\ 2^{\log_2(x)} &= 2^{-1} \\ x &= 2^{-1} \\ x &= 0.5 \end{aligned}$$

Ex 23: Find x such that $\log_2(x) = 0$.

$$x = \boxed{1}$$

Answer: Take $2^{\cdot}$ on both sides:

$$\begin{aligned} \log_2(x) &= 0 \\ 2^{\log_2(x)} &= 2^{0} \\ x &= 2^{0} \\ x &= 1 \end{aligned}$$

Ex 24: Find x such that $\log_2(x) = 7$.

$$x = \boxed{128}$$

Answer: Take $2^{\cdot}$ on both sides:

$$\begin{aligned} \log_2(x) &= 7 \\ 2^{\log_2(x)} &= 2^{7} \\ x &= 2^{7} \\ x &= 128 \end{aligned}$$

C NATURAL LOGARITHM

C.1 EVALUATING NATURAL LOGARITHMS

Ex 25: Evaluate:

$$\ln e^3 = \boxed{3}$$

Answer:

$$\ln(e^3) = 3$$

Ex 26: Evaluate:

$$\ln(1/e) = \boxed{-1}$$

Answer:

$$\begin{aligned} \ln(1/e) &= \ln(e^{-1}) \\ &= -1 \end{aligned}$$

Ex 27: Evaluate:

$$\ln\left(\frac{1}{e^2}\right) = \boxed{-2}$$

Answer:

$$\begin{aligned} \ln\left(\frac{1}{e^2}\right) &= \ln\left(e^{-2}\right) \\ &= -2 \end{aligned}$$

Ex 28: Evaluate:

$$\ln\sqrt{e} = \boxed{0.5}$$

Answer:

$$\begin{aligned} \ln(\sqrt{e}) &= \ln(e^{0.5}) \\ &= 0.5 \end{aligned}$$

Ex 29: Evaluate:

$$\ln 1 = \boxed{0}$$

Answer:

$$\begin{aligned} \ln(1) &= \ln\left(e^{0}\right) \\ &= 0 \end{aligned}$$

D LAWS OF LOGARITHMS

D.1 WRITING AS A SINGLE LOGARITHM: LEVEL 1

Ex 30: Write as a single logarithm

$$\log(5) + \log(3) = \boxed{\log(15)}$$

Answer:

$$\begin{aligned} \log(5) + \log(3) &= \log(5 \times 3) \\ &= \log 15 \end{aligned}$$

Ex 31: Write as a single logarithm in the form $\log k$:

$$\log(15) - \log(5) = \boxed{\log(3)}$$

Answer:

$$\begin{aligned} \log(15) - \log(5) &= \log\left(\frac{15}{5}\right) \\ &= \log(3) \end{aligned}$$

Ex 32: Write as a single logarithm in the form $\log k$:

$$\log(4) + \log\left(\frac{1}{2}\right) = \boxed{\log(2)}$$

Answer:

$$\begin{aligned} \log(4) + \log\left(\frac{1}{2}\right) &= \log\left(4 \times \frac{1}{2}\right) \\ &= \log(2) \end{aligned}$$

Ex 33: Write as a single logarithm in the form $\log k$:

$$\log(18) - \log(3) = \boxed{\log(6)}$$

Answer:

$$\begin{aligned} \log(18) - \log(3) &= \log\left(\frac{18}{3}\right) \\ &= \log(6) \end{aligned}$$

D.2 WRITING AS A SINGLE LOGARITHM: LEVEL 2

Ex 34: Write as a single logarithm in the form $\log k$:

$$\log(8) + 1 = \boxed{\log(80)}$$

Answer:

$$\begin{aligned}\log(8) + 1 &= \log(8) + \log(10)\\ &= \log(8 \times 10)\\ &= \log(80)\end{aligned}$$

Ex 35: Write as a single logarithm in the form $\log k$:

$$\log(3) + 2 = \boxed{\log(300)}$$

Answer:

$$\begin{aligned}\log(3) + 2 &= \log(3) + \log(10^2)\\ &= \log(3) + \log(100)\\ &= \log(3 \times 100)\\ &= \log(300)\end{aligned}$$

