LOGARITHM FUNCTIONS

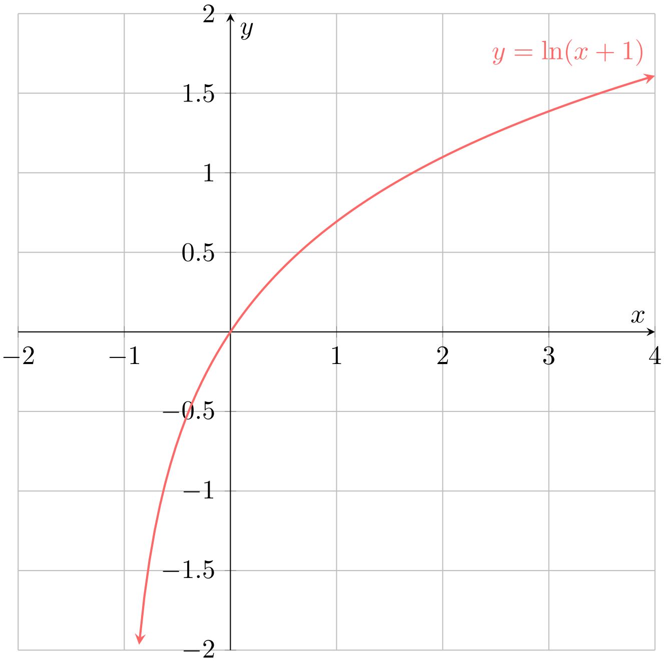
A NATURAL LOGARITHM FUNCTION

A.1 DETERMINING DOMAINS OF LOGARITHMIC FUNCTIONS

MCQ 1: What is the domain of the function $f(x) = \ln(x+1)$?

- ☐ $(-\infty, \infty)$
- ☐ $(-\infty, -1]$
- ☒ $(-1, \infty)$
- ☐ $[0, \infty)$

Answer: The natural logarithm $\ln y$ is defined for $y > 0$. For $f(x) = \ln(x+1)$, we need $x + 1 > 0 \implies x > -1$. Thus, the domain is $(-1, \infty)$.



MCQ 2: Find the domain of the function $f : x \mapsto \ln(2 - x)$.

- ☐ $\mathbb{R}$
- ☐ $[-2, +\infty)$
- ☐ $(2, +\infty)$
- ☒ $(-\infty, 2)$

Answer: The function $f(x) = \ln(2 - x)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $2-x > 0$. Solving this inequality:

$$\begin{aligned} 2 - x &> 0 \\ -x &> -2 \quad \text{(subtracting 2 from both sides)} \\ x &< 2 \quad \text{(multiplying both sides by } -1\text{, reversing the inequality)} \end{aligned}$$

Therefore, the function is defined for $x < 2$, so the domain is $(-\infty, 2)$.

MCQ 3: Find the domain of the function $f : x \mapsto \ln(2x - 6)$.

- ☐ $\mathbb{R}$
- ☐ $[3, +\infty)$
- ☒ $(3, +\infty)$
- ☐ $(-\infty, 3)$

Answer: The function $f(x) = \ln(2x - 6)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $2x-6 > 0$. Solving this inequality:

$$\begin{aligned} 2x - 6 &> 0 \\ 2x &> 6 \quad \text{(adding 6 to both sides)} \\ x &> 3 \quad \text{(dividing both sides by 2)} \end{aligned}$$

Therefore, the function is defined for $x > 3$, so the domain is $(3, +\infty)$.

MCQ 4: Find the domain of the function $f : x \mapsto \ln(9 - 3x)$.

- ☐ $\mathbb{R}$
- ☐ $[3, +\infty)$
- ☐ $(3, +\infty)$
- ☒ $(-\infty, 3)$

Answer: The function $f(x) = \ln(9 - 3x)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $9-3x > 0$. Solving this inequality:

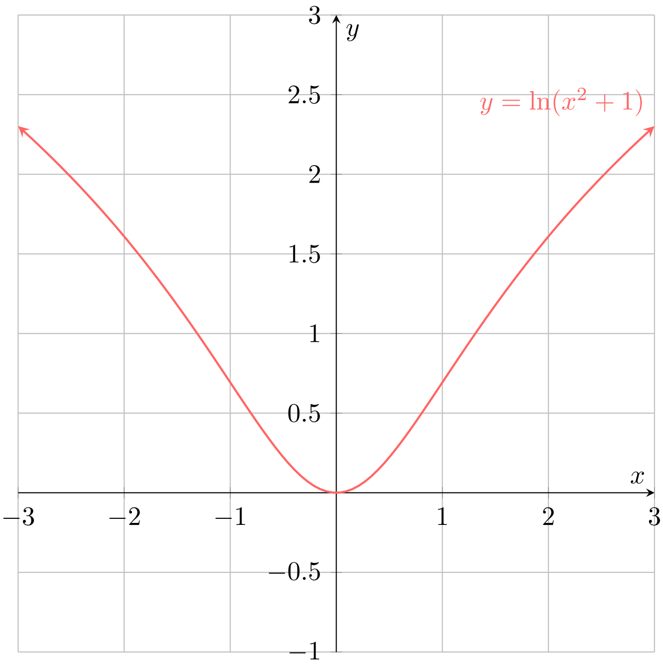
$$\begin{aligned} 9 - 3x &> 0 \\ -3x &> -9 \quad \text{(subtracting 9 from both sides)} \\ x &< 3 \quad \text{(dividing both sides by } -3\text{, reversing the inequality)} \end{aligned}$$

Therefore, the function is defined for $x < 3$, so the domain is $(-\infty, 3)$.

MCQ 5: What is the domain of the function $f(x) = \ln(x^2+1)$?

- ☒ $(-\infty, \infty)$
- ☐ $(-\infty, 0) \cup (0, \infty)$
- ☐ $(-1, 1)$
- ☐ $[-1, 1]$

Answer: The natural logarithm $\ln y$ is defined for $y > 0$. For $f(x) = \ln(x^2 + 1)$, we need $x^2 + 1 > 0$. Since $x^2 \geq 0$ for all real x, $x^2 + 1 \geq 1 > 0$. Thus, the domain is all real numbers, $(-\infty, \infty)$.



