A DISPLACEMENT

A.1 ANALYZING DISPLACEMENT FUNCTIONS

Ex 1: An object travels with displacement function $s(t) = 6 - 2t$ m, where $t \geq 0$ is the time in seconds.

1. Find the initial displacement of the object.

$$\boxed{6} \text{ m}$$

2. Find the displacement of the object at time $t = 2$ s.

$$\boxed{2} \text{ m}$$

3. At what time does the object reach the origin?

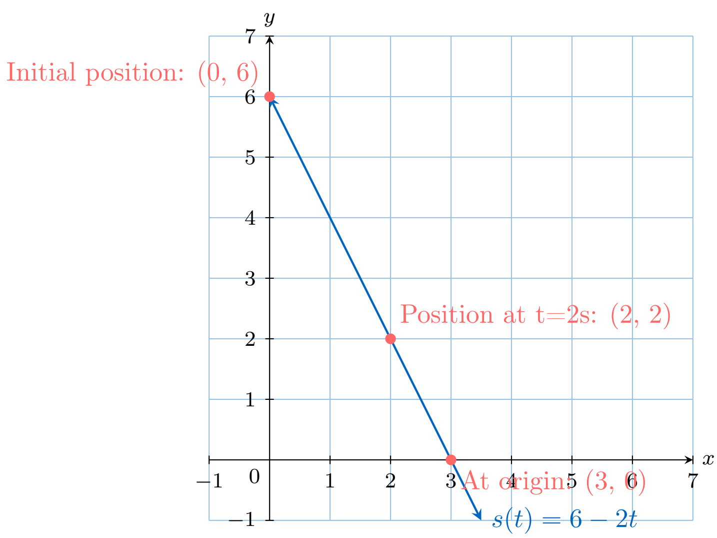
$$\boxed{3} \text{ s}$$

Answer:

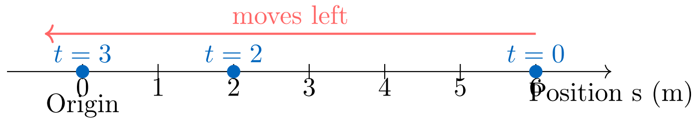
1. **Initial Displacement**
The initial displacement occurs at $t = 0$.
$$s(0) = 6 - 2(0) = 6 \text{ m}$$
The object starts 6 m to the right of the origin.

2. **Displacement at $t = 2$ s**
We substitute $t = 2$ into the function:
$$s(2) = 6 - 2(2) = 6 - 4 = 2 \text{ m}$$
At 2 seconds, the object is 2 m to the right of the origin.

3. **Time to Reach the Origin**
The object reaches the origin when $s(t) = 0$.
$$\begin{aligned} 6 - 2t &= 0 \\ 6 &= 2t \\ t &= 3 \text{ s} \end{aligned}$$
The object reaches the origin after 3 seconds.



Motion Diagram:
The object always moves to the left.



Ex 2: An object travels with displacement function $s(t) = -t^2 + 3t + 4$ m, where $t \geq 0$ is the time in seconds.

