VECTOR EQUATIONS OF LINES

A VECTOR EQUATION

A.1 LOCATING POINTS ON A LINE

Ex 1: For the vector equation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, locate the point on the line for which $\lambda = 0$.

$$A(\boxed{-2},\boxed{3})$$

Answer: For $\lambda = 0$:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} \end{aligned}$$

This gives the point $A(-2, 3)$.

Ex 2: For the vector equation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, locate the point on the line for which $\lambda = 2$.

$$A(\boxed{2},\boxed{1})$$

Answer: For $\lambda = 2$:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \times 2 \\ 2 \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} -2+4 \\ 3-2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the point $A(2, 1)$.

Ex 3: For the vector equation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, locate the point on the line for which $\lambda = \frac{1}{2}$.

$$A(\boxed{-1},\boxed{\frac{5}{2}})$$

Answer: For $\lambda = \frac{1}{2}$:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 1/2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \times 2 \\ \frac{1}{2} \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -2+1 \\ 3-\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \frac{6}{2}-\frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \frac{5}{2} \end{pmatrix} \end{aligned}$$

This gives the point $A(-1, \frac{5}{2})$.

Ex 4: For the vector equation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, locate the point on the line for which $\lambda = -\frac{3}{2}$.

$$A(\boxed{-5},\boxed{\frac{9}{2}})$$

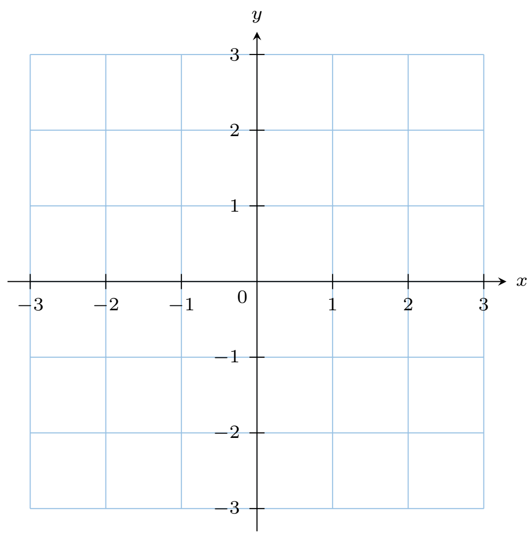
Answer: For $\lambda = -\frac{3}{2}$:

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + (-3/2) \begin{pmatrix} 2 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} -\frac{3}{2} \times 2 \\ -\frac{3}{2} \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \begin{pmatrix} -3 \\ \frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} -2-3 \\ 3+\frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ \frac{6}{2}+\frac{3}{2} \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ \frac{9}{2} \end{pmatrix} \end{aligned}$$

This gives the point $A(-5, \frac{9}{2})$.

A.2 PLOTTING A LINE FROM ITS VECTOR EQUATION

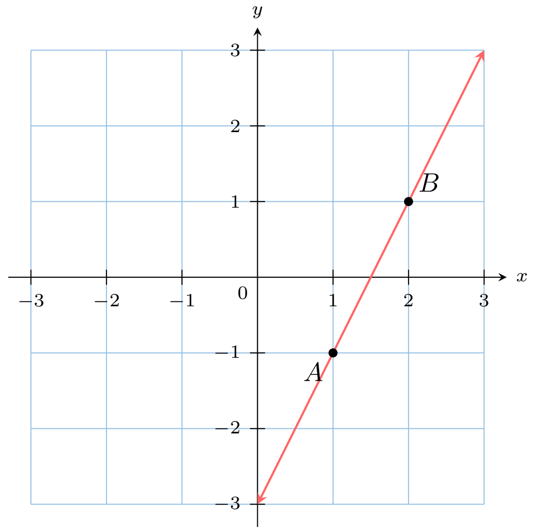
Ex 5: For the vector equation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, plot the line.