MCQ 6: What is the domain of the function $f(x) = \ln(-x^2 + 3x - 2)$?

- ☐ $(-\infty, \infty)$
- ☐ $(-\infty, 1] \cup [2, \infty)$
- ☒ $(1, 2)$
- ☐ $[1, 2]$

Answer: The natural logarithm $\ln y$ is defined for $y > 0$. For $f(x) = \ln(-x^2 + 3x - 2)$, we need $-x^2 + 3x - 2 > 0$. Solving: $x^2 - 3x + 2 < 0$. Factor: $(x-1)(x-2) < 0$. The inequality holds between the roots: $1 < x < 2$. Thus, the domain is $(1, 2)$.

A.2 CALCULATING $f(x)$

Ex 7: For $f : x \mapsto 3\ln(x)$, find in simplest form:

1. $f(1) = \boxed{0}$
2. $f(e) = \boxed{3}$

Answer:

1. $f(1) = 3\ln(1)$
   $= 3 \cdot 0 \quad (\text{since } \ln 1 = 0)$
   $= 0$
2. $f(e) = 3\ln(e)$
   $= 3 \cdot 1 \quad (\text{since } \ln e = 1)$
   $= 3$

Ex 8: For $f : x \mapsto \frac{1}{1+\ln(x)}$, find in simplest form:

1. $f(1) = \boxed{1}$
2. $f(e) = \boxed{\frac{1}{2}}$

Answer:

1. $f(1) = \dfrac{1}{1+\ln(1)}$
   $= \dfrac{1}{1+0} \quad (\text{since } \ln 1 = 0)$
   $= 1$
2. $f(e) = \dfrac{1}{1+\ln(e)}$
   $= \dfrac{1}{1+1} \quad (\text{since } \ln e = 1)$
   $= \dfrac{1}{2}$

Ex 9: For $f : x \mapsto x\ln(x+1)$, find in simplest form:

1. $f(0) = \boxed{0}$
2. $f(1) = \boxed{\ln(2)}$

Answer:

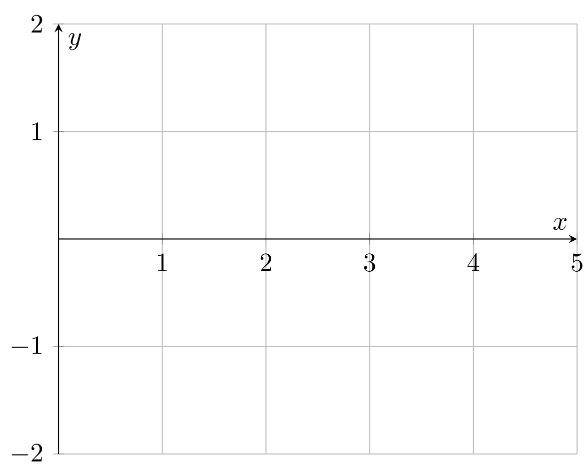
1. $f(0) = 0\ln(0+1)$
   $= 0 \cdot \ln(1)$
   $= 0 \cdot 0$
   $= 0$
2. $f(1) = 1\ln(1+1)$
   $= 1 \cdot \ln(2)$
   $= \ln(2)$

A.3 PLOTTING GRAPHS OF THE NATURAL LOGARITHM

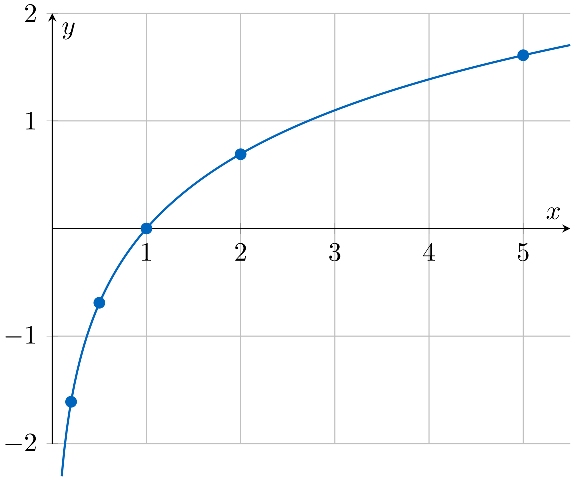
Ex 10: Here is a table of values for the function $f(x) = \ln(x)$:

x	0.2	0.5	1	2	5
$\ln(x)$	−1.61	−0.69	0	0.69	1.61

Plot the graph of the function.



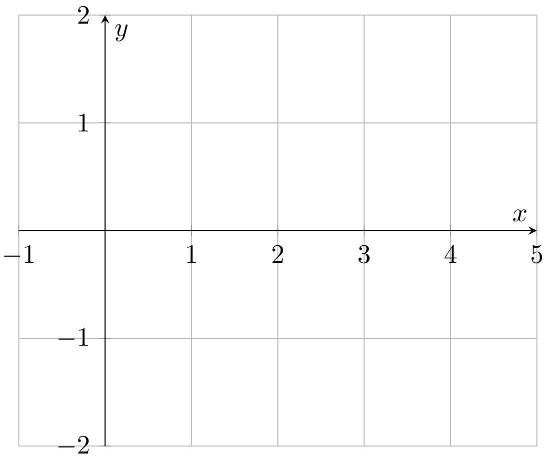
Answer:



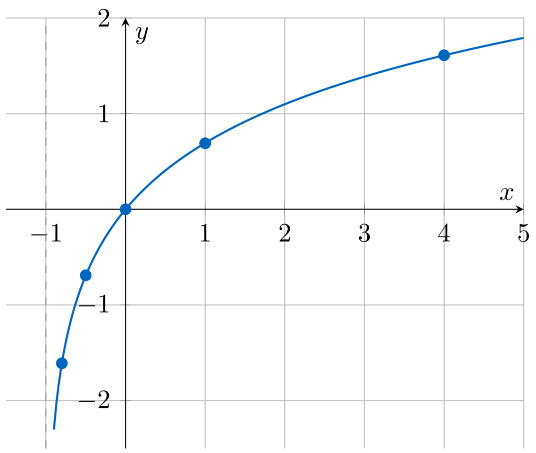
Ex 11: Here is a table of values for the function $f(x) = \ln(1+x)$:

x	-0.8	-0.5	0	1	4
$\ln(1+x)$	-1.61	-0.69	0	0.69	1.61

Plot the graph of the function.