1. Find the initial displacement of the object.

$$\boxed{4} \text{ m}$$

2. Find the displacement of the object at time $t = 3$ s.

$$\boxed{4} \text{ m}$$

3. At what time does the object reach the origin?

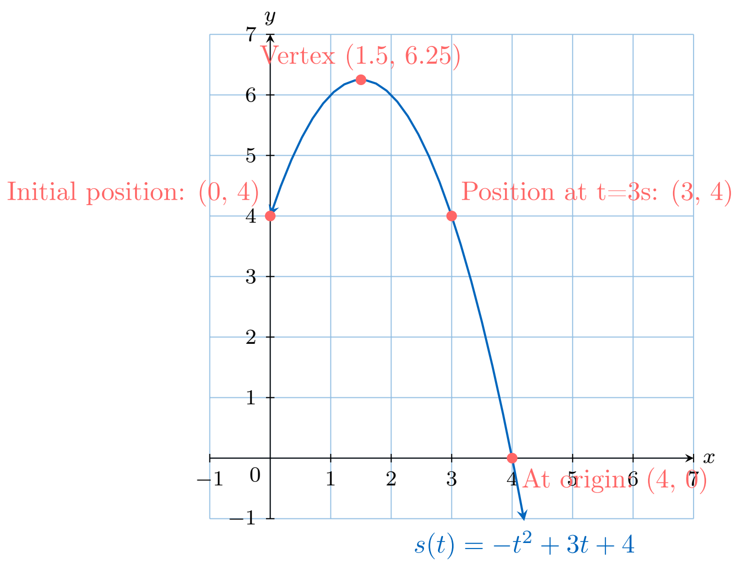
$$\boxed{4} \text{ s}$$

Answer:

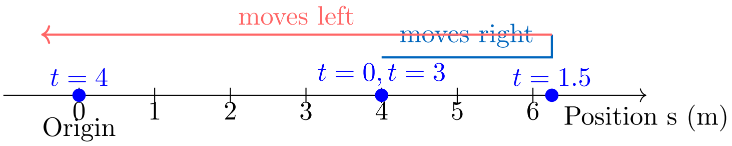
1. **Initial Displacement**
The initial displacement occurs at $t = 0$.
$$s(0) = -(0)^2 + 3(0) + 4 = 4 \text{ m}$$
The object starts 4 m to the right of the origin.

2. **Displacement at $t = 3$ s**
We substitute $t = 3$ into the function:
$$s(3) = -(3)^2 + 3(3) + 4 = -9 + 9 + 4 = 4 \text{ m}$$
At 3 seconds, the object is 4 m to the right of the origin.

3. **Time to Reach the Origin**
The object reaches the origin when $s(t) = 0$.
$$\begin{aligned} -t^2 + 3t + 4 &= 0 \\ t^2 - 3t - 4 &= 0 \\ (t - 4)(t + 1) &= 0 \end{aligned}$$
Since $t \geq 0$, the only valid solution is $t = 4$ s. The object reaches the origin after 4 seconds.



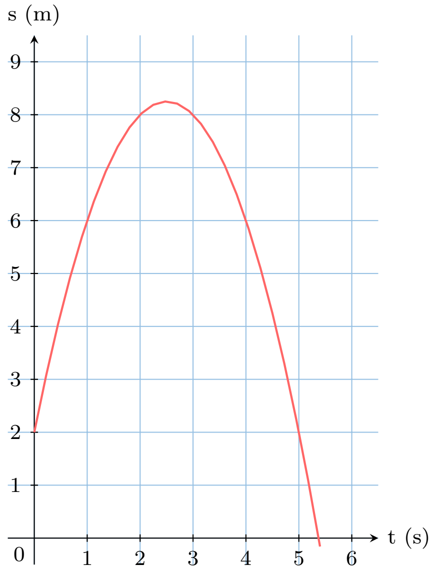
Motion Diagram:



B VELOCITY

B.1 FINDING THE AVERAGE VELOCITY ON A DISPLACEMENT–TIME GRAPH

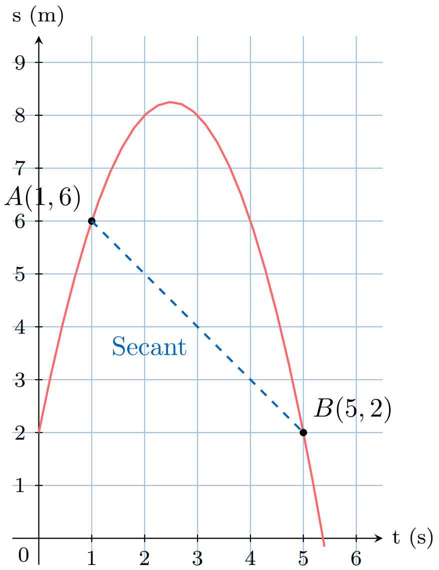
Ex 3: The displacement–time graph for a particle moving in a straight line is shown below.



