LIMITS

A DEFINITION

A.1 INVESTIGATING LIMITS NUMERICALLY

Ex 1: Complete the table of values:

x	$\dfrac{x^2-1}{x-1}$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

Hence conjecture:

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \boxed{2}$$

Answer: We evaluate the function $f(x)=\frac{x^2-1}{x-1}$ for each value of x.

- For $x=1.1$, $f(1.1)=\dfrac{(1.1)^2-1}{1.1-1}=2.1$.
- For $x=1.01$, $f(1.01)=\dfrac{(1.01)^2-1}{1.01-1}=2.01$.
- For $x=1.001$, $f(1.001)=\dfrac{(1.001)^2-1}{1.001-1}=2.001$.
- For $x=1.0001$, $f(1.0001)=\dfrac{(1.0001)^2-1}{1.0001-1}=2.0001$.

As x gets closer and closer to 1, the value of the function gets closer and closer to 2. Therefore, we conjecture that the limit is 2.
Note: We can see why this happens by simplifying the expression algebraically: $\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}=x+1$ for $x\neq 1$. As $x\to 1$, the value of $x+1$ clearly approaches 2.

Ex 2: Complete the table of values below (round to 5 decimal places where needed).

h	$\dfrac{(1+h)^3-1}{h}$
0.1	3.31
0.01	3.0301
0.001	3.003001
-0.01	2.9701

Hence conjecture:

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h} = \boxed{3}$$

Answer: We evaluate the function $f(h)=\frac{(1+h)^3-1}{h}$ for each value of h.

- For $h=0.1$, $f(0.1)=\dfrac{(1.1)^3-1}{0.1}=\dfrac{1.331-1}{0.1}=\dfrac{0.331}{0.1}=3.31$.
- For $h=0.01$, $f(0.01)=\dfrac{(1.01)^3-1}{0.01}=\dfrac{1.030301-1}{0.01}=\dfrac{0.030301}{0.01}=3.0301$.
- For $h=0.001$, $f(0.001)=\dfrac{(1.001)^3-1}{0.001}=\dfrac{1.003003001-1}{0.001}\approx 3.00300$.
- For $h=-0.01$, $f(-0.01)=\dfrac{(0.99)^3-1}{-0.01}=\dfrac{0.970299-1}{-0.01}=\dfrac{-0.029701}{-0.01}=2.9701$.

As h gets closer to 0 from both the positive and negative sides, the value of the function gets closer to 3. Therefore, we conjecture that the limit is 3.
Note: This limit can be confirmed algebraically by expanding the numerator:

$$\frac{(1+3h+3h^2+h^3)-1}{h}=\frac{3h+3h^2+h^3}{h}=3+3h+h^2$$

As $h\to 0$, this expression clearly approaches $3+0+0=3$.

Ex 3: Complete the table of values below, ensuring your calculator is in **radian mode** (round to 5 decimal places).

x	$\dfrac{\sin(x)}{x}$
0.1	0.99833
0.01	0.99998
-0.01	0.99998

Hence conjecture:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \boxed{1}$$

Answer: We evaluate the function $f(x)=\frac{\sin(x)}{x}$ for each value of x using a calculator in radian mode, rounding to 5 decimal places.

- For $x=0.1$, $f(0.1)=\dfrac{\sin(0.1)}{0.1}\approx 0.99833$.
- For $x=0.01$, $f(0.01)=\dfrac{\sin(0.01)}{0.01}\approx 0.99998$.
- For $x=-0.01$, $f(-0.01)=\dfrac{\sin(-0.01)}{-0.01}\approx 0.99998$.

As x gets closer to 0 from both the positive and negative sides, the value of the function gets closer to 1. Therefore, we conjecture that the limit is 1.
Note: This is a fundamental trigonometric limit. Unlike the previous example, it cannot be simplified by factoring, so numerical or geometric arguments are necessary to evaluate it.

A.2 EVALUATING LIMITS BY DIRECT SUBSTITUTION

Ex 4: Evaluate:

$$\lim_{x\to 2} x^2 = \boxed{4}$$

Answer: We can evaluate the limit by direct substitution:

$$\begin{aligned}\lim_{x\to 2} x^2 &= (2)^2\\ &= 4\end{aligned}$$

Ex 5: Evaluate:

$$\lim_{x\to 2}(x^2-3x+1) = \boxed{-1}$$

Answer: We can evaluate the limit by direct substitution:

$$\begin{aligned}\lim_{x\to 2}(x^2-3x+1) &= (2)^2-3(2)+1\\ &= 4-6+1\\ &= -1\end{aligned}$$

Ex 6: Evaluate:

$$\lim_{x\to 5} 7 = \boxed{7}$$

Answer: The function is $f(x)=7$. Since the value of the function is 7 for all values of x, the limit as x approaches any number is also 7.

$$\lim_{x\to 5} 7 = 7$$

Ex 7: Evaluate:

$$\lim_{x\to 1}\frac{x+3}{x+1} = \boxed{2}$$

Answer: The function $f(x)=\dfrac{x+3}{x+1}$ is a rational function. Since the denominator is not zero at $x=1$, we can evaluate the limit by direct substitution:

$$\begin{aligned}\lim_{x\to 1}\frac{x+3}{x+1} &= \frac{1+3}{1+1}\\ &= \frac{4}{2}\\ &= 2\end{aligned}$$