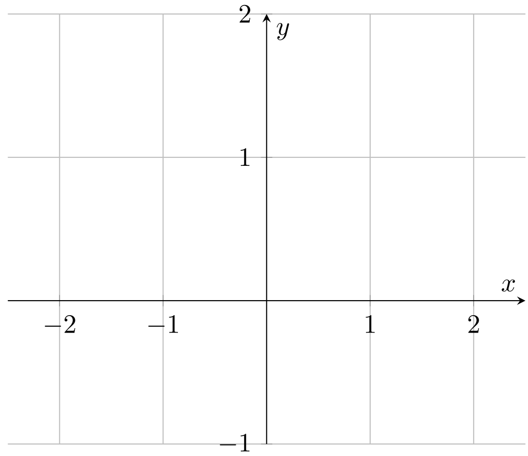
Answer:



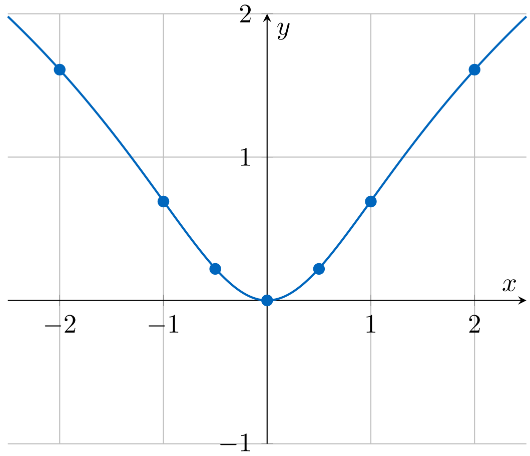
Ex 12: Here is a table of values for the function $f(x) = \ln(1+x^2)$:

x	-2	-1	-0.5	0	0.5	1	2
$\ln(1+x^2)$	1.61	0.69	0.22	0	0.22	0.69	1.61

Plot the graph of the function.



Answer:



A.4 FINDING INVERSE FUNCTIONS

Ex 13: For $f : x \mapsto 3\ln(x)$, find the inverse function:

$$f^{-1}(x) = \boxed{e^{\frac{x}{3}}}$$

Answer: Let $y = 3\ln(x)$. To find the inverse, we swap x and y and solve for y.

$$\begin{aligned} x &= 3\ln(y) \\ \frac{x}{3} &= \ln(y) \\ e^{\frac{x}{3}} &= e^{\ln(y)} \\ y &= e^{\frac{x}{3}} \end{aligned}$$

The inverse function is $\boldsymbol{f^{-1}(x) = e^{\frac{x}{3}}}$.

Ex 14: For $f : x \mapsto \ln(x+2) - 3$, find the inverse function:

$$f^{-1}(x) = \boxed{e^{x+3} - 2}$$

Answer: Let $y = \ln(x+2) - 3$. To find the inverse, we swap x and y and solve for y.

$$\begin{aligned} x &= \ln(y+2) - 3 \\ x + 3 &= \ln(y+2) \\ e^{x+3} &= e^{\ln(y+2)} \\ e^{x+3} &= y + 2 \\ y &= e^{x+3} - 2 \end{aligned}$$

The inverse function is $\boldsymbol{f^{-1}(x) = e^{x+3} - 2}$.

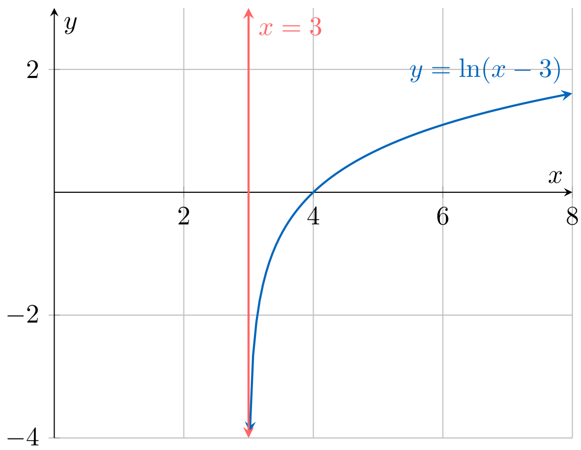
A.5 FINDING ASYMPTOTES

Ex 15: For the function $f(x) = \ln(x-3)$, find the equation of the vertical asymptote:

$$x = \boxed{3}$$

Answer: There is a **vertical asymptote** where the argument of the logarithm is zero

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

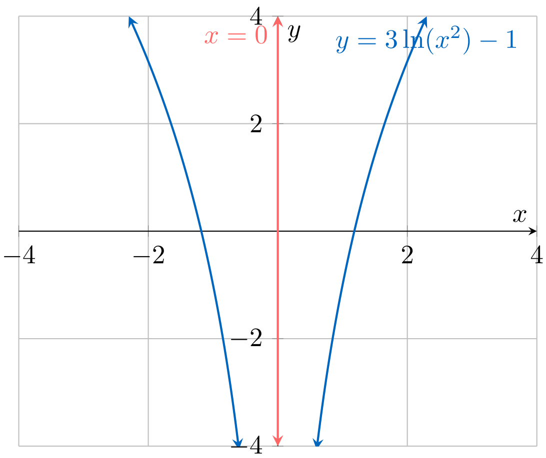


Ex 16: For the function $f(x) = 3\ln(x^2) - 1$, find the equation of the vertical asymptote:

$$x = \boxed{0}$$

Answer: There is a **vertical asymptote** where the argument of the logarithm is zero.