Ex 36: Write as a single logarithm in the form $\log k$:

$$2 - \log(25) = \boxed{\log(4)}$$

Answer:

$$\begin{aligned}2 - \log(25) &= \log(10^2) - \log(25)\\ &= \log\left(\frac{100}{25}\right)\\ &= \log(4)\end{aligned}$$

Ex 37: Write as a single logarithm in the form $\log k$:

$$\log(200) - 2 = \boxed{\log(2)}$$

Answer:

$$\begin{aligned}\log(200) - 2 &= \log(200) - \log(10^2)\\ &= \log\left(\frac{200}{100}\right)\\ &= \log(2)\end{aligned}$$

D.3 WRITING AS A SINGLE LOGARITHM: LEVEL 3

Ex 38: Write as a single logarithm in the form $\log k$:

$$2\log(3) + 1 = \boxed{\log(90)}$$

Answer:

$$\begin{aligned}2\log(3) + 1 &= \log(3^2) + \log(10^1)\\ &= \log(9) + \log(10)\\ &= \log(9 \times 10)\\ &= \log(90)\end{aligned}$$

Ex 39: Write as a single logarithm in the form $\log k$:

$$3\log(2) - \log(4) = \boxed{\log(2)}$$

Answer:

$$\begin{aligned}3\log(2) - \log(4) &= \log(2^3) - \log(4)\\ &= \log(8) - \log(4)\\ &= \log\left(\frac{8}{4}\right)\\ &= \log(2)\end{aligned}$$

Ex 40: Write as a single logarithm in the form $\log k$:

$$2\log(20) - 2 = \boxed{\log(4)}$$

Answer:

$$\begin{aligned}2\log(20) - 2 &= \log(20^2) - \log(10^2)\\ &= \log(400) - \log(100)\\ &= \log\left(\frac{400}{100}\right)\\ &= \log(4)\end{aligned}$$

Ex 41: Write as a single logarithm in the form $\log k$:

$$2\log(30) - 1 = \boxed{\log(90)}$$

Answer:

$$\begin{aligned}2\log(30) - 1 &= \log(30^2) - \log(10^1)\\ &= \log(900) - \log(10)\\ &= \log\left(\frac{900}{10}\right)\\ &= \log(90)\end{aligned}$$

D.4 WRITING AS A SINGLE NATURAL LOGARITHM: LEVEL 1

Ex 42: Write as a single logarithm

$$\ln(5) + \ln(3) = \boxed{\ln(15)}$$

Answer:

$$\begin{aligned}\ln(5) + \ln(3) &= \ln(5 \times 3)\\ &= \ln 15\end{aligned}$$

Ex 43: Write as a single logarithm in the form $\ln k$:

$$\ln(15) - \ln(5) = \boxed{\ln(3)}$$

Answer:

$$\begin{aligned}\ln(15) - \ln(5) &= \ln\left(\frac{15}{5}\right)\\ &= \ln(3)\end{aligned}$$

Ex 44: Write as a single logarithm

$$\ln(x) + \ln(x) = \boxed{\ln(x^2)}$$

Answer:

$$\begin{aligned}\ln(x) + \ln(x) &= \ln(x \times x)\\ &= \ln(x^2)\end{aligned}$$

Ex 45: Write as a single logarithm in the form $\ln k$:

$$\ln(20) - \ln(4) = \boxed{\ln(5)}$$

Answer:

$$\begin{aligned}\ln(20) - \ln(4) &= \ln\left(\frac{20}{4}\right)\\ &= \ln(5)\end{aligned}$$

D.5 WRITING AS A SINGLE NATURAL LOGARITHM: LEVEL 2

Ex 46: Write as a single logarithm in the form $\ln k$:

$$\ln(8) + 1 = \boxed{\ln(8e)}$$

Answer:

$$\begin{aligned} \ln(8) + 1 &= \ln(8) + \ln(e) \\ &= \ln(8 \times e) \\ &= \ln(8e) \end{aligned}$$