Find the average velocity of the particle from $t = 1$ s to $t = 5$ s.

$$\text{Average velocity} = \boxed{-1}\ \text{m/s}$$

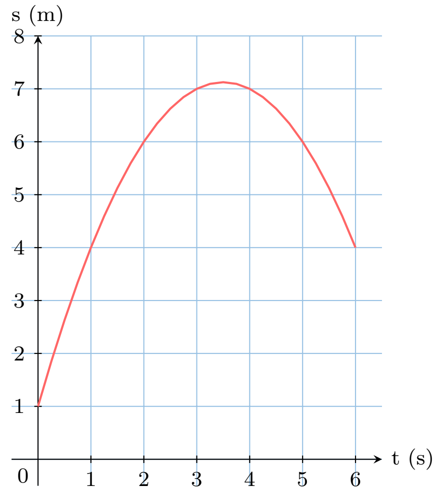
Answer:



The average velocity is the slope of the secant line through $A(1,6)$ and $B(5,2)$:

$$\begin{aligned}\text{Average velocity} &= \frac{s(5) - s(1)}{5 - 1}\\ &= \frac{2 - 6}{4}\\ &= \frac{-4}{4}\\ &= -1 \text{ m/s.}\end{aligned}$$

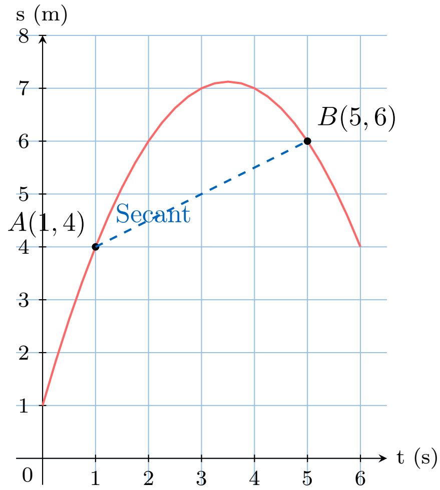
Ex 4: The displacement–time graph for a particle moving in a straight line is shown below.



Find the average velocity of the particle from $t = 1$ s to $t = 5$ s.

$$\text{Average velocity} = \boxed{0.5}\ \text{m/s}$$

Answer:



The average velocity is the slope of the secant line through $A(1,4)$ and $B(5,6)$:

$$\begin{aligned}\text{Average velocity} &= \frac{s(5) - s(1)}{5 - 1}\\ &= \frac{6 - 4}{4}\\ &= \frac{2}{4}\\ &= 0.5 \text{ m/s.}\end{aligned}$$

B.2 FINDING AVERAGE AND INSTANTANEOUS VELOCITY

Ex 5: A particle moves in a straight line with displacement from an origin O given by $s(t) = t^2 - 6t + 5$ metres at time t seconds. Find:

1. the average velocity for the time interval from $t = 1$ to $t = 4$ seconds.

$$\boxed{-1} \text{ m/s}$$

2. the instantaneous velocity at $t = 3$ seconds.

$$\boxed{0} \text{ m/s}$$

Answer:

1.

$$\begin{aligned}\text{average velocity} &= \frac{s(4) - s(1)}{4 - 1} \\ &= \frac{(4^2 - 6\cdot(4) + 5) - (1^2 - 6\cdot(1) + 5)}{3} \\ &= \frac{(16 - 24 + 5) - (1 - 6 + 5)}{3} \\ &= \frac{-3 - 0}{3} \\ &= -1 \text{ m/s}\end{aligned}$$

2.

$$\begin{aligned}v(t) &= \frac{d}{dt}(s(t)) \\ &= \frac{d}{dt}(t^2 - 6t + 5) \\ &= \frac{d}{dt}(t^2) - 6\frac{d}{dt}(t) + \frac{d}{dt}(5) \\ &= 2t - 6 \\ v(3) &= 2(3) - 6 \\ &= 0 \text{ m/s}\end{aligned}$$

The particle is instantaneously at rest.