$$\begin{aligned} x^2 &= 0 \\ x &= 0 \end{aligned}$$

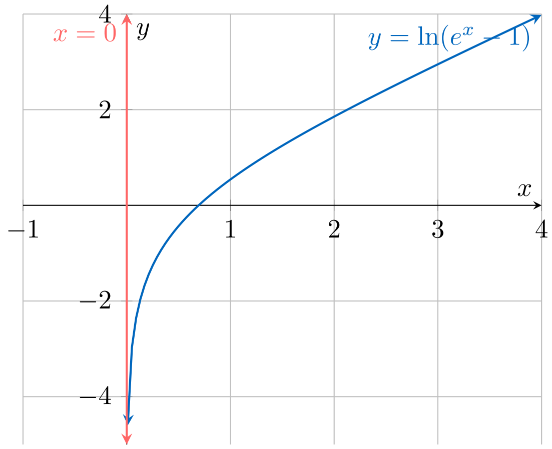


Ex 17: For the function $f(x) = \ln(e^x - 1)$, find the equation of the vertical asymptote:

$$x = \boxed{0}$$

Answer: There is a **vertical asymptote** where the argument of the logarithm is zero.

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ \ln(e^x) &= \ln(1) \\ x &= 0 \end{aligned}$$



A.6 FINDING $f(g(x))$

Ex 18: For the function $f(x) = e^x$ and $g(x) = \ln(x-3)$, find and simplify:

$$(f \circ g)(x) = \boxed{x - 3}$$

Answer:

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\ln(x-3)) \\ &= e^{\ln(x-3)} \\ &= x - 3 \end{aligned}$$

Ex 19: For the function $f(x) = \ln(x)$ and $g(x) = x^2 + 4$, find and simplify:

$$(f \circ g)(x) = \boxed{\ln(x^2+4)}$$

Answer:

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(x^2+4) \\ &= \ln(x^2+4) \end{aligned}$$

Ex 20: For the function $f(x) = \ln(x)$ and $g(x) = e^{2x}$, find and simplify:

$$(g \circ f)(x) = \boxed{x^2}$$

Answer:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= g(\ln(x)) \\ &= e^{2\ln(x)} \\ &= e^{\ln(x^2)} \\ &= x^2 \end{aligned}$$

A.7 ANALYZING LOGARITHMIC FUNCTIONS

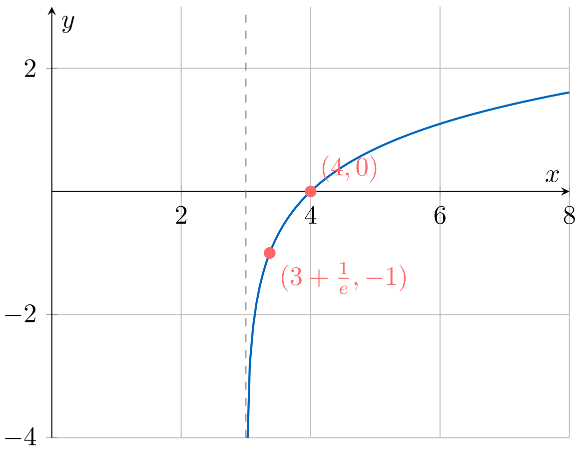
Ex 21: For the function $f(x) = \ln(x-3)$:

1. Find the domain and range.
2. Find any asymptotes and axes intercepts.
3. Sketch the graph of $y = f(x)$, showing all important features.

4. Solve $f(x) = -1$ algebraically and check the solution on your graph.
5. Find the inverse function f^{-1}.

Answer:

1. **Domain and Range**
   - **Domain:** The argument of the natural logarithm must be positive. Therefore, $x - 3 > 0$, which implies $x > 3$. The domain is $\mathbf{(3, \infty)}$.
   - **Range:** The range of the basic logarithmic function $\ln(u)$ is all real numbers. The horizontal shift does not affect the range. The range is $\mathbb{R}$.
2. **Asymptotes and Intercepts**
   - **Asymptotes:** There is a **vertical asymptote** where the argument of the logarithm is zero, which is at $\boldsymbol{x = 3}$. There is no horizontal asymptote.
   - **Axes Intercepts:**
     - **y-intercept:** To find the y-intercept, we set $x = 0$. $f(0) = \ln(0 - 3) = \ln(-3)$, which is undefined. There is **no y-intercept**.
     - **x-intercept:** To find the x-intercept, we set $f(x) = 0$. $\ln(x - 3) = 0 \implies e^0 = x - 3 \implies 1 = x - 3 \implies x = 4$. The x-intercept is at $\mathbf{(4, 0)}$.
3. **Graph of the function**



4. **Solve** $f(x) = -1$

$$\ln(x-3) = -1$$
$$e^{\ln(x-3)} = e^{-1}$$
$$x - 3 = \frac{1}{e}$$
$$x = 3 + \frac{1}{e}$$

The exact solution is $\boldsymbol{x = 3 + \frac{1}{e}}$. As $e \approx 2.718$, $x \approx 3 + 0.368 = 3.368$. This point is marked on the graph, confirming that for $y = -1$, x is slightly greater than 3.

5. **Inverse Function** Let $y = \ln(x - 3)$. To find the inverse, we swap x and y and solve for y.

$$x = \ln(y-3)$$
$$e^x = e^{\ln(y-3)}$$
$$e^x = y - 3$$
$$y = e^x + 3$$

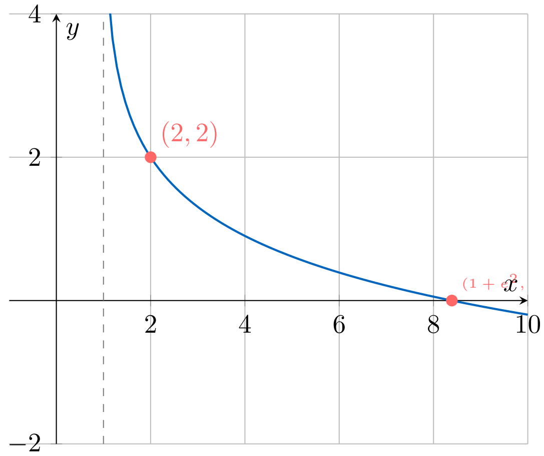
The inverse function is $\boldsymbol{f^{-1}(x) = e^x + 3}$.