A.3 EVALUATING LIMITS BY ALGEBRAIC SIMPLIFICATION

Ex 8: Evaluate:

$$\lim_{x\to 0}\frac{x+x^2}{2x} = \boxed{1/2}$$

Answer: Direct substitution of $x=0$ results in the indeterminate form $\frac{0}{0}$. We must first simplify the expression algebraically for $x\neq 0$.

$$\begin{aligned}\frac{x+x^2}{2x} &= \frac{x(1+x)}{2x}\\ &= \frac{1+x}{2} \quad (\text{for } x\neq 0)\\ &\xrightarrow[x\to 0]{} \frac{1+0}{2} = \frac{1}{2}\end{aligned}$$

Ex 9: Evaluate:

$$\lim_{x\to 0}\frac{3x^2-2x}{x^2+2x} = \boxed{-1}$$

Answer: Direct substitution of $x=0$ results in the indeterminate form $\frac{0}{0}$. We must first simplify the expression by factoring.

$$\begin{aligned}\frac{3x^2-2x}{x^2+2x} &= \frac{x(3x-2)}{x(x+2)}\\ &= \frac{3x-2}{x+2} \quad (\text{for } x\neq 0)\\ &\xrightarrow[x\to 0]{} \frac{3(0)-2}{0+2} = \frac{-2}{2} = -1\end{aligned}$$

Ex 10: Evaluate:

$$\lim_{x\to 3}\frac{x^2-9}{x-3} = \boxed{6}$$

Answer: Direct substitution of $x=3$ results in the indeterminate form $\frac{0}{0}$. We simplify by factoring the numerator as a difference of squares.

$$\begin{aligned}\frac{x^2-9}{x-3} &= \frac{(x-3)(x+3)}{x-3}\\ &= x+3 \quad (\text{for } x\neq 3)\\ &\xrightarrow[x\to 3]{} 3+3=6\end{aligned}$$

Ex 11: Evaluate:

$$\lim_{x\to -1}\frac{x^2+3x+2}{x+1} = \boxed{1}$$

Answer: Direct substitution of $x=-1$ results in the indeterminate form $\frac{0}{0}$. We must first simplify the expression by factoring the numerator.

$$\begin{aligned}\frac{x^2+3x+2}{x+1} &= \frac{(x+1)(x+2)}{x+1}\\ &= x+2 \quad (\text{for } x\neq -1)\\ &\xrightarrow[x\to -1]{} -1+2=1\end{aligned}$$

Note: To factor the quadratic x^2+3x+2, we look for two numbers that multiply to 2 and add to 3. These numbers are +1 and +2, which gives us the factors $(x+1)(x+2)$.

A.4 FINDING DERIVATIVES FROM FIRST PRINCIPLES

Ex 12: Evaluate:

$$\lim_{h\to 0}\frac{(2(x+h)+3)-(2x+3)}{h} = \boxed{2}$$

Answer: Direct substitution of $h=0$ results in the indeterminate form $\dfrac{(2x+3)-(2x+3)}{0} = \dfrac{0}{0}$. We must first simplify the expression by expanding the numerator.

$$\begin{aligned}\frac{(2(x+h)+3)-(2x+3)}{h} &= \frac{2x+2h+3-2x-3}{h}\\ &= \frac{2h}{h}\\ &= 2 \quad (\text{for } h\neq 0)\\ &\xrightarrow[h\to 0]{} 2\end{aligned}$$

(Note: This limit is the definition of the derivative of the function $f(x)=2x+3$.)

Ex 13: Evaluate:

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\boxed{2x}$$

Answer: Direct substitution of $h=0$ results in the indeterminate form $\frac{x^2-x^2}{0}=\frac{0}{0}$. We must first simplify the expression by expanding the numerator.

$$\begin{aligned}\frac{(x+h)^2-x^2}{h} &= \frac{(x^2+2xh+h^2)-x^2}{h}\\ &= \frac{2xh+h^2}{h}\\ &= \frac{h(2x+h)}{h}\\ &= 2x+h \quad (\text{for } h\neq 0)\\ &\xrightarrow[h\to 0]{} 2x+0=2x\end{aligned}$$

(Note: This limit is the definition of the derivative of the function $f(x)=x^2$.)

Ex 14: Evaluate for $x\neq 0$:

$$\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\boxed{-1/x^2}$$

Answer: Let $x\neq 0$.

Direct substitution of $h=0$ results in the indeterminate form $\frac{0}{0}$. We must first simplify the expression by combining the fractions in the main numerator.

$$\begin{aligned}\frac{\frac{1}{x+h}-\frac{1}{x}}{h} &= \frac{\frac{x-(x+h)}{x(x+h)}}{h}\\ &= \frac{\frac{-h}{x(x+h)}}{h}\\ &= \frac{-h}{h\cdot x(x+h)}\\ &= \frac{-1}{x(x+h)} \quad (\text{for } h\neq 0)\\ &\xrightarrow[h\to 0]{} \frac{-1}{x(x+0)}=-\frac{1}{x^2}\end{aligned}$$

(Note: This limit is the definition of the derivative of the function $f(x)=1/x$.)