Ex 47: Write as a single logarithm in the form $\ln k$:

$$2 - \ln(5) = \boxed{\ln(e^2/5)}$$

Answer:

$$\begin{aligned} 2 - \ln(5) &= \ln(e^2) - \ln(5) \\ &= \ln\left(\frac{e^2}{5}\right) \end{aligned}$$

Ex 48: Write as a single logarithm:

$$1 - \ln(x) = \boxed{\ln(e/x)}$$

Answer:

$$\begin{aligned} 1 - \ln(x) &= \ln(e) - \ln(x) \\ &= \ln\left(\frac{e}{x}\right) \end{aligned}$$

Ex 49: Write as a single logarithm:

$$\ln(2) + x = \boxed{\ln(2e^x)}$$

Answer:

$$\begin{aligned} \ln(2) + x &= \ln(2) + \ln(e^x) \\ &= \ln(2 \cdot e^x) \\ &= \ln(2e^x) \end{aligned}$$

D.6 WRITING AS A SINGLE NATURAL LOGARITHM: LEVEL 3

Ex 50: Write as a single logarithm in the form $\ln k$:

$$2\ln(3) + 1 = \boxed{\ln(9e)}$$

Answer:

$$\begin{aligned} 2\ln(3) + 1 &= \ln(3^2) + \ln(e) \\ &= \ln(9) + \ln(e) \\ &= \ln(9 \times e) \\ &= \ln(9e) \end{aligned}$$

Ex 51: Write as a single logarithm in the form $\ln k$:

$$3\ln(2) - \ln(4) = \boxed{\ln(2)}$$

Answer:

$$\begin{aligned} 3\ln(2) - \ln(4) &= \ln(2^3) - \ln(4) \\ &= \ln(8) - \ln(4) \\ &= \ln\left(\frac{8}{4}\right) \\ &= \ln(2) \end{aligned}$$

Ex 52: Write as a single logarithm:

$$2\ln(x) + \ln(4) = \boxed{\ln(4x^2)}$$

Answer:

$$\begin{aligned} 2\ln(x) + \ln(4) &= \ln(x^2) + \ln(4) \\ &= \ln(x^2 \cdot 4) \\ &= \ln(4x^2) \end{aligned}$$

Ex 53: Write as a single logarithm:

$$3\ln(x) - 2\ln(\sqrt{x}) = \boxed{\ln(x^2)}$$

Answer:

$$\begin{aligned} 3\ln(x) - 2\ln(\sqrt{x}) &= \ln(x^3) - \ln((\sqrt{x})^2) \\ &= \ln(x^3) - \ln(x) \\ &= \ln\left(\frac{x^3}{x}\right) \\ &= \ln(x^2) \end{aligned}$$

D.7 WRITING AS A SINGLE LOGARITHM IN BASE A: LEVEL 1

Ex 54: Write as a single logarithm

$$\log_2(5) + \log_2(3) = \boxed{\log_2(15)}$$

Answer:

$$\begin{aligned} \log_2(5) + \log_2(3) &= \log_2(5 \times 3) \\ &= \log_2 15 \end{aligned}$$

Ex 55: Write as a single logarithm in the form $\log_a k$:

$$\log_3(18) - \log_3(6) = \boxed{\log_3(3)}$$

Answer:

$$\begin{aligned} \log_3(18) - \log_3(6) &= \log_3\left(\frac{18}{6}\right) \\ &= \log_3(3) \end{aligned}$$

D.8 WRITING AS A SINGLE LOGARITHM IN BASE A: LEVEL 2

Ex 56: Write as a single logarithm in the form $\log_a k$:

$$\log_5(4) + 1 = \boxed{\log_5(20)}$$

Answer:

$$\begin{aligned} \log_5(4) + 1 &= \log_5(4) + \log_5(5) \\ &= \log_5(4 \times 5) \\ &= \log_5(20) \end{aligned}$$

