EQUATIONS OF LINES

There are many applications of vectors in geometry. While some of these applications can be addressed with other tools in 2-dimensional planar geometry, vector methods become particularly efficient and powerful in 3-dimensional space, especially when considering the relationships between lines and planes.

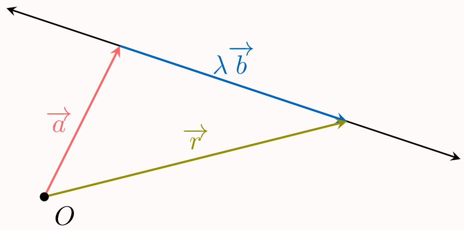
A VECTOR EQUATION

The position of any point on a line can be described by a starting point and a direction of travel. In vector terms, this means that the position vector of any point on the line can be reached by starting with the position vector of a known point and adding a scalar multiple of the line's direction vector. This principle allows us to define a line in both two and three dimensions.

Definition Vector Equation of a Line

The **vector equation of the line** is:

$$\overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{b}, \quad \lambda \in \mathbb{R}$$



B PARAMETRIC EQUATIONS

Let a line pass through point $A(a_1, a_2, a_3)$ with direction vector $\overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$. For any point $R(x, y, z)$ on the line, the vector equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ leads to **parametric equations:**

$$\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases}, \quad \lambda \in \mathbb{R}$$

In 2 dimensions, the z-components are simply omitted.

Definition Parametric Equations of a Line

The **parametric equations** of a line passing through a point $A(a_1, a_2, a_3)$ with direction vector $\overrightarrow{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are given by the system:

$$\begin{cases} x = a_1 + \lambda b_1 \\ y = a_2 + \lambda b_2 \\ z = a_3 + \lambda b_3 \end{cases}$$

where the variable $\lambda \in \mathbb{R}$ is the **parameter**.

C CARTESIAN EQUATION IN PLANE

In two dimensions, a line can also be defined by a normal vector. Let $\overrightarrow{n}$ be a normal vector to the line's direction, so that $\overrightarrow{b} \cdot \overrightarrow{n} = 0$.Starting from the vector equation ,$\overrightarrow{r} = \overrightarrow{a} + \lambda \overrightarrow{b}$, we take the scalar product of both sides with the normal vector $\overrightarrow{n}$:

$$\begin{aligned} \overrightarrow{r} \cdot \overrightarrow{n} &= (\lambda \overrightarrow{b} + \overrightarrow{a}) \cdot \overrightarrow{n} \\ \overrightarrow{r} \cdot \overrightarrow{n} &= \lambda \overrightarrow{b} \cdot \overrightarrow{n} + \overrightarrow{a} \cdot \overrightarrow{n} && \text{(distributivity)} \\ \overrightarrow{r} \cdot \overrightarrow{n} &= \overrightarrow{a} \cdot \overrightarrow{n} && (\overrightarrow{b} \cdot \overrightarrow{n} = 0) \end{aligned}$$

If we let $\overrightarrow{r} = \begin{pmatrix} x \\ y \end{pmatrix}$ and the normal vector $\overrightarrow{n} = \begin{pmatrix} a \\ b \end{pmatrix}$, the expression $\overrightarrow{r} \cdot \overrightarrow{n}$ becomes $ax + by$. The term $\overrightarrow{a} \cdot \overrightarrow{n}$ is a constant, which we can call C. This leads to the familiar Cartesian form.

Definition Cartesian Equation of a Line in 2D

The **Cartesian equation** of a line in 2D with normal vector $\overrightarrow{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ is given by:

$$ax + by = C$$

where C is a constant.