LIMITS

A DEFINITION

A.1 INVESTIGATING LIMITS NUMERICALLY

Ex 1: Complete the table of values:

x	$\dfrac{x^2-1}{x-1}$
1.1	$\square$
1.01	$\square$
1.001	$\square$
1.0001	$\square$

Hence conjecture:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\square$$

Ex 2: Complete the table of values below (round to 5 decimal places where needed).

h	$\dfrac{(1+h)^3-1}{h}$
0.1	$\square$
0.01	$\square$
0.001	$\square$
-0.01	$\square$

Hence conjecture:

$$\lim_{h\to 0}\frac{(1+h)^3-1}{h}=\square$$

Ex 3: Complete the table of values below, ensuring your calculator is in **radian mode** (round to 5 decimal places).

x	$\dfrac{\sin(x)}{x}$
0.1	$\square$
0.01	$\square$
-0.01	$\square$

Hence conjecture:

$$\lim_{x\to 0}\frac{\sin(x)}{x}=\square$$

A.2 EVALUATING LIMITS BY DIRECT SUBSTITUTION

Ex 4: Evaluate:

$$\lim_{x\to 2}x^2=\square$$

Ex 5: Evaluate:

$$\lim_{x\to 2}(x^2-3x+1)=\square$$

Ex 6: Evaluate:

$$\lim_{x\to 5}7=\square$$

Ex 7: Evaluate:

$$\lim_{x\to 1}\frac{x+3}{x+1}=\square$$

A.3 EVALUATING LIMITS BY ALGEBRAIC SIMPLIFICATION

Ex 8: Evaluate:

$$\lim_{x\to 0}\frac{x+x^2}{2x}=\square$$

Ex 9: Evaluate:

$$\lim_{x\to 0}\frac{3x^2-2x}{x^2+2x}=\square$$

Ex 10: Evaluate:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\square$$

Ex 11: Evaluate:

$$\lim_{x\to -1}\frac{x^2+3x+2}{x+1}=\square$$

A.4 FINDING DERIVATIVES FROM FIRST PRINCIPLES

Ex 12: Evaluate:

$$\lim_{h\to 0}\frac{(2(x+h)+3)-(2x+3)}{h}=\square$$

Ex 13: Evaluate:

$$\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\square$$

Ex 14: Evaluate for $x\neq 0$:

$$\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\square$$

Ex 15: Evaluate for $x>0$:

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\square$$

A.5 RESOLVING INDETERMINATE FORMS BY FACTORING

Ex 16: Evaluate the following limit algebraically:

$$\lim_{x\to 0}\frac{x+x^2}{2x}$$

Ex 17: Evaluate the following limit algebraically:

$$\lim_{x\to 0} \frac{5x^2+3x}{2x^2-x}$$

Ex 18: Evaluate the following limit algebraically:

$$\lim_{x\to 2} \frac{x^2-4}{x-2}$$

Ex 19: Evaluate the following limit algebraically:

$$\lim_{x\to 2} \frac{x^2+3x-10}{x-2}$$

B ALGEBRAIC EVALUATION OF LIMITS

B.1 APPLYING THE LIMIT LAWS

Ex 20: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -1$, evaluate:

$$\lim_{x\to a} [f(x)g(x)] = \boxed{}$$

Ex 21: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -1$, evaluate:

$$\lim_{x\to a} [f(x)+g(x)] = \boxed{}$$

Ex 22: Given that $\lim_{x\to a} f(x) = 3$, evaluate:

$$\lim_{x\to a} [5f(x)] = \boxed{}$$

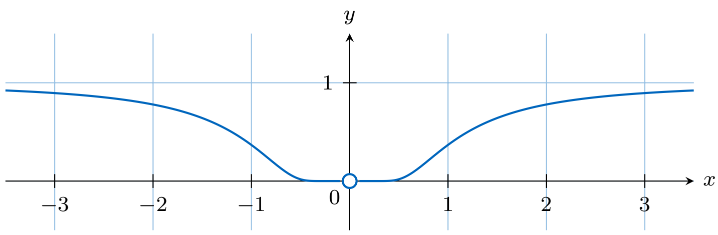
Ex 23: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -1$, evaluate:

$$\lim_{x\to a} \left[\frac{f(x)}{g(x)}\right] = \boxed{}$$

C EXISTENCE OF A LIMIT

C.1 EVALUATING LIMITS GRAPHICALLY

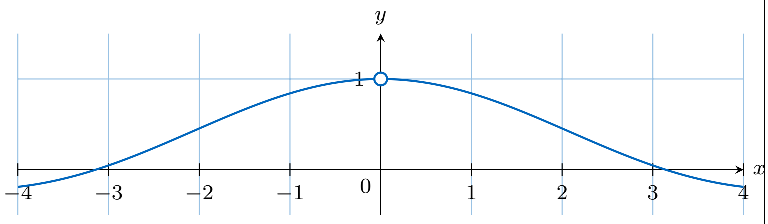
Ex 24: The graph of the function $f(x) = e^{-1/x^2}$ is shown below.