Ex 57: Write as a single logarithm in the form $\log_a k$:

$$2 - \log_3(5) = \boxed{\log_3(9/5)}$$

Answer:

$$\begin{aligned} 2 - \log_3(5) &= \log_3(3^2) - \log_3(5) \\ &= \log_3(9) - \log_3(5) \\ &= \log_3\left(\frac{9}{5}\right) \end{aligned}$$

Ex 58: Write as a single logarithm:

$$\log_2(3) + x = \boxed{\log_2(3 \cdot 2^x)}$$

Answer:

$$\begin{aligned} \log_2(3) + x &= \log_2(3) + \log_2(2^x) \\ &= \log_2(3 \cdot 2^x) \end{aligned}$$

D.9 WRITING AS A SINGLE LOGARITHM IN BASE A: LEVEL 3

Ex 59: Write as a single logarithm in the form $\log_a k$:

$$2\log_6(3)+\log_6(4)=\boxed{\log_6(36)}$$

Answer:

$$\begin{aligned}2\log_6(3)+\log_6(4)&=\log_6(3^2)+\log_6(4)\\&=\log_6(9)+\log_6(4)\\&=\log_6(9\times 4)\\&=\log_6(36)\end{aligned}$$

Ex 60: Write as a single logarithm in the form $\log_a k$:

$$3\log_2(4)-\log_2(8)=\boxed{\log_2(8)}$$

Answer:

$$\begin{aligned}3\log_2(4)-\log_2(8)&=\log_2(4^3)-\log_2(8)\\&=\log_2(64)-\log_2(8)\\&=\log_2\left(\frac{64}{8}\right)\\&=\log_2(8)\end{aligned}$$

Ex 61: Write as a single logarithm:

$$\log_2(5)-2x=\boxed{\log_2(5/4^x)}$$

Answer:

$$\begin{aligned}\log_2(5)-2x&=\log_2(5)-2x\log_2(2)\\&=\log_2(5)-\log_2(2^{2x})\\&=\log_2(5)-\log_2((2^2)^x)\\&=\log_2(5)-\log_2(4^x)\\&=\log_2\left(\frac{5}{4^x}\right)\end{aligned}$$

E CHANGE OF BASE RULE

E.1 EVALUATING LOGARITHMS USING CHANGE OF BASE FORMULA

Ex 62: Evaluate in changing to base 10(round to 2 decimal places).

$$\log_3(2)\approx\boxed{0.63}$$

Answer:

$$\begin{aligned}\log_3(2)&=\frac{\log_1 0(2)}{\log_1 0(3)}\\&\approx 0.63\end{aligned}$$

By entering $\log(2)\div log(3)$, on the calculator, it displays: 0.63.

Ex 63: Evaluate by changing to base 10 (round to 2 decimal places).

$$\log_5(8)\approx\boxed{1.29}$$

Answer:

$$\begin{aligned}\log_5(8)&=\frac{\log_{10}(8)}{\log_{10}(5)}\\&\approx 1.29\end{aligned}$$

By entering $\log(8)\div\log(5)$ on the calculator, it displays: 1.292....

Ex 64: Evaluate by changing to base 10 (round to 2 decimal places).

$$\log_7(0.5)\approx\boxed{-0.36}$$

Answer:

$$\begin{aligned}\log_7(0.5)&=\frac{\log_{10}(0.5)}{\log_{10}(7)}\\&\approx -0.36\end{aligned}$$

By entering $\log(0.5)\div\log(7)$ on the calculator, it displays: $-0.356....$

Ex 65: Evaluate by changing to base 10 (round to 2 decimal places).

$$\log_2(100)\approx\boxed{6.64}$$

Answer:

$$\begin{aligned}\log_2(100)&=\frac{\log_{10}(100)}{\log_{10}(2)}\\&=\frac{2}{\log_{10}(2)}\\&\approx 6.64\end{aligned}$$

By entering $2\div\log(2)$ on the calculator, it displays: 6.643....

E.2 PROVING CHANGE OF BASE IDENTITIES

Ex 66: Prove that $\log_a b\cdot\log_b a=1$.