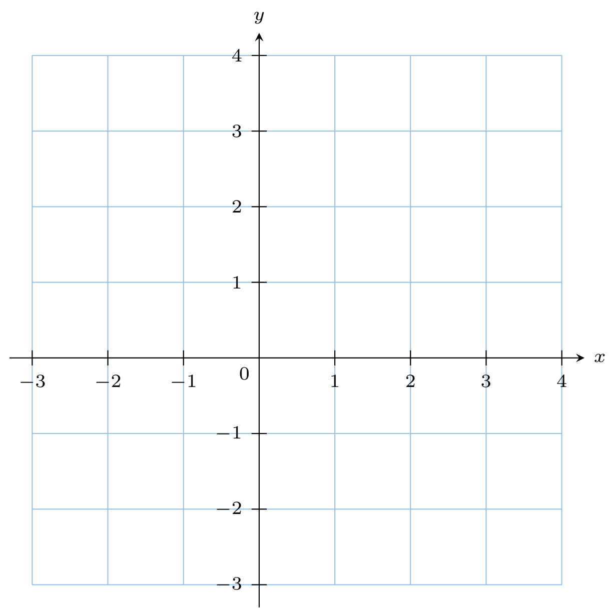
Answer: To plot the line, we can find any two distinct points on it by choosing two different values for the parameter λ.

- For $\lambda = 0$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This gives the point $A(1, -1)$.
- For $\lambda = 1$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This gives the point $B(2, 1)$.

We now plot these two points and draw the unique line that passes through them.



Ex 6: For the vector equation, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad \lambda \in \mathbb{R}$, plot the line.

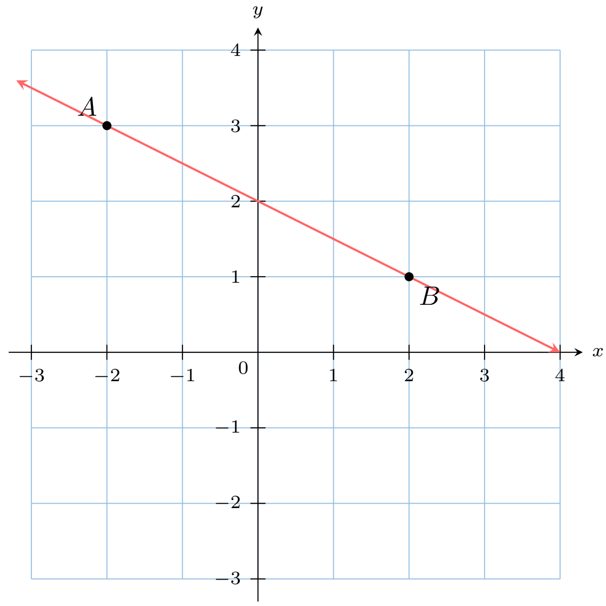


Answer: To plot the line, we can find any two distinct points on it by choosing two different values for the parameter λ.

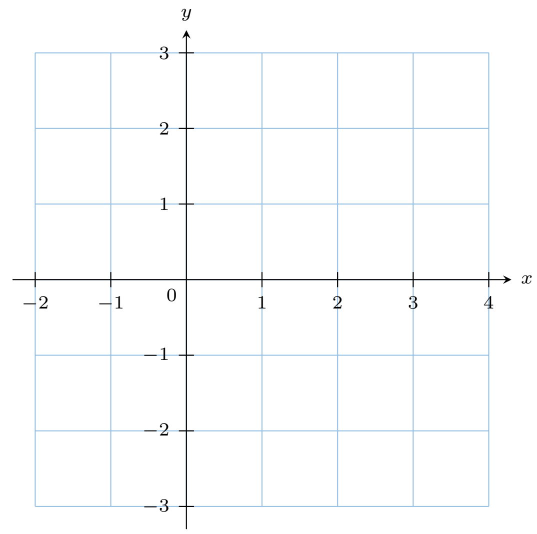
- For $\lambda = 0$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 0 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$. This gives the point $A(-2, 3)$.

- For $\lambda = 2$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -2+4 \\ 3-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. This gives the point $B(2, 1)$.