Ex 6: A particle moves in a straight line with displacement from an origin O given by $s(t) = t^3 - 3t^2 + 2t$ metres at time t seconds. Find:

1. the average velocity for the time interval from $t = 0$ to $t = 3$ seconds.

$$\boxed{2} \text{ m/s}$$

2. the instantaneous velocity at $t = 2$ seconds.

$$\boxed{2} \text{ m/s}$$

Answer:

1.

$$\begin{aligned}\text{average velocity} &= \frac{s(3) - s(0)}{3 - 0} \\ &= \frac{(3^3 - 3\cdot(3)^2 + 2\cdot(3)) - (0^3 - 3\cdot(0)^2 + 2\cdot(0))}{3} \\ &= \frac{(27 - 27 + 6) - (0)}{3} \\ &= \frac{6}{3} \\ &= 2 \text{ m/s}\end{aligned}$$

2.

$$\begin{aligned}v(t) &= \frac{d}{dt}(s(t)) \\ &= \frac{d}{dt}(t^3 - 3t^2 + 2t) \\ &= 3t^2 - 6t + 2 \\ v(2) &= 3(2)^2 - 6(2) + 2 \\ &= 12 - 12 + 2 \\ &= 2 \text{ m/s}\end{aligned}$$

The particle is moving to the right at 2 m/s.

Ex 7: A particle moves in a straight line with displacement from an origin O given by $s(t) = 10 - 8e^{-0.5t}$ metres at time t seconds, for $t \geq 0$.

1. Find the average velocity for the time interval from $t = 0$ to $t = 2$ seconds.

$$\boxed{4 - 4/e} \text{ m/s}$$

2. Find the instantaneous velocity at $t = 2$ seconds.

$$\boxed{4/e} \text{ m/s}$$

Answer:

1.

$$\begin{aligned}\text{average velocity} &= \frac{s(2) - s(0)}{2 - 0} \\ &= \frac{(10 - 8e^{-0.5\cdot 2}) - (10 - 8e^{-0.5\cdot 0})}{2} \\ &= \frac{(10 - 8e^{-1}) - (10 - 8e^{0})}{2} \\ &= \frac{10 - 8/e - 10 + 8}{2} \\ &= \frac{8 - 8/e}{2} \\ &= 4 - \frac{4}{e} \text{ m/s}\end{aligned}$$

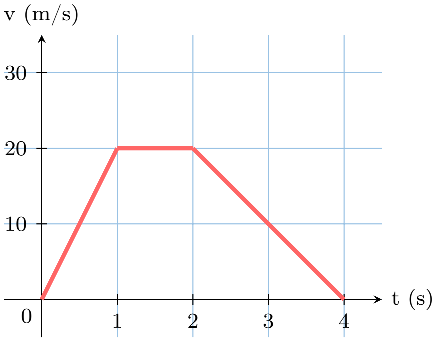
2.

$$\begin{aligned}v(t) &= \frac{d}{dt}(s(t)) \\ &= \frac{d}{dt}(10 - 8e^{-0.5t}) \\ &= 0 - 8\cdot(-0.5)e^{-0.5t} \\ &= 4e^{-0.5t} \\ v(2) &= 4e^{-0.5\cdot 2} \\ &= 4e^{-1} \\ &= \frac{4}{e} \text{ m/s}\end{aligned}$$

The particle is moving to the right at $\frac{4}{e}$ m/s.

B.3 FINDING DISPLACEMENT FROM A VELOCITY–TIME GRAPH

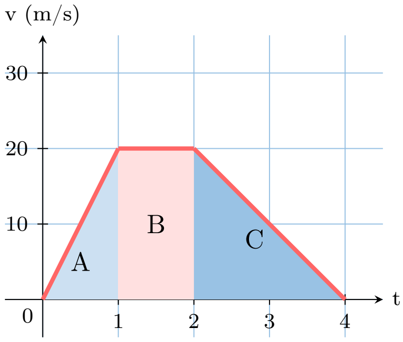
Ex 8: The velocity–time graph for a runner is shown below.