Answer:

$$\begin{aligned}\log_a b\cdot\log_b a&=\frac{\ln b}{\ln a}\cdot\frac{\ln a}{\ln b}\\&=1\end{aligned}$$

Ex 67: Prove that $\log_a(b)\log_b(c)=\log_a(c)$.

Answer:

$$\begin{aligned}\log_a(b)\cdot\log_b(c)&=\frac{\ln(b)}{\ln(a)}\cdot\frac{\ln(c)}{\ln(b)}\\&=\frac{\ln(c)}{\ln(a)}\\&=\log_a(c)\end{aligned}$$

Ex 68: Prove that $\log_{a^n}(b)=\frac{1}{n}\log_a(b)$.

Answer:

$$\begin{aligned}\log_{a^n}(b)&=\frac{\ln(b)}{\ln(a^n)}\\&=\frac{\ln(b)}{n\ln(a)}\\&=\frac{1}{n}\frac{\ln(b)}{\ln(a)}\\&=\frac{1}{n}\log_a(b)\end{aligned}$$

Ex 69: Prove that $\log_{1/a}(b)=-\log_a(b)$.

Answer:

$$\begin{aligned}\log_{1/a}(b) &= \frac{\ln(b)}{\ln(1/a)} \\ &= \frac{\ln(b)}{\ln(a^{-1})} \\ &= \frac{\ln(b)}{-\ln(a)} \\ &= -\frac{\ln(b)}{\ln(a)} \\ &= -\log_a(b)\end{aligned}$$

F USING LOGARITHMS TO SOLVE EXPONENTIAL EQUATIONS

F.1 SOLVING EXPONENTIAL EQUATIONS: LEVEL 1

Ex 70: Solve $2^x = 7$ (give your answer to 3 decimal places).

$x = \boxed{2.807}$

Answer:

$$\begin{aligned}2^x &= 7 \\ \log(2^x) &= \log 7 && \text{(taking log of both sides)} \\ x\log 2 &= \log 7 && \text{(power rule)} \\ x &= \frac{\log 7}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 2.807 && \text{(using calculator)}\end{aligned}$$

Ex 71: Solve $3^x = 15$ (give your answer to 3 decimal places).

$x = \boxed{2.465}$

Answer:

$$\begin{aligned}3^x &= 15 \\ \log(3^x) &= \log 15 && \text{(taking log of both sides)} \\ x\log 3 &= \log 15 && \text{(power rule)} \\ x &= \frac{\log 15}{\log 3} && \text{(dividing both sides by } \log 3\text{)} \\ x &\approx 2.465 && \text{(using calculator)}\end{aligned}$$

Ex 72: Solve $5^x = 100$ (give your answer to 3 decimal places).

$x = \boxed{2.861}$

Answer:

$$\begin{aligned}5^x &= 100 \\ \log(5^x) &= \log 100 && \text{(taking log of both sides)} \\ x\log 5 &= \log 100 && \text{(power rule)} \\ x &= \frac{\log 100}{\log 5} && \text{(dividing both sides by } \log 5\text{)} \\ x &\approx 2.861 && \text{(using calculator)}\end{aligned}$$

Ex 73: Solve $6^x = 80$ (give your answer to 3 decimal places).

$x = \boxed{2.446}$

Answer:

$$\begin{aligned}6^x &= 80 \\ \log(6^x) &= \log 80 && \text{(taking log of both sides)} \\ x\log 6 &= \log 80 && \text{(power rule)} \\ x &= \frac{\log 80}{\log 6} && \text{(dividing both sides by } \log 6\text{)} \\ x &\approx 2.446 && \text{(using calculator)}\end{aligned}$$

F.2 SOLVING EXPONENTIAL EQUATIONS: LEVEL 2

Ex 74: Solve $5 \cdot 2^x = 7$ (give your answer to 3 decimal places).