We now plot these two points and draw the unique line that passes through them.



Ex 7: A line passes through the point $A(1, -2)$ with direction vector $\overrightarrow{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Plot the point A, the direction vector originating from A, and the resulting line.



Answer: To plot the line, we start at the given point and use the direction vector to find a second point.

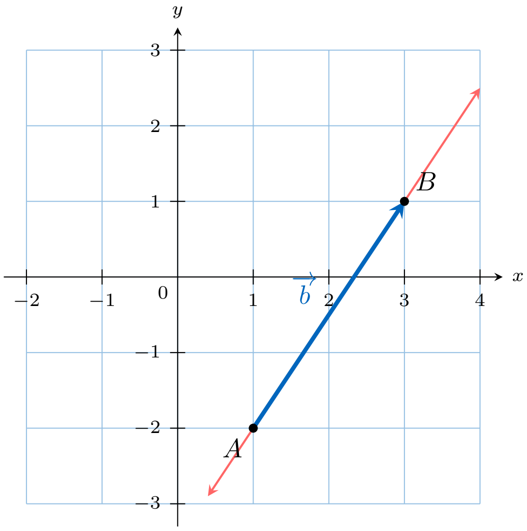
- The starting point is $A(1, -2)$.

- The direction vector $\overrightarrow{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ tells us to move 2 units in the positive x-direction and 3 units in the positive y-direction to find another point on the line.

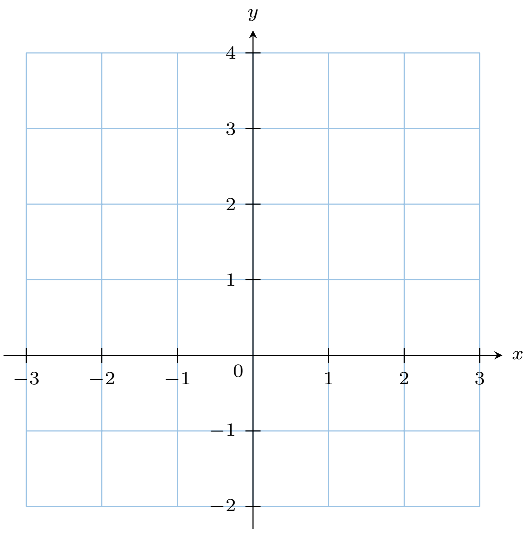
- Starting from A, the second point, let's call it B, will have coordinates:

$$B = (1 + 2, -2 + 3) = (3, 1)$$

We now plot point A, draw the vector from A to B, and then draw the line that passes through both points.



Ex 8: A line passes through the point $A(-2, 2)$ with direction vector $\vec{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$. Plot the point A, the direction vector originating from A, and the resulting line.

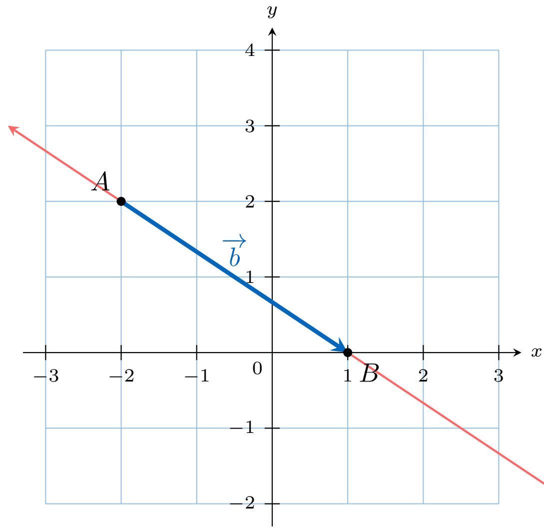


Answer: To plot the line, we start at the given point and use the direction vector to find a second point.

- The starting point is $A(-2, 2)$.
- The direction vector $\vec{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ tells us to move 3 units in the positive x-direction and 2 units in the negative y-direction to find another point on the line.
- Starting from A, the second point, let's call it B, will have coordinates:

$$B = (-2 + 3, 2 - 2) = (1, 0)$$

We now plot point A, draw the vector from A to B, and then draw the line that passes through both points.