Ex 22: For the function $f : x \mapsto 2 - \ln(x - 1)$:

1. Find the domain and range.
2. Find any asymptotes and axes intercepts.
3. Sketch the graph of $y = f(x)$, showing all important features.
4. Solve $f(x) = -1$ algebraically and check the solution on your graph.
5. Find the inverse function f^{-1}.

Answer:

1. **Domain and Range**
   - **Domain:** The argument of the natural logarithm must be positive. Therefore, $x - 1 > 0$, which implies $x > 1$. The domain is $\mathbf{(1, \infty)}$.
   - **Range:** The range of $\ln(x - 1)$ is $\mathbb{R}$. The transformations (reflection and vertical shift) do not change the range. The range is $\mathbb{R}$.
2. **Asymptotes and Intercepts**
   - **Asymptotes:** There is a **vertical asymptote** where the argument of the logarithm is zero, which is at $\boldsymbol{x = 1}$.
   - **Axes Intercepts:**
     - **y-intercept:** Set $x = 0$. $f(0) = 2 - \ln(0 - 1)$ is undefined. There is **no y-intercept**.
     - **x-intercept:** Set $f(x) = 0$. $2 - \ln(x - 1) = 0 \implies \ln(x - 1) = 2 \implies x - 1 = e^2 \implies x = 1 + e^2$. The x-intercept is at $\mathbf{(1 + e^2, 0)}$.
3. **Graph of the function**



4. **Solve** $f(x) = -1$

$$2 - \ln(x-1) = -1$$
$$3 = \ln(x-1)$$
$$e^3 = x - 1$$
$$x = 1 + e^3$$

The exact solution is $\boldsymbol{x = 1 + e^3}$. As $e^3 \approx 20.09$, $x \approx 21.09$. This is consistent with the graph, which shows that as y becomes more negative, x increases.

5. **Inverse Function** Let $y = 2 - \ln(x-1)$. Swap x and y.

$$\begin{aligned} x &= 2 - \ln(y-1) \\ \ln(y-1) &= 2 - x \\ e^{\ln(y-1)} &= e^{2-x} \\ y - 1 &= e^{2-x} \\ y &= 1 + e^{2-x} \end{aligned}$$

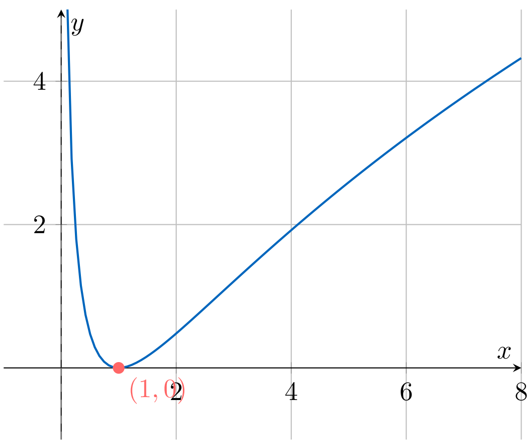
The inverse function is $\boldsymbol{f^{-1}(x) = 1 + e^{2-x}}$.

Ex 23: For the function $f(x) = (\ln(x))^2$:

1. Find the domain and range.
2. Find any asymptotes and axes intercepts.
3. Sketch the graph of $y = f(x)$, showing all important features.
4. Solve $f(x) = 4$ algebraically and check the solution on your graph.
5. Find the inverse function f^{-1}, for $\boldsymbol{x \geq 1}$

Answer:

1. **Domain and Range**
   - **Domain:** The function $\ln(x)$ is defined for $x > 0$. Squaring the function does not change this condition. The domain is $\boldsymbol{(0, \infty)}$.
   - **Range:** The range of $\ln(x)$ is $\mathbb{R}$. Since $f(x)$ is the square of $\ln(x)$, its values must be non-negative. The range is $\boldsymbol{[0, \infty)}$.
2. **Asymptotes and Intercepts**
   - **Asymptotes:** As $x \to 0^+$, $\ln(x) \to -\infty$, so $(\ln(x))^2 \to \infty$. There is a **vertical asymptote** at $\boldsymbol{x = 0}$. There is no horizontal asymptote as $f(x) \to \infty$ when $x \to \infty$.
   - **Axes Intercepts:**
     - **y-intercept:** Set $x = 0$. This is not in the domain. There is **no y-intercept**.
     - **x-intercept:** Set $f(x) = 0$. $(\ln(x))^2 = 0 \implies \ln(x) = 0 \implies x = e^0 = 1$. The x-intercept is at $\boldsymbol{(1, 0)}$. This is also the minimum point of the function.
3. **Graph of the function**



4. **Solve** $f(x) = 4$

$$\begin{aligned} (\ln(x))^2 &= 4 \\ \ln(x) &= \pm\sqrt{4} \\ \ln(x) = 2 \quad &\text{or} \quad \ln(x) = -2 \\ x = e^2 \quad &\text{or} \quad x = e^{-2} \end{aligned}$$

The solutions are $\boldsymbol{x = e^2 \approx 7.39}$ and $\boldsymbol{x = e^{-2} = \frac{1}{e^2} \approx 0.135}$. Both solutions are positive and therefore valid. This is confirmed on the graph, which shows the line $y = 4$ intersecting the curve at two points.

5. **Inverse Function** Let $y = (\ln(x))^2$. Swap x and y.

$$\begin{aligned} x &= (\ln(y))^2 \\ \sqrt{x} &= \ln(y) \\ e^{\sqrt{x}} &= y \end{aligned}$$

The inverse function is $\boldsymbol{f^{-1}(x) = e^{\sqrt{x}}}$.

B LOGARITHMIC FUNCTION IN BASE a

B.1 DETERMINING DOMAINS OF LOGARITHMIC FUNCTIONS

MCQ 24: Find the domain of the function $f : x \mapsto \log_2(x-4)$.