Evaluate graphically:

$$\lim_{x\to 0} e^{-1/x^2} = \boxed{}$$

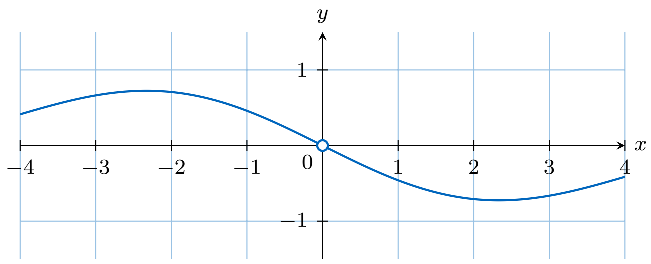
Ex 25: The graph of the function $f(x) = \dfrac{\sin(x)}{x}$ is shown below.



Evaluate graphically:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = \square$$

Ex 26: The graph of the function $f(x) = \dfrac{\cos(x)-1}{x}$ is shown below.



Evaluate graphically:

$$\lim_{x\to 0}\frac{\cos(x)-1}{x} = \square$$

D INFINITE LIMITS AND VERTICAL ASYMPTOTES

D.1 EVALUATING INFINITE LIMITS

Ex 27: Evaluate the following one-sided limit:

$$\lim_{x\to 1^+}\frac{1}{x-1}$$

Ex 28: Evaluate the following one-sided limit:

$$\lim_{x\to 1^-}\frac{1}{x-1}$$

Ex 29: Evaluate the following limit:

$$\lim_{x\to 2}\frac{-5}{(x-2)^2}$$

D.2 FINDING LIMITS AND VERTICAL ASYMPTOTES

Ex 30: Consider the function $f(x) = \dfrac{x+1}{x-2}$.

1. Evaluate the one-sided limits of $f(x)$ as x approaches 2:
   - $\lim_{x\to 2^+} f(x)$
   - $\lim_{x\to 2^-} f(x)$
2. Does $\lim_{x\to 2} f(x)$ exist? Justify your answer.
3. Hence, state the equation of any vertical asymptotes of the graph of $y = f(x)$.

Ex 31: Consider the function $f(x) = \dfrac{x}{(x-1)^2}$.

1. Evaluate the one-sided limits of $f(x)$ as x approaches 1:

   - $\lim_{x\to 1^+} f(x)$
   - $\lim_{x\to 1^-} f(x)$

2. Does $\lim_{x\to 1} f(x)$ exist? Justify your answer.

3. Hence, state the equation of any vertical asymptotes of the graph of $y = f(x)$.

E LIMITS AT INFINITY

E.1 EVALUATING LIMITS AT INFINITY

Ex 32: Evaluate:

$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 1} = \boxed{}$$

Ex 33: Evaluate:

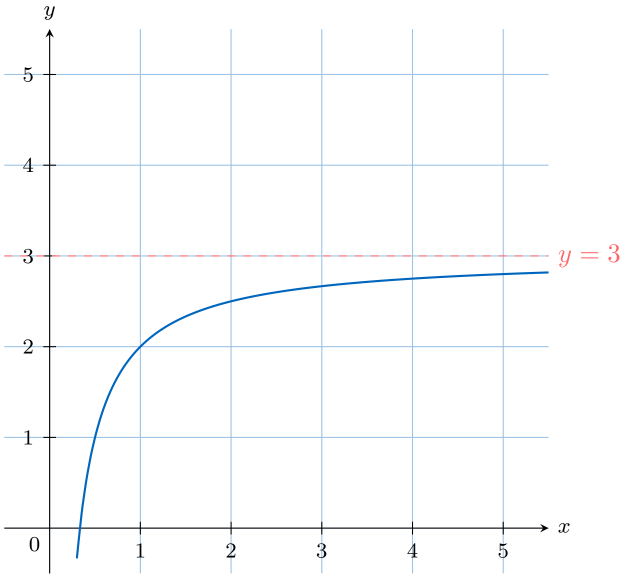
$$\lim_{x\to\infty} \frac{2x + 5}{x^2 - 3x + 1} = \boxed{}$$