$x = \boxed{0.485}$

Answer:

$$\begin{aligned}5 \cdot 2^x &= 7 \\ 2^x &= \frac{7}{5} && \text{(dividing both sides by 5)} \\ \log(2^x) &= \log\left(\frac{7}{5}\right) && \text{(taking log of both sides)} \\ x\log 2 &= \log\left(\frac{7}{5}\right) && \text{(power rule)} \\ x &= \frac{\log\left(\frac{7}{5}\right)}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 0.485 && \text{(using calculator)}\end{aligned}$$

Ex 75: Solve $-(2^x) = -10$ (give your answer to 3 decimal places).

$x = \boxed{3.322}$

Answer:

$$\begin{aligned}-(2^x) &= -10 \\ 2^x &= 10 && \text{(dividing both sides by } -1\text{)} \\ \log(2^x) &= \log 10 && \text{(taking log of both sides)} \\ x\log 2 &= \log 10 && \text{(power rule)} \\ x &= \frac{\log 10}{\log 2} && \text{(dividing both sides by } \log 2\text{)} \\ x &\approx 3.322 && \text{(using calculator)}\end{aligned}$$

Ex 76: Solve $4 \cdot 3^x = 60$ (give your answer to 3 decimal places).

$x = \boxed{2.465}$

Answer:

$$\begin{aligned} 4 \cdot 3^x &= 60 \\ 3^x &= \frac{60}{4} && \text{(dividing both sides by 4)} \\ 3^x &= 15 \\ \log(3^x) &= \log 15 && \text{(taking log of both sides)} \\ x \log 3 &= \log 15 && \text{(power rule)} \\ x &= \frac{\log 15}{\log 3} && \text{(dividing both sides by } \log 3) \\ x &\approx 2.465 && \text{(using calculator)} \end{aligned}$$

Ex 77: Solve $-2 \cdot (0.5)^x = -4$.

$$x = \boxed{-1}$$

Answer:

$$\begin{aligned} -2 \cdot (0.5)^x &= -4 \\ (0.5)^x &= \frac{-4}{-2} && \text{(dividing both sides by } -2) \\ (0.5)^x &= 2 \\ \log\left((0.5)^x\right) &= \log(2) \\ x \cdot \log(0.5) &= \log(2) \\ x &= \frac{\log(2)}{\log(0.5)} \\ x &= -1 \end{aligned}$$

So, $x = -1$.

F.3 APPLYING EXPONENTIAL FUNCTIONS

Ex 78: The population of a town, P, is growing exponentially. The population can be modelled by the function $P(t) = 25000 \times 1.035^t$, where t is the number of years after the 1st of January 2020.

1. Write down the population of the town on the 1st of January 2020.
2. Calculate the population of the town after 5 years, giving your answer to the nearest whole number.
3. Determine the number of years it will take for the population to double. Give your answer to the nearest year.
4. Another town's population is modelled by the function $Q(t) = 40000 \times 1.018^t$. After how many years will the population of both towns be equal?

Answer:

1. The population on 1st January 2020 corresponds to $t = 0$. $P(0) = 25000 \times 1.035^0 = 25000 \times 1 = \mathbf{25000}$.
2. The population after 5 years corresponds to $t = 5$. $P(5) = 25000 \times 1.035^5 \approx 29692.15$. To the nearest whole number, the population is **29692**.
3. We need to find the time t when the population is double its initial value, which is $2 \times 25000 = 50000$.
$$\begin{aligned} 50000 &= 25000 \times 1.035^t \\ 2 &= 1.035^t \end{aligned}$$
Now, we take the logarithm of both sides to solve for t:
$$\begin{aligned} \log(2) &= \log(1.035^t) \\ \log(2) &= t \cdot \log(1.035) \\ t &= \frac{\log(2)}{\log(1.035)} \\ t &\approx 20.15 \text{ years} \end{aligned}$$
To the nearest year, it will take **20** years for the population to double.
4. We need to find the time t when $P(t) = Q(t)$.
$$25000 \times 1.035^t = 40000 \times 1.018^t$$
First, we group the terms with t on one side and the constants on the other.
$$\begin{aligned} \frac{1.035^t}{1.018^t} &= \frac{40000}{25000} \\ \left(\frac{1.035}{1.018}\right)^t &= 1.6 \end{aligned}$$
Now, take the logarithm of both sides:
$$\begin{aligned} t \cdot \log\left(\frac{1.035}{1.018}\right) &= \log(1.6) \\ t &= \frac{\log(1.6)}{\log(1.035/1.018)} \\ t &\approx \mathbf{28} \text{ years} \end{aligned}$$

Ex 79: The mass, M, in grams of a radioactive substance is modelled by the function $M(t) = 150 \times (0.88)^t$, where t is the time in years.

