LIMIT THEOREMS OF RANDOM VARIABLES

Many practical situations involve random variables that are defined as the sum or average of other random variables. For instance, estimating a population mean from a sample mean relies heavily on the properties of these sums. Limit theorems provide the mathematical foundation for such estimations.
Limit theorems are fundamental results in probability theory that describe the behavior of sums of random variables as the number of terms increases. We will focus on two key theorems:

- **The Law of Large Numbers (LLN)**: this theorem explains why averages stabilize. It states that as you perform an experiment many times, the average of the results (experimental mean) gets closer and closer to the expected value (theoretical mean).
- **The Central Limit Theorem (CLT)**: this theorem explains the shape of the distribution. It states that if you sum a large number of independent random variables, the distribution of their sum (or mean) tends toward a normal distribution, regardless of the original distribution of the variables (under mild technical conditions).

These theorems are crucial for **statistical inference**, allowing us to draw conclusions about entire populations based on limited samples. They help answer two fundamental questions:

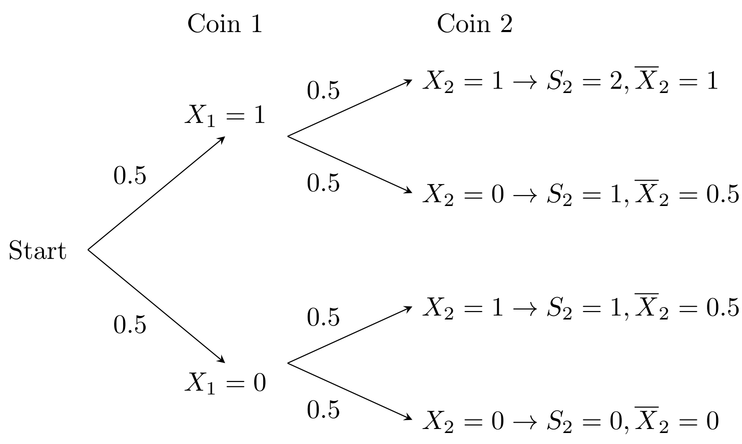
- Does the experimental probability (or observed frequency) approach the theoretical probability as the number of trials increases? **Yes**, the Law of Large Numbers guarantees this.
- Can we quantify the error between our sample estimate and the true value? **Yes**, the Central Limit Theorem allows us to define confidence intervals and error margins using the normal distribution.

A LINEAR COMBINATION OF RANDOM VARIABLES

Discover: Double Coin Flip Consider flipping a fair coin two times. Let the random variables X_1 and X_2 represent the outcome of the first and second coin flips, respectively. We code success (Heads) as 1 and failure (Tails) as 0. We assume the two flips are independent.
Thus, X_1 and X_2 follow a Bernoulli distribution with probability of success $p = \frac{1}{2}$. We are interested in two new random variables:

- $S_2 = X_1 + X_2$: the total number of heads.
- $\overline{X}_2 = \frac{X_1 + X_2}{2}$: the average proportion of heads.

1. **Probability distribution:** We can visualize the outcomes using a probability tree.



The probability distributions are:

s (Sum)	0	1	2
$P(S_2 = s)$	1/4	1/2	1/4

$\bar{x}$ (Mean)	0	0.5	1
$P(\overline{X}_2 = \bar{x})$	1/4	1/2	1/4

2. **Expectation:**

$$E(S_2) = 0\left(\tfrac{1}{4}\right) + 1\left(\tfrac{1}{2}\right) + 2\left(\tfrac{1}{4}\right) = 1,$$

$$E(\overline{X}_2) = 0\left(\tfrac{1}{4}\right) + 0.5\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{4}\right) = 0.5.$$

3. **Connection to individual variables:**
Since $E(X_1) = E(X_2) = p = 0.5$, we have:

$$E(S_2) = 1 = 2 \times 0.5 = 2E(X_1),$$

$$E(\overline{X}_2) = 0.5 = E(X_1).$$

4. **Variance:**

$$V(S_2) = (0-1)^2\left(\tfrac{1}{4}\right) + (1-1)^2\left(\tfrac{1}{2}\right) + (2-1)^2\left(\tfrac{1}{4}\right) = \tfrac{1}{4} + 0 + \tfrac{1}{4} = 0.5,$$

$$V(\overline{X}_2) = (0-0.5)^2\left(\tfrac{1}{4}\right) + (0.5-0.5)^2\left(\tfrac{1}{2}\right) + (1-0.5)^2\left(\tfrac{1}{4}\right) = \tfrac{0.25}{4} + 0 + \tfrac{0.25}{4} = 0.125.$$

5. **Connection to individual variance:**
Since $V(X_1) = p(1-p) = 0.25$, it follows that:

$$V(S_2) = 0.5 = 2 \times 0.25 = 2V(X_1),$$

$$V(\overline{X}_2) = 0.125 = \frac{0.25}{2} = \frac{V(X_1)}{2}.$$

This example suggests general rules for sums and means of independent random variables.