Ex 34: Evaluate:

$$\lim_{x\to -\infty} \frac{4 - 3x}{2x + 1} = \boxed{}$$

E.2 DETERMINING END BEHAVIOR GRAPHICALLY

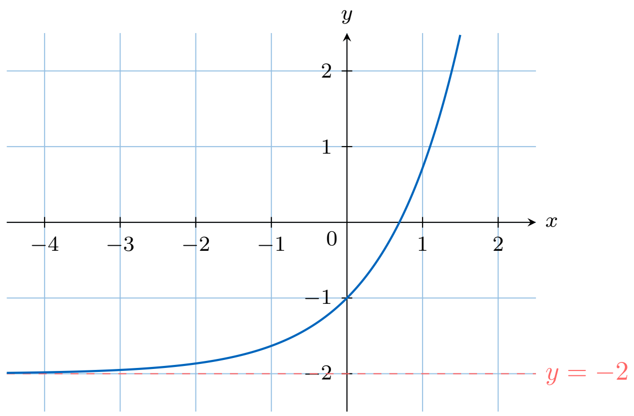
Ex 35: The graph of the function $f(x) = -\dfrac{1}{x} + 3$ is shown below for $x > 0$.



Evaluate graphically:

$$\lim_{x\to\infty}\left(-\frac{1}{x}+3\right)=\boxed{}$$

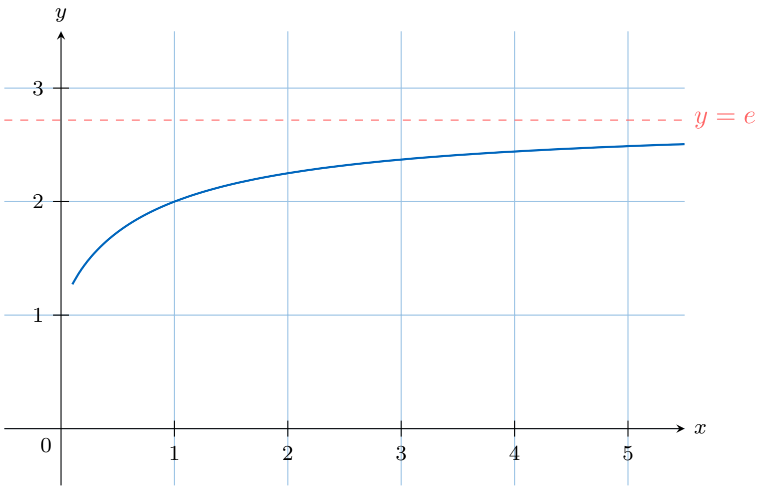
Ex 36: The graph of the function $f(x)=e^x-2$ is shown below.



Evaluate graphically:

$$\lim_{x\to-\infty}(e^x-2)=\boxed{}$$

Ex 37: The graph of the function $f(x)=\left(1+\frac{1}{x}\right)^x$ is shown below for $x>0$.



Evaluate graphically:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=\boxed{}$$

E.3 FINDING LIMITS AT INFINITY WITH RADICAL FUNCTIONS

Ex 38: Consider the function $f(x)=\dfrac{2x}{\sqrt{x^2+1}}$.

1. Find $\lim\limits_{x\to\infty} f(x)$.
2. Find $\lim\limits_{x\to-\infty} f(x)$.
3. Hence, write down the equations of any horizontal asymptotes of the graph of $y=f(x)$.

Ex 39: Consider the function $f(x)=\dfrac{\sqrt{9x^2+4}}{x-1}$.

1. Find $\lim\limits_{x\to\infty} f(x)$.
2. Find $\lim\limits_{x\to-\infty} f(x)$.
3. Hence, write down the equations of any horizontal asymptotes of the graph of $y=f(x)$.

F THE SQUEEZE THEOREM

F.1 APPLYING THE SQUEEZE THEOREM

Ex 40: Evaluate $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$.

Ex 41: Evaluate $\lim_{x\to 0} x \cos\left(\frac{1}{x^2}\right)$.

Ex 42: Evaluate $\lim_{x\to \infty} \frac{\cos(x)}{x}$.

G CONTINUITY

G.1 EVALUATING LIMITS USING CONTINUITY

Ex 43: Evaluate:

$$\lim_{x\to 1} e^{2x} = \boxed{}$$

Ex 44: Evaluate:

$$\lim_{x\to 3} \sqrt{x^2+7} = \boxed{}$$

Ex 45: Evaluate:

$$\lim_{x\to \pi} \cos(x+\pi) = \boxed{}$$

Ex 46: Evaluate:

$$\lim_{x\to 2} \sin\left(\frac{x^2-4}{x-2}\pi\right) = \boxed{}$$

Ex 47: Evaluate:

$$\lim_{x\to \infty} [\ln(x+1)-\ln(x)] = \boxed{}$$