- ☐ $\mathbb{R}$
- ☐ $[-4, +\infty)$
- ☒ $(4, +\infty)$
- ☐ $(-\infty, 4)$

Answer: The function $f(x) = \log_2(x-4)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $x-4 > 0$. Solving this inequality:

$$\begin{aligned} x - 4 &> 0 \\ x &> 4 \quad \text{(adding 4 to both sides)} \end{aligned}$$

Therefore, the function is defined for $x > 4$, so the domain is $(4, +\infty)$.

MCQ 25: Find the domain of the function $f : x \mapsto \log_5(2-x)$.

- ☐ $\mathbb{R}$
- ☐ $[-2, +\infty)$
- ☐ $(2, +\infty)$
- ☒ $(-\infty, 2)$

Answer: The function $f(x) = \log_5(2-x)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $2-x > 0$. Solving this inequality:

$$\begin{aligned} 2 - x &> 0 \\ -x &> -2 \quad \text{(subtracting 2 from both sides)} \\ x &< 2 \quad \text{(multiplying both sides by } -1\text{, reversing the inequality)} \end{aligned}$$

Therefore, the function is defined for $x < 2$, so the domain is $(-\infty, 2)$.

MCQ 26: Find the domain of the function $f : x \mapsto \log(2x-6)$.

- ☐ $\mathbb{R}$
- ☐ $[3, +\infty)$
- ☒ $(3, +\infty)$
- ☐ $(-\infty, 3)$

Answer: The function $f(x) = \log(2x - 6)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $2x - 6 > 0$. Solving this inequality:

$$\begin{aligned} 2x - 6 &> 0 \\ 2x &> 6 \quad \text{(adding 6 to both sides)} \\ x &> 3 \quad \text{(dividing both sides by 2)} \end{aligned}$$

Therefore, the function is defined for $x > 3$, so the domain is $(3, +\infty)$.

MCQ 27: Find the domain of the function $f : x \mapsto \log_{10}(9 - 3x)$.

- ☐ $\mathbb{R}$
- ☐ $[3, +\infty)$
- ☐ $(3, +\infty)$
- ☒ $(-\infty, 3)$

Answer: The function $f(x) = \log_{10}(9-3x)$ is defined only when the argument of the logarithm is strictly positive, i.e., when $9-3x > 0$. Solving this inequality:

$$\begin{aligned} 9 - 3x &> 0 \\ -3x &> -9 \quad \text{(subtracting 9 from both sides)} \\ x &< 3 \quad \text{(dividing both sides by } -3\text{, reversing the inequality)} \end{aligned}$$

Therefore, the function is defined for $x < 3$, so the domain is $(-\infty, 3)$.

B.2 CALCULATING $f(x)$

Ex 28: For $f : x \mapsto 3\log(x)$, find in simplest form:

1. $f(1) = \boxed{0}$
2. $f(10) = \boxed{3}$

Answer:

1. $f(1) = 3\log(1) = 3 \cdot 0 \quad (\text{since } \log 1 = 0) = 0$
2. $f(10) = 3\log(10) = 3 \cdot 1 \quad (\text{since } \log 10 = 1) = 3$

Ex 29: For $f : x \mapsto \frac{1}{1+\log_2(x)}$, find in simplest form:

1. $f(1) = \boxed{1}$
2. $f(2) = \boxed{\frac{1}{2}}$

Answer:

1. $f(1) = \dfrac{1}{1+\log_2(1)} = \dfrac{1}{1+0} \quad (\text{since } \log_2 1 = 0) = 1$
2. $f(2) = \dfrac{1}{1+\log_2(2)} = \dfrac{1}{1+1} \quad (\text{since } \log_2 2^1 = 1) = \dfrac{1}{2}$

Ex 30: For $f : x \mapsto x\log(x+1)$, find in simplest form:

1. $f(0) = \boxed{0}$
2. $f(1) = \boxed{\log(2)}$

Answer:

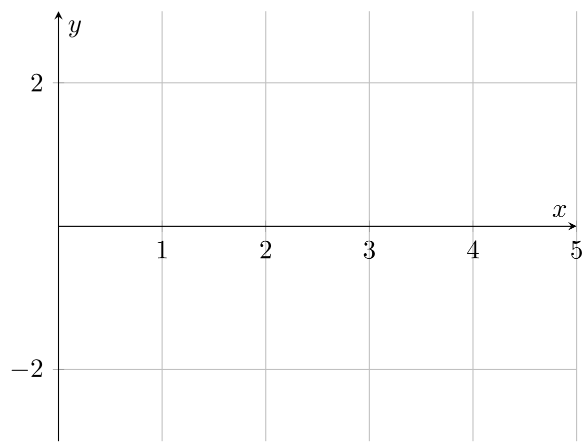
1. $f(0) = 0\log(0+1) = 0 \cdot \log(1) = 0 \cdot 0 = 0$
2. $f(1) = 1\log(1+1) = 1 \cdot \log(2) = \log(2)$

B.3 PLOTTING LOGARITHMIC GRAPHS

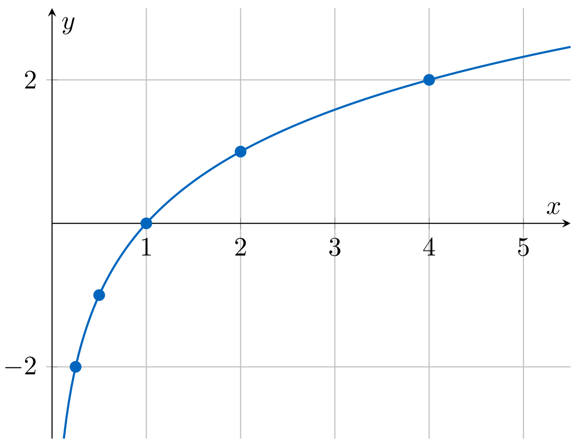
Ex 31: Here is a table of values for the function $f(x) = \log_2(x)$:

x	0.25	0.5	1	2	4
$\log_2(x)$	-2	-1	0	1	2

Plot the graph of the function.