Definition Linear Combination of Random Variables

A **linear combination** of random variables $X_1, X_2, \ldots, X_n$ is a new random variable Y defined as

$$Y = a_1X_1 + a_2X_2 + \ldots + a_nX_n,$$

where $a_1, a_2, \ldots, a_n$ are constant coefficients.

Definition Sum and Mean of Random Variables

Given n random variables $X_1, X_2, \ldots, X_n$:

- The **sum** is denoted $S_n = X_1 + X_2 + \ldots + X_n$.
- The **sample mean** is denoted $\overline{X}_n = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$.

Proposition Properties of Expectation and Variance

For random variables $X_1, \ldots, X_n$ and constants $a_1, \ldots, a_n$:

- **Expectation is linear:**

$$E\left(\sum_{i=1}^{n} a_iX_i\right) = \sum_{i=1}^{n} a_iE(X_i).$$

- **Variance adds for independent variables:** if $X_1, \ldots, X_n$ are independent,

$$V\left(\sum_{i=1}^{n} a_iX_i\right) = \sum_{i=1}^{n} a_i^2V(X_i).$$

Proposition Expectation of Sums and Means

If $X_1, X_2, \ldots, X_n$ are identically distributed with mean μ, then:

$$E(S_n) = n\mu \qquad \text{and} \qquad E(\overline{X}_n) = \mu.$$

Proof

- **For the sum S_n:**
Using linearity of expectation,

$$\begin{aligned} E(S_n) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= \mu + \mu + \cdots + \mu \quad (n \text{ times}) \\ &= n\mu. \end{aligned}$$

- **For the mean $\overline{X}_n$:**

Using $E(aX) = aE(X)$,

$$\begin{aligned} E(\overline{X}_n) &= E\left(\frac{S_n}{n}\right) \\ &= \frac{1}{n}E(S_n) \\ &= \frac{1}{n}(n\mu) \\ &= \mu. \end{aligned}$$

Ex: Let $X_1, \ldots, X_9$ be random variables with mean $\mu = 5$. Find the expected value of their sum S_9.

Answer:

$$E(S_9) = 9 \times \mu = 9 \times 5 = 45.$$

Proposition **Variance of Sums and Means**

If $X_1, X_2, \ldots, X_n$ are independent and identically distributed with variance σ^2 (and standard deviation σ), then:

- **For the sum:**
$$V(S_n) = n\sigma^2 \quad \text{and} \quad \sigma(S_n) = \sigma\sqrt{n};$$
- **For the mean:**
$$V(\overline{X}_n) = \frac{\sigma^2}{n} \quad \text{and} \quad \sigma(\overline{X}_n) = \frac{\sigma}{\sqrt{n}}.$$

Proof

- **For the sum S_n:**
Since the variables $X_1, \ldots, X_n$ are **independent**, the variance of the sum is the sum of the variances:
$$\begin{aligned} V(S_n) &= V(X_1 + X_2 + \cdots + X_n) \\ &= V(X_1) + V(X_2) + \cdots + V(X_n) \\ &= \sigma^2 + \sigma^2 + \cdots + \sigma^2 \quad (n \text{ times}) \\ &= n\sigma^2. \end{aligned}$$
The standard deviation is the square root of the variance:
$$\sigma(S_n) = \sqrt{V(S_n)} = \sqrt{n\sigma^2} = \sigma\sqrt{n}.$$
- **For the mean $\overline{X}_n$:**
Using the property $V(aX) = a^2V(X)$ with $a = \frac{1}{n}$,
$$\begin{aligned} V(\overline{X}_n) &= V\left(\frac{S_n}{n}\right) \\ &= \left(\frac{1}{n}\right)^2 V(S_n) \\ &= \frac{1}{n^2}(n\sigma^2) \\ &= \frac{\sigma^2}{n}. \end{aligned}$$
Thus,
$$\sigma(\overline{X}_n) = \sqrt{V(\overline{X}_n)} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}.$$

Ex: Sample Proportion Items coming off an assembly line are defective with probability p. Let X_i be an indicator variable where $X_i = 1$ if the i-th item is defective and 0 otherwise. Then X_i follows a Bernoulli distribution. Find the mean and standard deviation of the sample proportion $\overline{X}_n$.