Given the initial displacement is 10 m, find the displacement after 4 seconds.

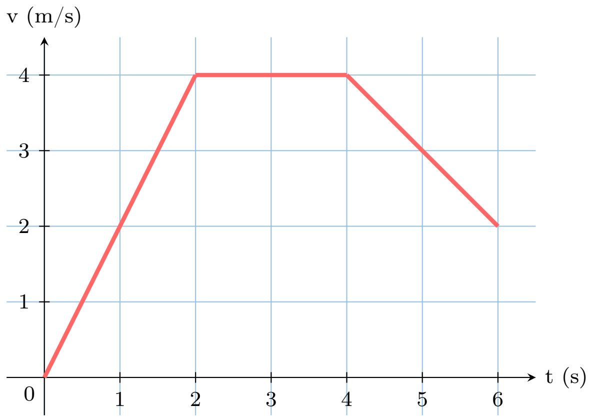
$$s(4) = \boxed{60} \text{ m}$$

Answer:



$$s(4) - s(0) = \int_0^4 v(t)\,dt$$
$$s(4) - s(0) = \int_0^1 v(t)\,dt + \int_1^2 v(t)\,dt + \int_2^4 v(t)\,dt$$
$$s(4) - s(0) = \underbrace{\tfrac{1}{2}\cdot 1\cdot 20}_{\text{triangle A}} + \underbrace{1\cdot 20}_{\text{rectangle B}} + \underbrace{\tfrac{1}{2}\cdot 2\cdot 20}_{\text{triangle C}}$$
$$s(4) - s(0) = 10 + 20 + 20$$
$$s(4) - s(0) = 50$$
$$s(4) = 50 + s(0)$$
$$s(4) = 60 \text{ m}$$

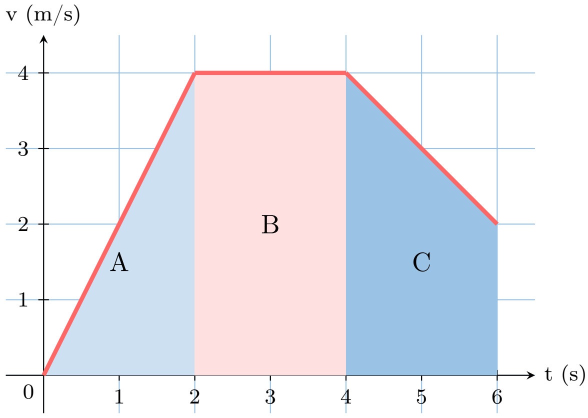
Ex 9: The velocity–time graph for a cyclist is shown below.



Given the initial displacement is 15 m, find the displacement after 6 seconds.

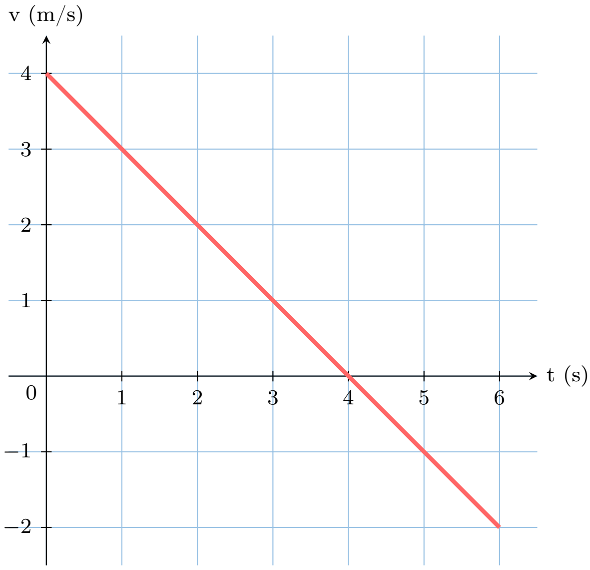
$$s(6) = \boxed{33} \text{ m}$$

Answer:



$$s(6) - s(0) = \int_0^6 v(t)\,dt$$
$$s(6) - s(0) = \int_0^2 v(t)\,dt + \int_2^4 v(t)\,dt + \int_4^6 v(t)\,dt$$
$$s(6) - s(0) = \underbrace{\tfrac{1}{2}\cdot 2\cdot 4}_{\text{triangle A}} + \underbrace{2\cdot 4}_{\text{rectangle B}} + \underbrace{\tfrac{1}{2}(4+2)\cdot 2}_{\text{trapezium C}}$$
$$s(6) - s(0) = 4 + 8 + 6$$
$$s(6) - s(0) = 18$$
$$s(6) = 18 + s(0)$$
$$s(6) = 33 \text{ m}$$

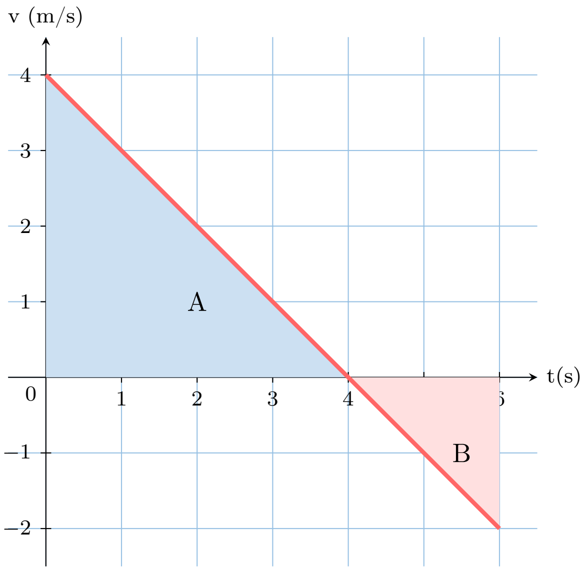
Ex 10: The velocity–time graph for a particle is shown below.



Given the initial displacement is 5 m, find the displacement after 6 seconds.

$$s(6) = \boxed{11} \text{ m}$$

Answer:



$$\begin{aligned} s(6) - s(0) &= \int_0^6 v(t)\,dt \\ &= \int_0^4 v(t)\,dt + \int_4^6 v(t)\,dt \\ &= \underbrace{\tfrac{1}{2}\cdot 4\cdot 4}_{\text{triangle A}} - \underbrace{\tfrac{1}{2}2\cdot 2}_{\text{triangle B}} \\ &= 8-2 \\ &= 6 \\ s(6) &= 6 + s(0) \\ s(6) &= 6+5 = 11 \text{ m.} \end{aligned}$$

Note: the area under the time axis (negative velocity) must be counted negatively when calculating displacement.

B.4 FINDING DISPLACEMENT BY INTEGRATION

Ex 11: A particle moves along a straight line with velocity

$$v(t) = 4 - t \quad \text{m/s}, \quad 0 \le t \le 5.$$

Given the initial displacement is $s(0) = 2$ m, find the displacement after 5 seconds.

$$s(5) = \boxed{9.5} \text{ m}$$

Answer:

$$\begin{aligned} s(5) - s(0) &= \int_0^5 v(t)\,dt \\ s(5) - s(0) &= \int_0^5 (4-t)\,dt \\ s(5) - s(0) &= \left[4t - \tfrac{1}{2}t^2\right]_0^5 \\ s(5) - s(0) &= (20 - 12.5) - (0) \\ &= 7.5 \text{ m} \end{aligned}$$

$$s(5) = s(0) + 7.5 = 2 + 7.5 = 9.5 \text{ m}$$

So, after 5 seconds, the particle is at 9.5 m from the origin.