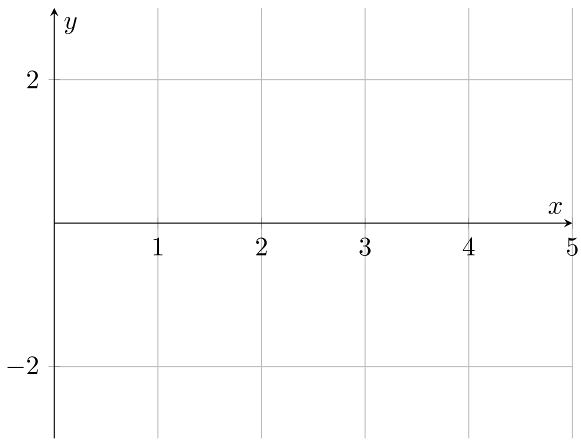
Answer:



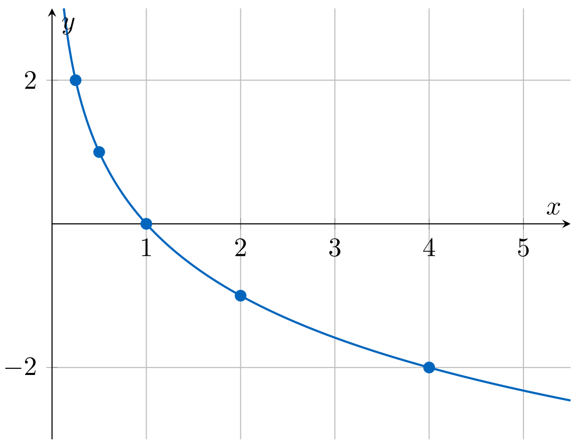
Ex 32: Here is a table of values for the function $f(x) = \log_{0.5}(x)$:

x	0.25	0.5	1	2	4
$\log_{0.5}(x)$	2	1	0	-1	-2

Plot the graph of the function.



Answer:



B.4 FINDING INVERSE FUNCTIONS

Ex 33: For $f : x \mapsto \log_2(x-1)$, find the inverse function:

$$f^{-1}(x) = \boxed{2^x + 1}$$

Answer: Let $y = \log_2(x-1)$. To find the inverse, we swap x and y and solve for y.

$$\begin{aligned} x &= \log_2(y-1) \\ 2^x &= 2^{\log_2(y-1)} \\ 2^x &= y - 1 \\ y &= 2^x + 1 \end{aligned}$$

The inverse function is $\boldsymbol{f^{-1}(x) = 2^x + 1}$.

Ex 34: For $f : x \mapsto 5\log_3(2x)$, find the inverse function:

$$f^{-1}(x) = \boxed{\frac{1}{2} \cdot 3^{\frac{x}{5}}}$$

Answer: Let $y = 5\log_3(2x)$. To find the inverse, we swap x and y and solve for y.

$$\begin{aligned} x &= 5\log_3(2y) \\ \frac{x}{5} &= \log_3(2y) \\ 3^{\frac{x}{5}} &= 2y \\ y &= \frac{1}{2} \cdot 3^{\frac{x}{5}} \end{aligned}$$

The inverse function is $\boldsymbol{f^{-1}(x) = \frac{1}{2} \cdot 3^{\frac{x}{5}}}$.

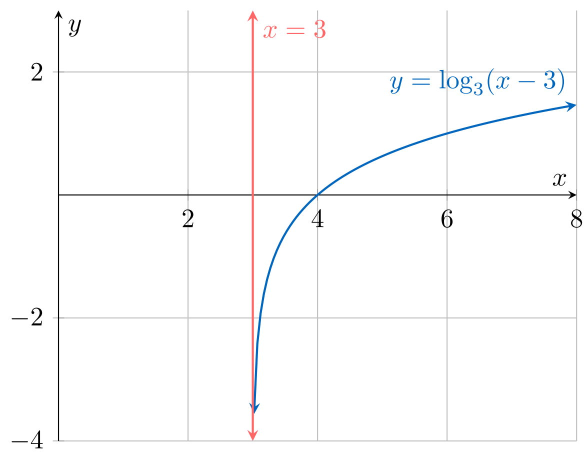
B.5 FINDING ASYMPTOTES

Ex 35: For the function $f(x) = \log_3(x-3)$, find the equation of the vertical asymptote:

$$x = \boxed{3}$$

Answer: There is a **vertical asymptote** where the argument of the logarithm is zero

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

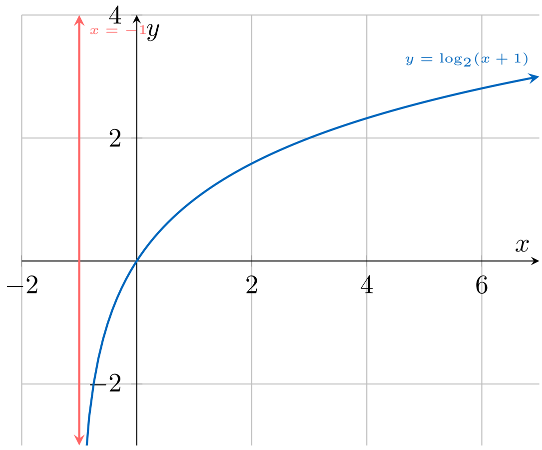


Ex 36: For the function $f(x) = \log_2(x+1)$, find the equation of the vertical asymptote:

$$x = \boxed{-1}$$

Answer: There is a **vertical asymptote** where the argument of the logarithm is zero

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned}$$



B.6 FINDING $f(g(x))$

Ex 37: For the function $f(x) = \log_2(x)$ and $g(x) = 4^x$, find and simplify:

$$(f \circ g)(x) = \boxed{2x}$$

Answer:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(4^x) \\ &= \log_2(4^x) \\ &= \log_2((2^2)^x) \\ &= \log_2(2^{2x}) \\ &= 2x\end{aligned}$$

Ex 38: For the function $f(x) = 2^x$ and $g(x) = \log_4(x)$, find and simplify:

$$(f \circ g)(x) = \boxed{\sqrt{x}}$$

Answer:

