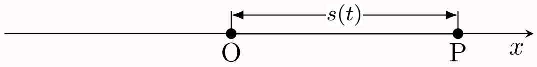
KINEMATICS

Kinematics is the study of the motion of objects without considering the forces that cause the motion. In this chapter, we use calculus to formally describe motion along a straight line. We will develop functions for displacement, velocity, and acceleration, and explore the relationships between them.

A DISPLACEMENT

Definition Displacement Function

For an object P in motion along a straight line, the **displacement function** $s(t)$ gives the position of the object (its signed distance from a fixed origin) at any time $t \geq 0$.



Ex: A particle's displacement from an origin O is given by $s(t) = t^2 - 4t + 3$ cm for $0 \leq t \leq 5$ seconds.

1. Find the initial displacement.
2. Find when the particle is at the origin.

Answer:

1. **Initial Displacement** ($t = 0$):
   $s(0) = 0^2 - 4(0) + 3 = 3$ cm. The particle starts 3 cm to the right of the origin (in the positive direction).
2. **At the Origin** ($s(t) = 0$):
   $t^2 - 4t + 3 = 0 \implies (t-1)(t-3) = 0$.
   The particle is at the origin at $t = 1$ s and $t = 3$ s.

B VELOCITY

Definition Average Velocity

The **average velocity** of an object moving in the time interval from $t = t_1$ to $t = t_2$ is given by

$$\begin{aligned} \text{average velocity} &= \frac{\text{change in displacement}}{\text{change in time}} \\ &= \frac{s(t_2) - s(t_1)}{t_2 - t_1}. \end{aligned}$$

Definition Velocity Function

The **instantaneous velocity** $v(t)$ of an object is the rate of change of its displacement. It is found by differentiating the displacement function $s(t)$:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Note

- If $v(t) > 0$, the object is moving to the right (in the positive direction).
- If $v(t) < 0$, the object is moving to the left (in the negative direction).
- If $v(t) = 0$, the object is instantaneously at rest.

Proposition Displacement from Velocity

The **change in displacement** over the interval $[t_1, t_2]$ is the definite integral of the velocity function:

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\, dt.$$

C SPEED

Definition Speed

The **speed** of an object is the magnitude of its velocity: $S(t) = |v(t)|$.

Proposition Distance

The **total distance travelled** over $[t_1, t_2]$ is the definite integral of the speed:

$$\text{Distance} = \int_{t_1}^{t_2} |v(t)|\, dt.$$

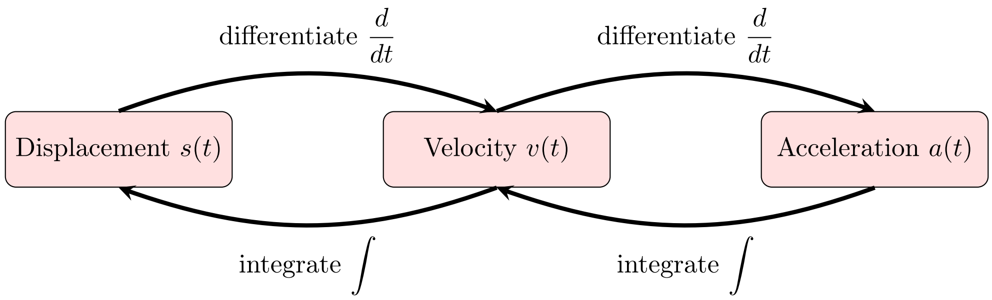
D ACCELERATION

Definition Acceleration Function

The **acceleration** $a(t)$ of an object is the rate of change of its velocity. It is the second derivative of the displacement function:

$$a(t) = v'(t) = s''(t)$$

The kinematic functions are related through differentiation and integration.



Proposition Sign Test for Speed

- Speed is **increasing** when $v(t)$ and $a(t)$ have the same sign.
- Speed is **decreasing** when $v(t)$ and $a(t)$ have opposite signs.

Ex: The displacement of a particle is given by $s(t) = t^3 - 6t^2 + 9t$ cm, for $t \geq 0$.

1. Find expressions for the velocity and acceleration.
2. Find when the particle's speed is increasing.

Answer:

1. **Velocity and Acceleration**:

$$v(t) = s'(t) = 3t^2 - 12t + 9 \text{ cm/s},$$
$$a(t) = v'(t) = 6t - 12 \text{ cm/s}^2.$$

2. **Increasing Speed**:
We need $v(t)$ and $a(t)$ to have the same sign. First, find the roots:

$$v(t) = 3(t^2 - 4t + 3) = 3(t-1)(t-3) = 0 \implies t = 1,\ 3,$$
$$a(t) = 6(t-2) = 0 \implies t = 2.$$

We build a sign table for both functions:

t	0		1		2		3		$+\infty$
Sign of $v(t)$		$+$	0	$-$		$-$	0	$+$	
Sign of $a(t)$		$-$		$-$	0	$+$		$+$	
Variations of Speed		↘		↗		↘		↗	

Speed is increasing when $v(t)$ and $a(t)$ have the same sign, that is for

$$1 < t < 2 \quad \text{and} \quad t > 3.$$