Ex 15: Evaluate for $x>0$:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\boxed{1/(2*sqrt(x))}$$

Answer: Let $x>0$.Direct substitution of $h=0$ results in the indeterminate form $\frac{0}{0}$. To simplify, we multiply the numerator and denominator by the conjugate of the numerator, which is $\sqrt{x+h}+\sqrt{x}$.

$$\begin{aligned}\frac{\sqrt{x+h}-\sqrt{x}}{h} &= \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\cdot\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)\\ &= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{h}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}} \quad (\text{for } h\neq 0)\\ &\xrightarrow[h\to 0]{} \frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}\end{aligned}$$

(Note: This limit is the definition of the derivative of the function $f(x)=\sqrt{x}$.)

A.5 RESOLVING INDETERMINATE FORMS BY FACTORING

Ex 16: Evaluate the following limit algebraically:

$$\lim_{x\to 0}\frac{x+x^2}{2x}$$

Answer: Direct substitution of $x=0$ results in the indeterminate form $\frac{0}{0}$. We must first simplify the expression algebraically for $x\neq 0$.

$$\begin{aligned}\frac{x+x^2}{2x} &= \frac{x(1+x)}{2x}\\ &= \frac{1+x}{2} \quad (\text{for } x\neq 0)\\ &\xrightarrow[x\to 0]{} \frac{1+0}{2}=\frac{1}{2}\end{aligned}$$

Ex 17: Evaluate the following limit algebraically:

$$\lim_{x\to 0}\frac{5x^2+3x}{2x^2-x}$$

Answer: Direct substitution of $x=0$ results in the indeterminate form $\frac{0}{0}$. We must first simplify the expression by factoring.

$$\begin{aligned}\frac{5x^2+3x}{2x^2-x} &= \frac{x(5x+3)}{x(2x-1)}\\ &= \frac{5x+3}{2x-1} \quad (\text{for } x\neq 0)\\ &\xrightarrow[x\to 0]{} \frac{5(0)+3}{2(0)-1}=\frac{3}{-1}=-3\end{aligned}$$

Ex 18: Evaluate the following limit algebraically:

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

Answer: Direct substitution of $x=2$ results in the indeterminate form $\frac{0}{0}$. We simplify by factoring the numerator as a difference of squares.

$$\begin{aligned}\frac{x^2-4}{x-2} &= \frac{(x-2)(x+2)}{x-2}\\ &= x+2 \quad (\text{for } x\neq 2)\\ &\xrightarrow[x\to 2]{} 2+2=4\end{aligned}$$

Ex 19: Evaluate the following limit algebraically:

$$\lim_{x\to 2}\frac{x^2+3x-10}{x-2}$$

Answer: Direct substitution of $x=2$ results in the indeterminate form $\frac{0}{0}$. We simplify by factoring the numerator.

$$\begin{aligned}\frac{x^2+3x-10}{x-2} &= \frac{(x-2)(x+5)}{x-2}\\ &= x+5 \quad (\text{for } x\neq 2)\\ &\xrightarrow[x\to 2]{} 2+5=7\end{aligned}$$

(Note: To factor $x^2+3x-10$, we look for two numbers that multiply to -10 and add to +3. These numbers are +5 and -2, giving the factors $(x-2)(x+5)$.)

B ALGEBRAIC EVALUATION OF LIMITS

B.1 APPLYING THE LIMIT LAWS

Ex 20: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -1$, evaluate:

$$\lim_{x\to a}[f(x)g(x)] = \boxed{-3}$$

Answer: We use the Product Law for limits, which states that the limit of a product is the product of the limits (provided the individual limits exist).

$$\begin{aligned}\lim_{x\to a}[f(x)g(x)] &= \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)\\ &= (3)\cdot(-1)\\ &= -3\end{aligned}$$

Ex 21: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -1$, evaluate:

$$\lim_{x\to a}[f(x)+g(x)] = \boxed{2}$$

Answer: We use the Sum Law for limits, which states that the limit of a sum is the sum of the limits.

$$\begin{aligned}\lim_{x\to a}[f(x)+g(x)] &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x)\\ &= 3+(-1)\\ &= 2\end{aligned}$$

Ex 22: Given that $\lim_{x\to a} f(x) = 3$, evaluate:

$$\lim_{x\to a}[5f(x)] = \boxed{15}$$

Answer: We use the Constant Multiple Law for limits, which allows us to move a constant factor outside of the limit.

$$\begin{aligned}\lim_{x\to a}[5f(x)] &= 5\cdot\lim_{x\to a} f(x)\\ &= 5\cdot(3)\\ &= 15\end{aligned}$$

Ex 23: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -1$, evaluate:

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \boxed{-3}$$

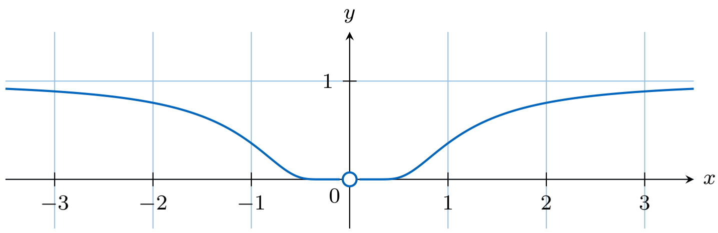
Answer: We use the Quotient Law for limits, which states that the limit of a quotient is the quotient of the limits (provided the limit of the denominator is not zero).

$$\begin{aligned}\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}\\ &= \frac{3}{-1}\\ &= -3\end{aligned}$$

C EXISTENCE OF A LIMIT

C.1 EVALUATING LIMITS GRAPHICALLY

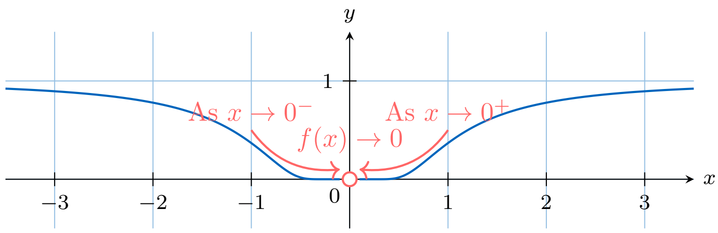
Ex 24: The graph of the function $f(x) = e^{-1/x^2}$ is shown below.