Ex 12: A particle moves along a straight line with velocity

$$v(t) = 3t^2 - 4t + 1 \quad \text{m/s}, \quad t \ge 0.$$

Given the initial displacement is $s(0) = -5$ m, find the displacement after 3 seconds.

$$s(3) = \boxed{7} \text{ m}$$

Answer: The change in displacement is the integral of the velocity function.

$$\begin{aligned} s(3) - s(0) &= \int_0^3 v(t)\,dt \\ &= \int_0^3 (3t^2 - 4t + 1)\,dt \\ &= \left[t^3 - 2t^2 + t\right]_0^3 \\ &= ((3)^3 - 2(3)^2 + 3) - (0) \\ &= (27 - 18 + 3) \\ &= 12 \text{ m} \end{aligned}$$

The final displacement is the initial displacement plus the change in displacement.

$$s(3) = s(0) + 12 = -5 + 12 = 7 \text{ m}$$

So, after 3 seconds, the particle is at 7 m from the origin.

Ex 13: A particle moves along a straight line with velocity

$$v(t) = 2\cos(t) \quad \text{m/s}, \quad t \ge 0.$$

Given the initial displacement is $s(0) = 4$ m, find the displacement at $t = \frac{\pi}{2}$ seconds.

$$s(\pi/2) = \boxed{6} \text{ m}$$

Answer: The change in displacement is the integral of the velocity function.

$$\begin{aligned} s(\pi/2) - s(0) &= \int_0^{\pi/2} v(t)\,dt \\ &= \int_0^{\pi/2} 2\cos(t)\,dt \\ &= [2\sin(t)]_0^{\pi/2} \\ &= (2\sin(\pi/2)) - (2\sin(0)) \\ &= (2\cdot 1) - (2\cdot 0) \\ &= 2 \text{ m} \end{aligned}$$

The final displacement is the initial displacement plus the change in displacement.

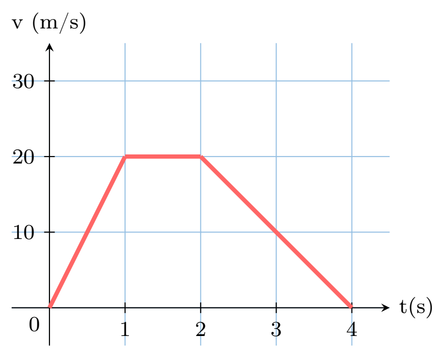
$$s(\pi/2) = s(0) + 2 = 4 + 2 = 6 \text{ m}$$

So, at $t = \frac{\pi}{2}$ seconds, the particle is at 6 m from the origin.

C SPEED

C.1 FINDING THE TOTAL DISTANCE FROM A VELOCITY-TIME GRAPH

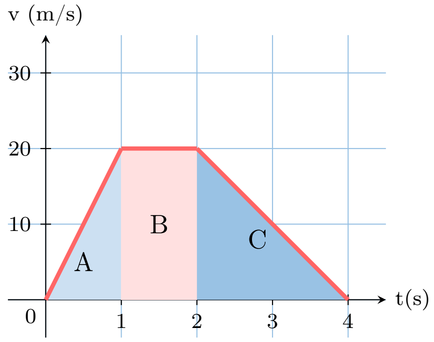
Ex 14: The velocity-time graph for a runner is shown below.



Find the total distance travelled by the runner.

$$\text{Total distance} = \boxed{50}\ \text{m}$$

Answer: The total distance travelled is the area under the velocity-time graph. We can split the area into three shapes: a triangle (A), a rectangle (B), and another triangle (C).