1. Write down the initial mass of the substance.
2. Calculate the mass of the substance remaining after 10 years, giving your answer to two decimal places.
3. Find the half-life of the substance. Give your answer to the nearest year.
4. Another radioactive substance has its mass modelled by the function $N(t) = 200 \times (0.85)^t$. Find the time it takes for the mass of both substances to be equal.

Answer:

1. The initial mass corresponds to $t = 0$. $M(0) = 150 \times (0.88)^0 = 150 \times 1 = \mathbf{150}$ grams.
2. The mass after 10 years corresponds to $t = 10$. $M(10) = 150 \times (0.88)^{10} \approx 41.78$ To two decimal places, the mass is **41.78** grams.

3. The half-life is the time t when the mass is half of its initial value, which is $150/2 = 75$ grams.

$$75 = 150 \times (0.88)^t$$
$$0.5 = 0.88^t$$

Take the logarithm of both sides to solve for t:

$$\log(0.5) = \log(0.88^t)$$
$$\log(0.5) = t \cdot \log(0.88)$$
$$t = \frac{\log(0.5)}{\log(0.88)}$$
$$t \approx 5.42 \text{ years}$$

To the nearest year, the half-life is **5** years.

4. We need to find the time t when $M(t) = N(t)$.

$$150 \times (0.88)^t = 200 \times (0.85)^t$$

Group the terms with t on one side and the constants on the other.

$$\frac{0.88^t}{0.85^t} = \frac{200}{150}$$
$$\left(\frac{0.88}{0.85}\right)^t = \frac{4}{3}$$

Now, take the logarithm of both sides:

$$t \cdot \log\left(\frac{0.88}{0.85}\right) = \log\left(\frac{4}{3}\right)$$
$$t = \frac{\log(4/3)}{\log(0.88/0.85)}$$
$$t \approx \mathbf{8.29} \text{ years}$$

Ex 80: Laura invests \$8000 in a savings account that pays a nominal annual interest rate of 4.2%, compounded annually. The value of her investment, V, after t years is given by the formula $V(t) = 8000(1.042)^t$.

1. Find the value of Laura's investment after 7 years. Give your answer to two decimal places.
2. Determine the number of years it will take for the value of the investment to exceed \$15,000.
3. Marco also invests in an account with an initial amount of \$7500. After 10 years, his investment is worth \$11,000. Assuming the interest is also compounded annually, find the annual interest rate for Marco's investment.

Answer:

1. The value after 7 years corresponds to $t = 7$.

$$V(7) = 8000(1.042)^7 \approx 10669.99$$

To two decimal places, the value is **\$10669.99**.

2. We need to find the smallest integer t for which $V(t) > 15000$.

$$8000(1.042)^t > 15000$$
$$(1.042)^t > \frac{15000}{8000}$$
$$(1.042)^t > 1.875$$

Take the logarithm of both sides to solve for t:

$$\log(1.042^t) > \log(1.875)$$
$$t \cdot \log(1.042) > \log(1.875)$$
$$t > \frac{\log(1.875)}{\log(1.042)}$$
$$t > 15.36...$$

Since the interest is compounded annually, we need to wait for the next full year. It will take **16** years.