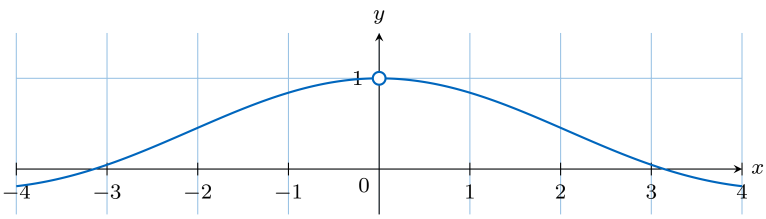
Evaluate graphically:

$$\lim_{x\to 0} e^{-1/x^2} = \boxed{0}$$

Answer: By observing the graph, as x approaches 0 from both the left side ($x\to 0^-$) and the right side ($x\to 0^+$), the curve gets closer and closer to the y-value of 0. Even though the function is not defined at $x=0$ (indicated by the open circle), the limit exists and is equal to 0.



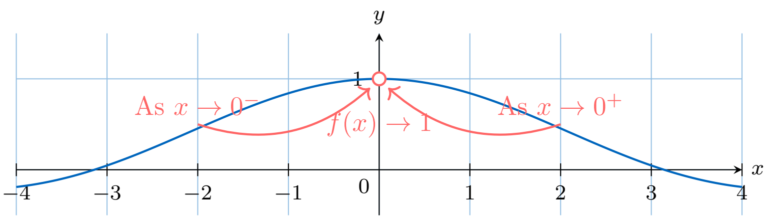
Ex 25: The graph of the function $f(x) = \dfrac{\sin(x)}{x}$ is shown below.



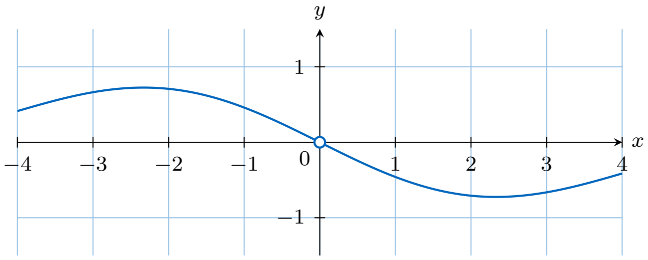
Evaluate graphically:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \boxed{1}$$

Answer: By observing the graph, as x approaches 0 from both the left side ($x\to 0^-$) and the right side ($x\to 0^+$), the curve gets closer and closer to the y-value of 1. Even though the function is not defined at $x=0$ (indicated by the open circle at $(0,1)$), the limit exists and is equal to 1.



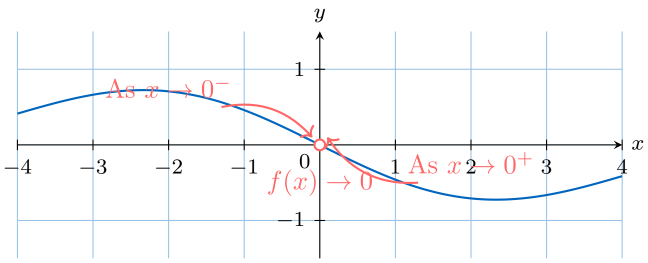
Ex 26: The graph of the function $f(x) = \dfrac{\cos(x)-1}{x}$ is shown below.



Evaluate graphically:

$$\lim_{x\to 0}\frac{\cos(x)-1}{x}=\boxed{0}$$

Answer: By observing the graph, as x approaches 0 from both the left side ($x \to 0^-$) and the right side ($x \to 0^+$), the curve gets closer and closer to the y-value of 0. Even though the function is not defined at $x = 0$ (indicated by the open circle at the origin), the limit exists and is equal to 0.



D INFINITE LIMITS AND VERTICAL ASYMPTOTES

D.1 EVALUATING INFINITE LIMITS

Ex 27: Evaluate the following one-sided limit:

$$\lim_{x\to 1^+}\frac{1}{x-1}$$

Answer: To evaluate the limit as x approaches 1 from the right ($x \to 1^+$), we consider values of x that are slightly greater than 1. For such values, the denominator $(x-1)$ is a small positive, which we can denote as 0^+ : $\lim_{x\to 1^+}(x-1)=0^+$.

$$\begin{aligned}\lim_{x\to 1^+}\frac{1}{x-1}&=\frac{1}{0^+}\\&=+\infty\end{aligned}$$

Ex 28: Evaluate the following one-sided limit:

$$\lim_{x\to 1^-}\frac{1}{x-1}$$

Answer: To evaluate the limit as x approaches 1 from the left ($x \to 1^-$), we consider values of x that are slightly less than 1. For such values, the denominator $(x-1)$ is a small negative number, which we can denote as 0^-: $\lim_{x\to 1^-}(x-1)=0^-$.