A.3 WRITING EQUATIONS OF LINES

Ex 9: A line in space passes through the point $(1, -2, 3)$ in the direction $\begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}$.
Describe the line using a vector equation.

Answer: The vector equation is $\vec{r} = \vec{a} + \lambda \vec{b}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 5 \\ -6 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Ex 10: A line in space passes through the points $A(2, -1, 4)$ and $B(-1, 0, 2)$.
Describe the line using a vector equation.

Answer:

1. **Direction Vector** We can use the vector $\overrightarrow{AB}$.

$$\vec{b} = \overrightarrow{AB} = \begin{pmatrix} -1 - 2 \\ 0 - (-1) \\ 2 - 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}$$

2. **Vector Equation** Using point A, the vector equation is $\vec{r} = \vec{a} + \lambda \vec{b}$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Ex 11: A line in a plane passes through the point $A(1, 4)$ and is perpendicular to the vector $\vec{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
Describe the line using a vector equation.

Answer:

1. **direction vector** The given vector $\vec{n}$ is perpendicular to the line, which means $\vec{n}$ is a **normal vector** to the line. The **direction vector** $\vec{b}$ of the line must be perpendicular to $\vec{n}$.
For two vectors in 2D, if $\vec{n} = \begin{pmatrix} a \\ b \end{pmatrix}$, a perpendicular vector is $\vec{b} = \begin{pmatrix} -b \\ a \end{pmatrix}$ because

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -b \\ a \end{pmatrix} = -ab + ba = 0$$

Given $\overrightarrow{n} = \begin{pmatrix}2\\-1\end{pmatrix}$, a suitable direction vector is $\overrightarrow{b} = \begin{pmatrix}-(-1)\\2\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$.

2. **Vector Equation** The vector equation is $\overrightarrow{r} = \overrightarrow{a} + \lambda\overrightarrow{b}$.

$$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\2\end{pmatrix}, \quad \lambda \in \mathbb{R}$$

B PARAMETRIC EQUATIONS

B.1 FINDING INTERSECTIONS WITH COORDINATE AXES AND PLANES

Ex 12: A line has parametric equations:

$$\begin{cases} x = -4 + 2\lambda \\ y = 9 - 3\lambda \end{cases}, \quad \lambda \in \mathbb{R}$$

The line intersects the x-axis at point P.

1. What is the value of the y-coordinate at point P?
2. Find the value of the parameter λ at point P.
3. Determine the coordinates of the point of intersection P.

Answer:

1. Any point on the x-axis has a y-coordinate of zero. So, at point P, $y = 0$.
2. We use the parametric equation for y and set it to zero to find the corresponding value of λ.

$$\begin{aligned} y &= 9 - 3\lambda \\ 0 &= 9 - 3\lambda \\ 3\lambda &= 9 \\ \lambda &= 3 \end{aligned}$$

The value of the parameter at the point of intersection is $\lambda = 3$.

3. Now we substitute $\lambda = 3$ into the parametric equation for x to find the x-coordinate of P.

$$x = -4 + 2(3) = -4 + 6 = 2$$

The coordinates of the point of intersection are $P(2, 0)$.

Ex 13: A line has parametric equations:

$$\begin{cases} x = 5 + \lambda \\ y = 1 - 2\lambda \\ z = -3 + 4\lambda \end{cases}, \quad \lambda \in \mathbb{R}$$

The line intersects the YZ-plane at point Q.

1. What is the value of the x-coordinate at point Q?
2. Find the value of the parameter λ at point Q.
3. Determine the coordinates of the point of intersection Q.

Answer:

1. Any point on the YZ-plane has an x-coordinate of zero. So, at point Q, $x = 0$.
2. We use the parametric equation for x and set it to zero to find the corresponding value of λ.

$$\begin{aligned} x &= 5 + \lambda \\ 0 &= 5 + \lambda \\ \lambda &= -5 \end{aligned}$$

The value of the parameter at the point of intersection is $\lambda = -5$.