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(\log_4(x)) \\ &= 2^{\log_4(x)} \\ &= 2^{\frac{\log_2(x)}{\log_2(4)}} && \text{(Change of base formula)} \\ &= 2^{\frac{\log_2(x)}{2}} \\ &= (2^{\log_2(x)})^{\frac{1}{2}} && \text{(Exponent law)} \\ &= x^{\frac{1}{2}} \\ &= \sqrt{x}\end{aligned}$$

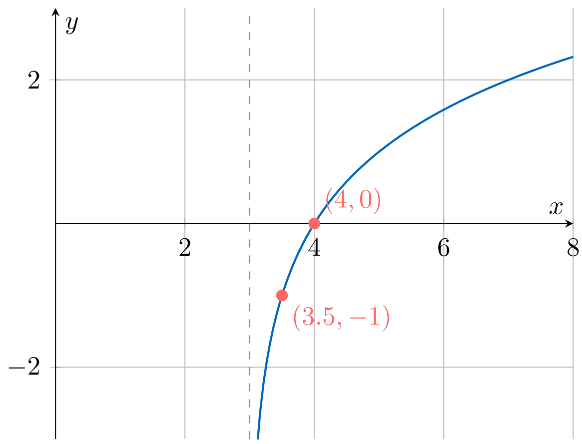
B.7 ANALYZING LOGARITHMIC FUNCTIONS

Ex 39: For the function $f(x) = \log_2(x-3)$:

1. Find the domain and range.
2. Find any asymptotes and axes intercepts.
3. Sketch the graph of $y = f(x)$, showing all important features.
4. Solve $f(x) = -1$ algebraically and check the solution on your graph.
5. Find the inverse function f^{-1}.

Answer:

1. **Domain and Range**
   - **Domain:** The argument of the logarithm must be positive. Therefore, $x - 3 > 0$, which implies $x > 3$. The domain is $\boldsymbol{(3, \infty)}$.
   - **Range:** The range of the logarithmic function $\log_2(u)$ is all real numbers. The range is $\mathbb{R}$.
2. **Asymptotes and Intercepts**
   - **Asymptotes:** There is a **vertical asymptote** at $\boldsymbol{x = 3}$.
   - **Axes Intercepts:**
     - **y-intercept:** For $x = 0$, $f(0) = \log_2(-3)$, which is undefined. There is **no y-intercept**.
     - **x-intercept:** Set $f(x) = 0$. $\log_2(x-3) = 0 \implies 2^0 = x - 3 \implies 1 = x - 3 \implies x = 4$. The x-intercept is at $\boldsymbol{(4, 0)}$.
3. **Graph of the function**



4. **Solve** $f(x) = -1$

$$\begin{aligned}\log_2(x-3) &= -1 \\ 2^{\log_2(x-3)} &= 2^{-1} \\ x - 3 &= \frac{1}{2} \\ x &= 3 + \frac{1}{2} = 3.5\end{aligned}$$

   The exact solution is $\boldsymbol{x = 3.5}$. This is confirmed on the graph.

5. **Inverse Function** Let $y = \log_2(x-3)$. Swap x and y.

$$\begin{aligned}x &= \log_2(y-3) \\ 2^x &= 2^{\log_2(y-3)} \\ 2^x &= y - 3 \\ y &= 2^x + 3\end{aligned}$$

   The inverse function is $\boldsymbol{f^{-1}(x) = 2^x + 3}$.

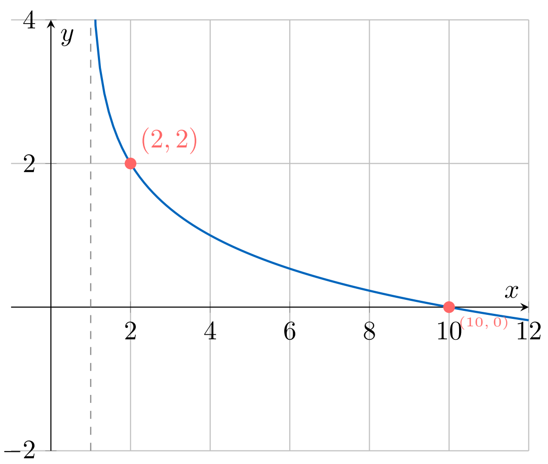
Ex 40: For the function $f(x) = 2 - \log_3(x-1)$:

1. Find the domain and range.

2. Find any asymptotes and axes intercepts.
3. Sketch the graph of $y = f(x)$, showing all important features.
4. Solve $f(x) = -1$ algebraically and check the solution on your graph.
5. Find the inverse function f^{-1}.

Answer:

1. **Domain and Range**
   - **Domain:** The argument of the logarithm must be positive. Therefore, $x - 1 > 0$, which implies $x > 1$. The domain is $\mathbf{(1, \infty)}$.
   - **Range:** The range of $\log_3(x-1)$ is $\mathbb{R}$. The transformations do not change the range. The range is $\mathbb{R}$.
2. **Asymptotes and Intercepts**
   - **Asymptotes:** There is a **vertical asymptote** at $\boldsymbol{x = 1}$.
   - **Axes Intercepts:**
     - **y-intercept:** For $x = 0$, $f(0) = 2 - \log_3(-1)$, which is undefined. There is **no y-intercept**.
     - **x-intercept:** Set $f(x) = 0$. $2 - \log_3(x-1) = 0 \implies \log_3(x-1) = 2 \implies x - 1 = 3^2 \implies x = 1 + 9 = 10$. The x-intercept is at $\mathbf{(10, 0)}$.
3. **Graph of the function**



4. **Solve** $f(x) = -1$

$$2 - \log_3(x-1) = -1$$
$$3 = \log_3(x-1)$$
$$3^3 = x - 1$$
$$x = 1 + 27 = 28$$

   The exact solution is $\boldsymbol{x = 28}$.

5. **Inverse Function** Let $y = 2 - \log_3(x-1)$. Swap x and y.

$$x = 2 - \log_3(y-1)$$
$$\log_3(y-1) = 2 - x$$
$$3^{\log_3(y-1)} = 3^{2-x}$$
$$y - 1 = 3^{2-x}$$
$$y = 1 + 3^{2-x}$$

   The inverse function is $\boldsymbol{f^{-1}(x) = 1 + 3^{2-x}}$.