$$\begin{aligned}\lim_{x\to 1^-}\frac{1}{x-1}&=\frac{1}{0^-}\\&=-\infty\end{aligned}$$

Ex 29: Evaluate the following limit:

$$\lim_{x\to 2}\frac{-5}{(x-2)^2}$$

Answer: As x approaches 2 from either the left or the right, the term $(x-2)$ becomes a small number that is squared. The square of any small non-zero number is a small positive number. We can denote this as 0^+: $\lim_{x\to 2}(x-2)^2=0^+$.

$$\begin{aligned}\lim_{x\to 2}\frac{-5}{(x-2)^2}&=\frac{-5}{0^+}\\&=-\infty\end{aligned}$$

D.2 FINDING LIMITS AND VERTICAL ASYMPTOTES

Ex 30: Consider the function $f(x)=\dfrac{x+1}{x-2}$.

1. Evaluate the one-sided limits of $f(x)$ as x approaches 2:
   - $\lim_{x\to 2^+} f(x)$
   - $\lim_{x\to 2^-} f(x)$
2. Does $\lim_{x\to 2} f(x)$ exist? Justify your answer.
3. Hence, state the equation of any vertical asymptotes of the graph of $y = f(x)$.

Answer:

1. We analyze the signs of the numerator and denominator as x approaches 2.
   - As $x \to 2^+$, the numerator $(x+1)$ approaches 3. The denominator $(x-2)$ is a small positive number (0^+).
     $$\lim_{x\to 2^+}\frac{x+1}{x-2}=\frac{3}{0^+}=+\infty$$
   - As $x \to 2^-$, the numerator $(x+1)$ approaches 3. The denominator $(x-2)$ is a small negative number (0^-).
     $$\lim_{x\to 2^-}\frac{x+1}{x-2}=\frac{3}{0^-}=-\infty$$
2. The limit $\lim_{x\to 2} f(x)$ does not exist. For a limit to exist, the function must approach a single, finite value from both the left and the right. Here, the function's behavior is unbounded.
3. The definition of a vertical asymptote is a line $x = a$ where the function approaches $\pm\infty$. Since the one-sided limits as $x \to 2$ are infinite, the equation of the vertical asymptote is $x = 2$.

Ex 31: Consider the function $f(x)=\dfrac{x}{(x-1)^2}$.

1. Evaluate the one-sided limits of $f(x)$ as x approaches 1:
   - $\lim_{x\to 1^+} f(x)$
   - $\lim_{x\to 1^-} f(x)$
2. Does $\lim_{x\to 1} f(x)$ exist? Justify your answer.
3. Hence, state the equation of any vertical asymptotes of the graph of $y = f(x)$.

Answer:

1. We analyze the signs of the numerator and denominator as x approaches 1.
   - As $x \to 1^+$, the numerator (x) approaches 1. The denominator $(x-1)^2$ is the square of a small positive number, so it approaches 0^+.
   $$\lim_{x\to 1^+} \frac{x}{(x-1)^2} = \frac{1}{0^+} = +\infty$$
   - As $x \to 1^-$, the numerator (x) approaches 1. The denominator $(x-1)^2$ is the square of a small negative number, so it also approaches 0^+.
   $$\lim_{x\to 1^-} \frac{x}{(x-1)^2} = \frac{1}{0^+} = +\infty$$
2. The limit $\lim_{x\to 1} f(x)$ does not exist. Although the function approaches $+\infty$ from both sides, for a limit to exist, it must approach a single, **finite** value.
3. Since the one-sided limits as $x \to 1$ are infinite, the definition of a vertical asymptote is met. The equation of the vertical asymptote is $x = 1$.

E LIMITS AT INFINITY

E.1 EVALUATING LIMITS AT INFINITY

Ex 32: Evaluate:

$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 1} = \boxed{\frac{3}{2}}$$

Answer: We manipulate the expression by factoring out the highest power of x from the numerator and the denominator, which is x^2.

$$\begin{aligned}\frac{3x^2 - x + 4}{2x^2 + 5x - 1} &= \frac{x^2\left(3 - \frac{1}{x} + \frac{4}{x^2}\right)}{x^2\left(2 + \frac{5}{x} - \frac{1}{x^2}\right)} \\ &= \frac{3 - \frac{1}{x} + \frac{4}{x^2}}{2 + \frac{5}{x} - \frac{1}{x^2}} \quad \text{for } x \neq 0 \\ &\xrightarrow[x\to+\infty]{} \frac{3 - 0 + 0}{2 + 0 - 0} = \frac{3}{2}\end{aligned}$$

Ex 33: Evaluate:

$$\lim_{x\to\infty} \frac{2x + 5}{x^2 - 3x + 1} = \boxed{0}$$

Answer: We manipulate the expression by factoring out the highest power of x from the denominator, which is x^2.

$$\begin{aligned}\frac{2x + 5}{x^2 - 3x + 1} &= \frac{x^2\left(\frac{2}{x} + \frac{5}{x^2}\right)}{x^2\left(1 - \frac{3}{x} + \frac{1}{x^2}\right)} \\ &= \frac{\frac{2}{x} + \frac{5}{x^2}}{1 - \frac{3}{x} + \frac{1}{x^2}} \quad \text{for } x \neq 0 \\ &\xrightarrow[x\to\infty]{} \frac{0 + 0}{1 - 0 + 0} = \frac{0}{1} = 0\end{aligned}$$