Answer: For a Bernoulli variable, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$. Therefore, for the sample mean $\overline{X}_n$:

$$E(\overline{X}_n) = p,$$

$$\sigma(\overline{X}_n) = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}}.$$

B LAW OF LARGE NUMBERS

The **Law of Large Numbers (LLN)** describes the result of performing the same experiment a large number of times. It states that the average of the results obtained from many trials should be close to the expected value and tends to become closer as more trials are performed.
In practice, this justifies our intuition that *experimental probability estimates theoretical probability*. While short-term results are unpredictable (for example, a casino might lose money on a single spin), long-term averages are highly predictable (in the long run, the casino wins).

Definition Limit Mean

If the limit exists, we denote the limit of the sample mean as n approaches infinity by $\overline{X}_\infty$:

$$\overline{X}_\infty = \lim_{n\to\infty} \overline{X}_n.$$

Theorem Law of Large Numbers

For independent, identically distributed random variables $X_1, X_2, \ldots$ with mean μ and standard deviation σ, the sample mean converges to the population mean as the number of observations goes to infinity. In more advanced notation,

$$P\big(\overline{X}_\infty = \mu\big) = 1.$$

This means that the sample mean converges to the population mean with (probabilistic) certainty.

Proof

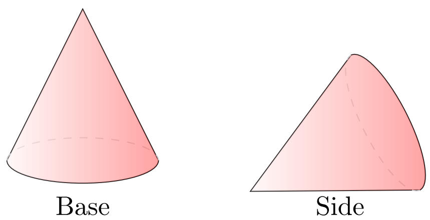
Recall that the variance of the sample mean is

$$V(\overline{X}_n) = \frac{\sigma^2}{n}.$$

As $n \to \infty$, the term $\frac{\sigma^2}{n}$ approaches 0.
A random variable with variance 0 has no dispersion; it is almost surely constant. Therefore, as n grows, the distribution of $\overline{X}_n$ becomes more and more concentrated around its expected value μ, and in the limit it collapses to this single value.

The limiting distribution is deterministic: in the limit the sample mean takes the value μ with probability 1.

Ex: Experimental vs Theoretical Probability Consider tossing a cone. The probability p of it landing on its base is not known a priori.



We repeat the experiment n times. Let X_i be 1 if the cone lands on its base on trial i, and 0 otherwise. Then

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n}$$

is the proportion of times it landed on its base. By the Law of Large Numbers, as $n \to \infty$, this proportion converges to the true probability $E(X_1) = p$. In other words, for large n,

$$\text{Experimental probability} \approx \text{Theoretical probability}.$$

In practice, we can never repeat an experiment infinitely many times. The sample mean is always an approximation of the true mean. The next theorem, the Central Limit Theorem, allows us to quantify the error of this approximation and to describe how the sample mean fluctuates around μ for large but finite n.

C CENTRAL LIMIT THEOREM

The **Central Limit Theorem (CLT)** is one of the most remarkable results in mathematics. Informally, it states that if you take a sufficiently large sample size from a population with *any* distribution (not necessarily normal), the distribution of the sample mean (or sum) is *approximately* normal.
This normal approximation is what justifies using the normal distribution to build confidence intervals and perform hypothesis tests, even when the underlying population is not itself normal.

Proposition **Sum and Mean of Normal Variables**

If $X_1, \ldots, X_n$ are independent and **normally distributed** with mean μ and standard deviation σ, then:

- The sum S_n is exactly normally distributed with mean $n\mu$ and variance $n\sigma^2$:
$$S_n \sim N(n\mu, n\sigma^2).$$
- The sample mean $\overline{X}_n$ is exactly normally distributed with mean μ and variance $\dfrac{\sigma^2}{n}$:
$$\overline{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Ex: Let $X_1, \ldots, X_5$ be normally distributed random variables with mean 90 and standard deviation 20.

1. Is $\overline{X}_5$ normally distributed?
2. Find $P(80 \leqslant \overline{X}_5 \leqslant 100)$.

Answer:

1. Yes, because any linear combination of normal variables is also normal. In particular, the mean $\overline{X}_5$ is normally distributed.
2. The mean of $\overline{X}_5$ is $E(\overline{X}_5) = 90$. Its standard deviation is
$$\sigma(\overline{X}_5) = \frac{20}{\sqrt{5}} \approx 8.94.$$
Using a normal distribution calculator for $N(90, 8.94^2)$, we obtain
$$P(80 \leqslant \overline{X}_5 \leqslant 100) \approx 0.737.$$

Proposition **Standardized Mean**

For n independent random variables with common mean μ and standard deviation σ, the sample mean $\overline{X}_n$ has
$$E(\overline{X}_n) = \mu, \qquad \sigma(\overline{X}_n) = \frac{\sigma}{\sqrt{n}}.$$
The standardized variable
$$Z_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$$
has mean 0 and standard deviation 1.