3. Let the annual interest rate for Marco's investment be r. The value of his investment is $W(t) = 7500(1+r)^t$. We are given that $W(10) = 11000$.

$$11000 = 7500(1+r)^{10}$$
$$\frac{11000}{7500} = (1+r)^{10}$$
$$\frac{22}{15} = (1+r)^{10}$$

To solve for r, we first take the 10th root of both sides.

$$\left(\frac{22}{15}\right)^{1/10} = 1 + r$$
$$r = \left(\frac{22}{15}\right)^{1/10} - 1$$
$$r \approx 0.03904$$

The annual interest rate is approximately 3.90%.

G APPLICATIONS OF LOGARITHMS

G.1 APPLYING LOGARITHMS IN SCIENCE

Ex 81: The pH scale in chemistry is $\text{pH} = -\log_{10}[H^+]$ where $[H^+]$ is the hydrogen ion concentration in moles per litre. The pH of a solution is 3.2. Find the hydrogen ion concentration $[H^+]$ (give your answer in scientific notation with 3 significant digits).

$\boxed{6.31} \times \boxed{10^{-4}}$ mol/L

Answer: We know:

$\text{pH} = -\log_{10}[H^+]$	
$3.2 = -\log_{10}[H^+]$	(substituting the value)
$-3.2 = \log_{10}[H^+]$	(multiplying both sides by -1)
$10^{-3.2} = 10^{\log_{10}[H^+]}$	(exponentiating both sides)
$10^{-3.2} = [H^+]$	($10^{\log_{10} x} = x$)
$[H^+] \approx 0.00063096$	(using calculator)
$[H^+] \approx 6.31 \times 10^{-4}$ mol/L	(in scientific notation with 3 significant digits)

Ex 82: The Richter scale measures earthquake intensity using the formula $M = \log_{10}\left(\frac{I}{I_0}\right)$, where M is the magnitude, I is the intensity of the earthquake, and I_0 is the intensity of a standard earthquake.
An earthquake has a magnitude of 4.5 on the Richter scale. Find the intensity ratio $\frac{I}{I_0}$ (give your answer in scientific notation with 3 significant digits).

$$\frac{I}{I_0} = \boxed{3.16} \times \boxed{10^4}$$

Answer: We know:

$$\begin{aligned}
M &= \log_{10}\left(\frac{I}{I_0}\right) \\
4.5 &= \log_{10}\left(\frac{I}{I_0}\right) && \text{(substituting the value)} \\
10^{4.5} &= 10^{\log_{10}\left(\frac{I}{I_0}\right)} && \text{(exponentiating both sides)} \\
10^{4.5} &= \frac{I}{I_0} && (10^{\log_{10} x} = x) \\
\frac{I}{I_0} &\approx 31622.7766 && \text{(using calculator)} \\
\frac{I}{I_0} &\approx 3.16 \times 10^4 && \text{(in scientific notation with 3 significant digits)}
\end{aligned}$$

Ex 83: The intensity of sound is measured in decibels (dB) using the formula $L = 10\log_{10}\left(\frac{I}{I_0}\right)$, where L is the sound level in decibels, I is the intensity of the sound, and I_0 is the reference intensity (threshold of human hearing).
A sound has a level of 75 decibels. Find the intensity ratio $\frac{I}{I_0}$ (give your answer in scientific notation with 3 significant digits).

$$\frac{I}{I_0} = \boxed{3.16} \times \boxed{10^7}$$

Answer: We know:

$$\begin{aligned}
L &= 10\log_{10}\left(\frac{I}{I_0}\right) \\
75 &= 10\log_{10}\left(\frac{I}{I_0}\right) && \text{(substituting the value)} \\
7.5 &= \log_{10}\left(\frac{I}{I_0}\right) && \text{(dividing both sides by 10)} \\
10^{7.5} &= 10^{\log_{10}\left(\frac{I}{I_0}\right)} && \text{(exponentiating both sides)} \\
10^{7.5} &= \frac{I}{I_0} && (10^{\log_{10} x} = x) \\
\frac{I}{I_0} &\approx 3162277.66 && \text{(using calculator)} \\
\frac{I}{I_0} &\approx 3.16 \times 10^7 && \text{(in scientific notation with 3 significant digits)}
\end{aligned}$$