Ex 34: Evaluate:

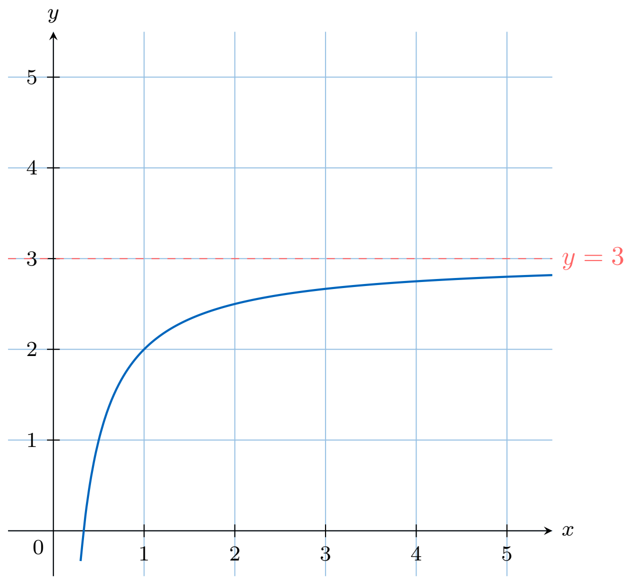
$$\lim_{x\to-\infty} \frac{4 - 3x}{2x + 1} = \boxed{-\frac{3}{2}}$$

Answer: We manipulate the expression by factoring out the highest power of x from the denominator, which is x.

$$\begin{aligned}\frac{4 - 3x}{2x + 1} &= \frac{x\left(\frac{4}{x} - 3\right)}{x\left(2 + \frac{1}{x}\right)} \\ &= \frac{\frac{4}{x} - 3}{2 + \frac{1}{x}} \quad \text{for } x \neq 0 \\ &\xrightarrow[x\to-\infty]{} \frac{0 - 3}{2 + 0} = -\frac{3}{2}\end{aligned}$$

E.2 DETERMINING END BEHAVIOR GRAPHICALLY

Ex 35: The graph of the function $f(x) = -\frac{1}{x} + 3$ is shown below for $x > 0$.

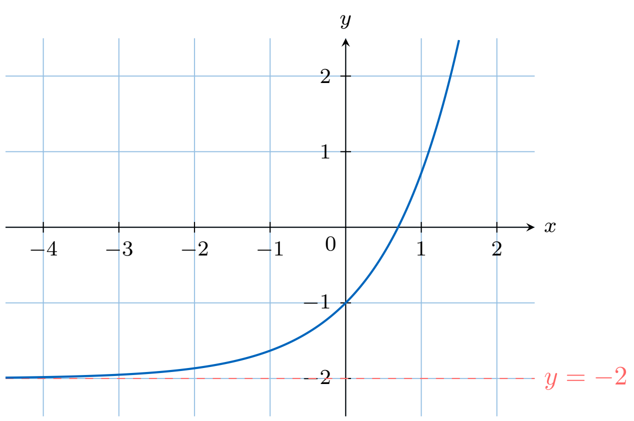


Evaluate graphically:

$$\lim_{x\to\infty}\left(-\frac{1}{x} + 3\right) = \boxed{3}$$

Answer: By observing the graph, as x becomes very large ($x \to \infty$), the curve gets closer and closer to the dashed horizontal line. The graph shows that this horizontal asymptote is the line $y = 3$. Therefore, we conclude from the graph that the limit is 3.

Ex 36: The graph of the function $f(x) = e^x - 2$ is shown below.

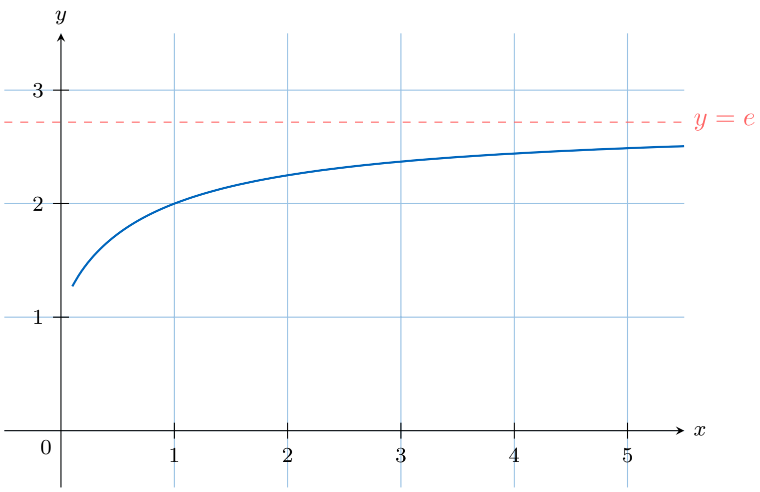


Evaluate graphically:

$$\lim_{x\to-\infty}(e^x - 2) = \boxed{-2}$$

Answer: By observing the graph, as x becomes very large and negative ($x \to -\infty$), the curve flattens out and gets closer and closer to the dashed horizontal line. The graph shows that this horizontal asymptote is the line $y = -2$. Therefore, we conclude from the graph that the limit is -2.