3. Now we substitute $\lambda = -5$ into the parametric equations for y and z to find the coordinates of Q.

$$y = 1 - 2(-5) = 1 + 10 = 11$$

$$z = -3 + 4(-5) = -3 - 20 = -23$$

The coordinates of the point of intersection are $Q(0, 11, -23)$.

Ex 14: A line has parametric equations:

$$\begin{cases} x = 7 - 2\lambda \\ y = -4 + 3\lambda \\ z = 10 - 5\lambda \end{cases}, \quad \lambda \in \mathbb{R}$$

The line intersects the XY-plane at point R.

1. What is the value of the z-coordinate at point R?
2. Find the value of the parameter λ at point R.
3. Determine the coordinates of the point of intersection R.

Answer:

1. Any point on the XY-plane has a z-coordinate of zero. So, at point R, $z = 0$.
2. We use the parametric equation for z and set it to zero to find the corresponding value of λ.

$$\begin{aligned} z &= 10 - 5\lambda \\ 0 &= 10 - 5\lambda \\ 5\lambda &= 10 \\ \lambda &= 2 \end{aligned}$$

The value of the parameter at the point of intersection is $\lambda = 2$.

3. Now we substitute $\lambda = 2$ into the parametric equations for x and y to find the coordinates of R.

$$x = 7 - 2(2) = 7 - 4 = 3$$

$$y = -4 + 3(2) = -4 + 6 = 2$$

The coordinates of the point of intersection are $R(3, 2, 0)$.

B.2 VERIFYING IF A POINT LIES ON A LINE

Ex 15: A line is defined by the parametric equations:

$$\begin{cases} x = 2 - t \\ y = 3 + 2t \end{cases} , \quad t \in \mathbb{R}$$

Determine if the point $Q(-1, 9)$ lies on the line.

Answer: To determine if the point Q lies on the line, we substitute its coordinates ($x = -1, y = 9$) into the parametric equations and solve for the parameter t. The point is on the line if there is a consistent value of t for both equations.

- For the x-coordinate:

$$\begin{aligned} -1 &= 2 - t \\ t &= 2 - (-1) \\ t &= 3 \end{aligned}$$

- For the y-coordinate:

$$\begin{aligned} 9 &= 3 + 2t \\ 6 &= 2t \\ t &= 3 \end{aligned}$$

Since we obtain the same value, $t = 3$, for both equations, the point $Q(-1, 9)$ does lie on the line.

Ex 16: A line is defined by the parametric equations:

$$\begin{cases} x = 4 - 2t \\ y = 1 + 3t \end{cases} , \quad t \in \mathbb{R}$$

Determine if the point $P(2, 5)$ lies on the line.

Answer: To determine if the point P lies on the line, we substitute its coordinates ($x = 2, y = 5$) into the parametric equations and check if we get a consistent value for the parameter t.

- For the x-coordinate:

$$\begin{aligned} 2 &= 4 - 2t \\ 2t &= 4 - 2 \\ 2t &= 2 \\ t &= 1 \end{aligned}$$

- For the y-coordinate:

$$\begin{aligned} 5 &= 1 + 3t \\ 4 &= 3t \\ t &= \frac{4}{3} \end{aligned}$$

The value of t from the x-coordinate ($t = 1$) is different from the value of t from the y-coordinate ($t = \frac{4}{3}$). Since there is no single value of t that satisfies both equations, the point $P(2, 5)$ does not lie on the line.

Ex 17: A line is defined by the parametric equations:

$$\begin{cases} x = 1 + 2t \\ y = 5 - t \\ z = -2 + 4t \end{cases} , \quad t \in \mathbb{R}$$

Determine if the point $P(5, 3, 6)$ lies on the line.

Answer: To determine if the point P lies on the line, we substitute its coordinates ($x = 5, y = 3, z = 6$) into the parametric equations and solve for the parameter t. The point is on the line if and only if there is a single, consistent value of t that satisfies all three equations.