Proof

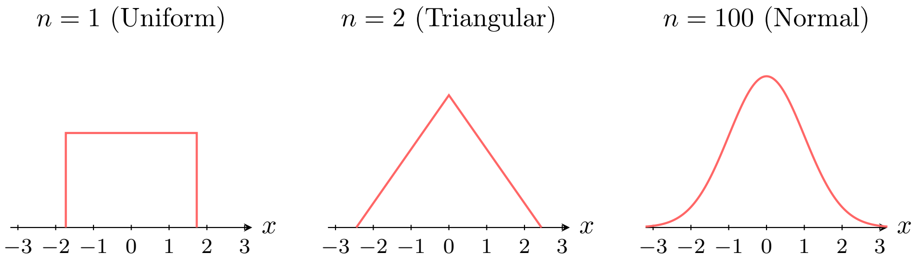
First, recall the properties of the sample mean $\overline{X}_n$:
$$E(\overline{X}_n) = \mu \quad \text{and} \quad \sigma(\overline{X}_n) = \frac{\sigma}{\sqrt{n}}.$$

- **Mean of Z_n:**
Using linearity of expectation, $E(aX + b) = aE(X) + b$:
$$\begin{aligned} E(Z_n) &= E\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}\right) \\ &= \frac{1}{\sigma/\sqrt{n}}\left(E(\overline{X}_n) - \mu\right) \\ &= \frac{1}{\sigma/\sqrt{n}}(\mu - \mu) \\ &= 0. \end{aligned}$$
- **Standard deviation of Z_n:**

Using the property $\sigma(aX+b) = |a|\sigma(X)$:

$$\begin{aligned}\sigma(Z_n) &= \sigma\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}\right) \\ &= \frac{1}{\sigma/\sqrt{n}} \cdot \sigma(\overline{X}_n) \\ &= \frac{1}{\sigma/\sqrt{n}} \cdot \frac{\sigma}{\sqrt{n}} \\ &= 1.\end{aligned}$$

The power of the Central Limit Theorem is especially visible when we start from a distribution that is not normal. For example, suppose each X_i follows a uniform distribution. The figures below show the density of the standardized sum (or mean) Z_n for increasing n.



When $n = 1$, we see a flat rectangular density (uniform distribution). For $n = 2$, the density becomes triangular. For large n (for example $n = 100$), the curve is visually indistinguishable from the bell curve of a standard normal distribution.

Theorem Central Limit Theorem

Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with mean μ and standard deviation σ. If the sample size n is sufficiently large (typically $n \geq 30$), then:

- The sample mean $\overline{X}_n$ is approximately normally distributed:
$$\overline{X}_n \approx N\left(\mu, \frac{\sigma^2}{n}\right).$$
- The sum S_n is approximately normally distributed:
$$S_n \approx N(n\mu, n\sigma^2).$$
- The standardized variable
$$Z_n = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}}$$
is approximately distributed as a standard normal $N(0,1)$ for large n. More rigorously, the distribution of Z_n converges to $N(0,1)$ as $n \to \infty$.

Ex: A population of random variables has mean 174 and standard deviation 6. A sample of size 64 is taken.

1. Is the sample mean $\overline{X}_{64}$ approximately normally distributed? Explain.
2. Find the mean and standard deviation of $\overline{X}_{64}$.
3. Find the probability that the sample mean is between 172 and 176.

Answer:

1. Yes. Since the sample size is $n = 64$ (which is large, $n \geq 30$), the Central Limit Theorem applies, and the sampling distribution of $\overline{X}_{64}$ is approximately normal.
2. Mean and standard deviation:
$$E(\overline{X}_{64}) = \mu = 174,$$
$$\sigma(\overline{X}_{64}) = \frac{\sigma}{\sqrt{n}} = \frac{6}{\sqrt{64}} = \frac{6}{8} = 0.75.$$
3. Using a normal distribution calculator with $\mu = 174$ and $\sigma = 0.75$:
$$P(172 \leqslant \overline{X}_{64} \leqslant 176) \approx 0.9923.$$