Ex 37: The graph of the function $f(x) = \left(1 + \frac{1}{x}\right)^x$ is shown below for $x > 0$.



Evaluate graphically:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \boxed{e}$$

Answer: By observing the graph, as x becomes very large ($x \to \infty$), the curve flattens out and approaches the horizontal asymptote. The graph shows that this horizontal asymptote is the line $y = e$. Therefore, we conclude from the graph that the limit is e.

E.3 FINDING LIMITS AT INFINITY WITH RADICAL FUNCTIONS

Ex 38: Consider the function $f(x) = \dfrac{2x}{\sqrt{x^2+1}}$.

1. Find $\lim_{x\to\infty} f(x)$.
2. Find $\lim_{x\to-\infty} f(x)$.
3. Hence, write down the equations of any horizontal asymptotes of the graph of $y = f(x)$.

Answer:

1. To find the limit as $x \to \infty$, we factor out the highest power of x from the denominator.

$$\frac{2x}{\sqrt{x^2+1}} = \frac{2x}{\sqrt{x^2(1+\frac{1}{x^2})}} = \frac{2x}{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}} = \frac{2x}{|x|\sqrt{1+\frac{1}{x^2}}}$$

Since $x \to \infty$, x is positive, so $|x| = x$.

$$\frac{2x}{x\sqrt{1+\frac{1}{x^2}}} = \frac{2}{\sqrt{1+\frac{1}{x^2}}} \quad (\text{for } x \neq 0)$$
$$\xrightarrow[x\to\infty]{} \frac{2}{\sqrt{1+0}} = 2$$

2. We use the same factored expression. Since $x \to -\infty$, x is negative, so $|x| = -x$.

$$\frac{2x}{|x|\sqrt{1+\frac{1}{x^2}}} = \frac{2x}{-x\sqrt{1+\frac{1}{x^2}}} = \frac{-2}{\sqrt{1+\frac{1}{x^2}}} \quad (\text{for } x \neq 0)$$
$$\xrightarrow[x\to-\infty]{} \frac{-2}{\sqrt{1+0}} = -2$$

3. The limits as $x \to \infty$ and $x \to -\infty$ define the horizontal asymptotes. Therefore, the equations of the horizontal asymptotes are $y = 2$ and $y = -2$.

Ex 39: Consider the function $f(x) = \dfrac{\sqrt{9x^2+4}}{x-1}$.

1. Find $\lim_{x\to\infty} f(x)$.
2. Find $\lim_{x\to-\infty} f(x)$.
3. Hence, write down the equations of any horizontal asymptotes of the graph of $y = f(x)$.

Answer:

1. To find the limit as $x \to \infty$, we factor out the highest power of x from the numerator and denominator.

$$\frac{\sqrt{9x^2+4}}{x-1} = \frac{\sqrt{x^2(9+\frac{4}{x^2})}}{x(1-\frac{1}{x})} = \frac{\sqrt{x^2}\sqrt{9+\frac{4}{x^2}}}{x(1-\frac{1}{x})} = \frac{|x|\sqrt{9+\frac{4}{x^2}}}{x(1-\frac{1}{x})}$$

Since $x \to \infty$, x is positive, so $|x| = x$.

$$\frac{x\sqrt{9+\frac{4}{x^2}}}{x(1-\frac{1}{x})} = \frac{\sqrt{9+\frac{4}{x^2}}}{1-\frac{1}{x}} \quad (\text{for } x \neq 0)$$
$$\xrightarrow[x\to\infty]{} \frac{\sqrt{9+0}}{1-0} = 3$$

2. We use the same factored expression. Since $x \to -\infty$, x is negative, so $|x| = -x$.

$$\frac{|x|\sqrt{9+\frac{4}{x^2}}}{x(1-\frac{1}{x})} = \frac{-x\sqrt{9+\frac{4}{x^2}}}{x(1-\frac{1}{x})} = \frac{-\sqrt{9+\frac{4}{x^2}}}{1-\frac{1}{x}} \quad (\text{for } x \neq 0)$$
$$\xrightarrow[x\to-\infty]{} \frac{-\sqrt{9+0}}{1-0} = -3$$

3. The limits as $x \to \infty$ and $x \to -\infty$ define the horizontal asymptotes. Therefore, the equations of the horizontal asymptotes are $y = 3$ and $y = -3$.

F THE SQUEEZE THEOREM

F.1 APPLYING THE SQUEEZE THEOREM

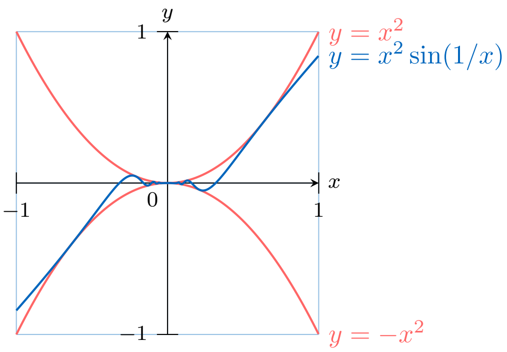
Ex 40: Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Answer:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$
$$-x^2 \leq x^2\sin\left(\frac{1}{x}\right) \leq x^2 \quad (x^2 \geq 0)$$

As $\lim_{x\to 0}(-x^2) = 0$ and $\lim_{x\to 0}(x^2) = 0$, by the Squeeze Theorem,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$



Ex 41: Evaluate $\lim_{x\to 0} x \cos\left(\frac{1}{x^2}\right)$.