- For the x-coordinate:

$$\begin{aligned} 5 &= 1 + 2t \\ 4 &= 2t \\ t &= 2 \end{aligned}$$

- For the y-coordinate:

$$\begin{aligned} 3 &= 5 - t \\ t &= 5 - 3 \\ t &= 2 \end{aligned}$$

- For the z-coordinate:

$$\begin{aligned} 6 &= -2 + 4t \\ 8 &= 4t \\ t &= 2 \end{aligned}$$

Since we obtain the same value, $t = 2$, for all three equations, the point $P(5, 3, 6)$ does lie on the line.

C CARTESIAN EQUATION IN PLANE

C.1 FINDING THE NORMAL VECTOR FROM A CARTESIAN EQUATION

Ex 18: State the normal vector for the line with equation $2x - 5y = 8$.

$$\overrightarrow{n} = \begin{pmatrix} \boxed{2} \\ \boxed{-5} \end{pmatrix}$$

Answer: For a line with the Cartesian equation $ax + by = C$, the normal vector is $\overrightarrow{n} = \begin{pmatrix} a \\ b \end{pmatrix}$.By comparing $2x - 5y = 8$ to the general form, we can identify $a = 2$ and $b = -5$.Therefore, the normal vector is $\overrightarrow{n} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

Ex 19: State the normal vector for the line with equation $y = 4x - 1$.

$$\overrightarrow{n} = \begin{pmatrix} \boxed{-4} \\ \boxed{1} \end{pmatrix}$$

Answer: We first need to write the equation in the general form $ax + by = C$.

$$\begin{aligned} y &= 4x - 1 \\ -4x + y &= -1 \end{aligned}$$

Therefore, the normal vector is $\overrightarrow{n} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$.

Ex 20: State the normal vector for the line with equation $y = -\frac{2}{3}x + 5$.

$$\overrightarrow{n} = \begin{pmatrix} \boxed{2} \\ \boxed{3} \end{pmatrix}$$

Answer: We first need to write the equation in the general form $ax + by = C$.

$$\begin{aligned} y &= -\frac{2}{3}x + 5 \\ 3y &= -2x + 15 \\ 2x + 3y &= 15 \end{aligned}$$

By comparing this to the general form, we can identify $a = 2$ and $b = 3$.Therefore, a normal vector is $\overrightarrow{n} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

C.2 USING THE NORMAL VECTOR TO FIND THE CARTESIAN EQUATION

Ex 21: Find the Cartesian equation of the line that passes through the point $P(-3, 5)$ and has a normal vector of $\overrightarrow{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

Answer: The Cartesian equation of a line with normal vector $\overrightarrow{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $ax + by = C$.

1. **Use the normal vector** With $\overrightarrow{n} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, the equation of the line is of the form:

$$1x - 2y = C \quad \text{or} \quad x - 2y = C$$

2. **Use the point to find C** The line passes through the point $P(-3, 5)$. We substitute $x = -3$ and $y = 5$:

$$\begin{aligned} (-3) - 2(5) &= C \\ -3 - 10 &= C \\ -13 &= C \end{aligned}$$

The Cartesian equation of the line is $x - 2y = -13$.

Ex 22: Find the Cartesian equation of the line that passes through the point $P(1, 2)$ and has a normal vector of $\overrightarrow{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Answer: The Cartesian equation of a line with normal vector $\overrightarrow{n} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $ax + by = C$.

1. **Use the normal vector** With $\overrightarrow{n} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, the equation of the line is of the form:

$$3x + 4y = C$$

2. **Use the point to find C** The line passes through the point $P(1, 2)$. Therefore, the coordinates of P must satisfy the equation. We substitute $x = 1$ and $y = 2$:

$$\begin{aligned} 3(1) + 4(2) &= C \\ 3 + 8 &= C \\ 11 &= C \end{aligned}$$

The Cartesian equation of the line is $3x + 4y = 11$.