Answer:

$$-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$$
$$-|x| \leq x\cos\left(\frac{1}{x^2}\right) \leq |x|$$

As $\lim_{x\to 0}(-|x|) = 0$ and $\lim_{x\to 0}|x| = 0$, by the Squeeze Theorem,

$$\lim_{x\to 0} x \cos\left(\frac{1}{x^2}\right) = 0$$

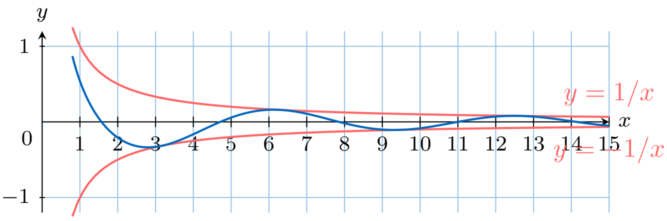
Ex 42: Evaluate $\lim_{x\to\infty} \frac{\cos(x)}{x}$.

Answer:

$$-1 \leq \cos(x) \leq 1$$
$$-\frac{1}{x} \leq \frac{\cos(x)}{x} \leq \frac{1}{x} \quad (\text{for } x > 0)$$

As $\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0$ and $\lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$, by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{\cos(x)}{x} = 0$$



G CONTINUITY

G.1 EVALUATING LIMITS USING CONTINUITY

Ex 43: Evaluate:

$$\lim_{x\to 1} e^{2x} = \boxed{e^2}$$

Answer: We apply the Limit of a Composite Function rule. The outer function is $f(u) = e^u$ and the inner function is $g(x) = 2x$. Since the exponential function is continuous everywhere, we can move the limit inside the function.

$$\begin{aligned}\lim_{x\to 1} e^{2x} &= e^{\left(\lim_{x\to 1} 2x\right)} && (\text{since } e^u \text{ is continuous})\\ &= e^{(2(1))} && (\text{evaluating the inner limit})\\ &= e^2\end{aligned}$$

Ex 44: Evaluate:

$$\lim_{x\to 3} \sqrt{x^2+7} = \boxed{4}$$

Answer: We apply the Limit of a Composite Function rule. The outer function is $f(u) = \sqrt{u}$ and the inner function is $g(x) = x^2+7$. Since the square root function is continuous on its domain, we can move the limit inside the function.

$$\begin{aligned}\lim_{x\to 3} \sqrt{x^2+7} &= \sqrt{\lim_{x\to 3}(x^2+7)} && (\text{since } \sqrt{u} \text{ is continuous})\\ &= \sqrt{3^2+7} && (\text{evaluating the inner limit})\\ &= \sqrt{9+7} = \sqrt{16}\\ &= 4\end{aligned}$$

Ex 45: Evaluate:

$$\lim_{x\to\pi} \cos(x+\pi) = \boxed{1}$$

Answer: We apply the Limit of a Composite Function rule. The outer function is $f(u) = \cos(u)$ and the inner function is $g(x) = x+\pi$. Since the cosine function is continuous everywhere, we can move the limit inside the function.

$$\begin{aligned}\lim_{x\to\pi} \cos(x+\pi) &= \cos\left(\lim_{x\to\pi}(x+\pi)\right) && (\text{since } \cos(u) \text{ is continuous})\\ &= \cos(\pi+\pi) && (\text{evaluating the inner limit})\\ &= \cos(2\pi)\\ &= 1\end{aligned}$$

Ex 46: Evaluate:

$$\lim_{x\to 2} \sin\left(\frac{x^2-4}{x-2}\pi\right) = \boxed{0}$$

Answer: We apply the Limit of a Composite Function rule. The outer function is $f(u) = \sin(u)$ and the inner function is $g(x) = \frac{x^2-4}{x-2}$. Since the sine function is continuous everywhere, we can move the limit inside the function.

$$\begin{aligned}\lim_{x\to 2} \sin\left(\frac{x^2-4}{x-2}\pi\right) &= \sin\left(\lim_{x\to 2}\frac{x^2-4}{x-2}\pi\right) && (\sin(u) \text{ is continuous})\\ &= \sin\left(\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}\pi\right)\\ &= \sin\left(\lim_{x\to 2}(x+2)\pi\right)\\ &= \sin(4\pi)\\ &= 0\end{aligned}$$

Ex 47: Evaluate:

$$\lim_{x\to\infty}[\ln(x+1)-\ln(x)] = \boxed{0}$$

Answer: Direct evaluation leads to the indeterminate form $\infty - \infty$. We must first use the laws of logarithms to combine the expression into a single term.

$$\ln(x+1)-\ln(x) = \ln\left(\frac{x+1}{x}\right)$$

Now we can evaluate the limit of this new expression. We apply the Limit of a Composite Function rule since $\ln(u)$ is continuous.

$$\begin{aligned}
\lim_{x\to\infty}\ln\left(\frac{x+1}{x}\right) &= \ln\left(\lim_{x\to\infty}\frac{x+1}{x}\right) \quad \text{(since } \ln(u) \text{ is continuous)} \\
&= \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)\right) \quad \text{(by algebraic simplification)} \\
&= \ln(1+0) \quad \text{(evaluating the inner limit)} \\
&= \ln(1) \\
&= 0
\end{aligned}$$