- **Area A (Triangle)**: $0 \leq t \leq 1$

$$A_A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 1 \times 20 = 10 \text{ m}$$

- **Area B (Rectangle)**: $1 \leq t \leq 2$

$$A_B = \text{base} \times \text{height} = 1 \times 20 = 20 \text{ m}$$

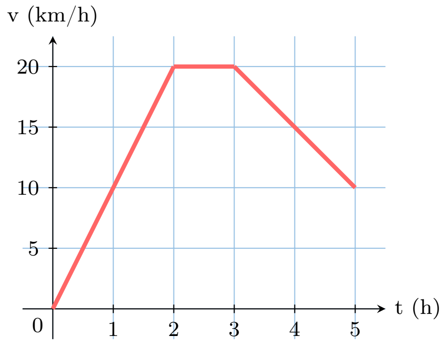
- **Area C (Triangle)**: $2 \leq t \leq 4$

$$A_C = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times (4-2) \times 20 = 20 \text{ m}$$

Total Distance:

$$\text{Distance} = A_A + A_B + A_C = 10 + 20 + 20 = 50 \text{ m}$$

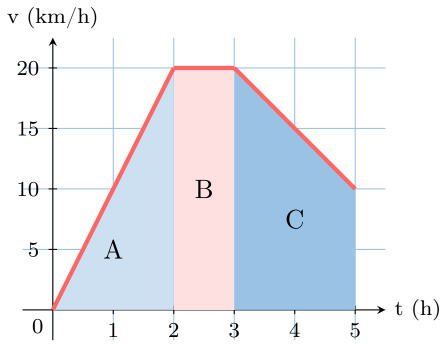
Ex 15: The velocity-time graph for a cyclist is shown below.



Find the total distance travelled by the cyclist.

$$\text{Total distance} = \boxed{70}\ \text{km}$$

Answer: The total distance travelled is the area under the velocity-time graph. We can split the area into three shapes: a triangle (A), a rectangle (B), and a trapezium (C).



- **Area A (Triangle)**: $0 \leq t \leq 2$

$$A_A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 2 \times 20 = 20 \text{ km}$$

- **Area B (Rectangle)**: $2 \leq t \leq 3$

$$A_B = \text{base} \times \text{height} = (3-2) \times 20 = 20 \text{ km}$$

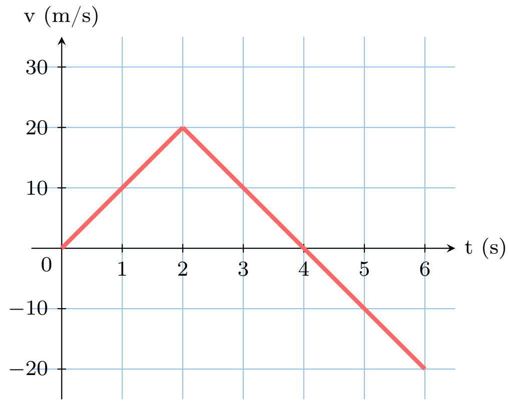
- **Area C (Trapezium)**: $3 \leq t \leq 5$ The heights of the parallel sides are $v(3) = 20$ and $v(5) = 10$.

$$A_C = \frac{1}{2}(a+b)h = \frac{1}{2}(20+10) \times (5-3) = \frac{1}{2} \times 30 \times 2 = 30 \text{ km}$$

Total Distance:

$$\text{Distance} = A_A + A_B + A_C = 20 + 20 + 30 = 70 \text{ km}$$

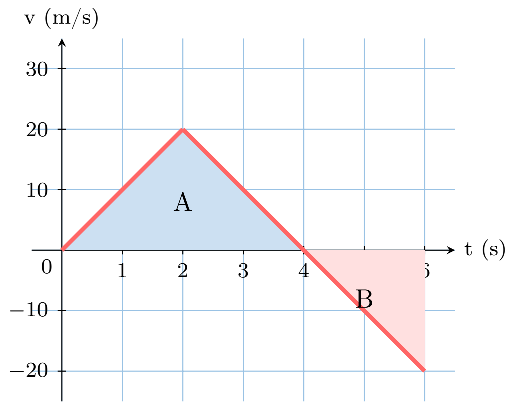
Ex 16: A car travels along a straight road. Its velocity is shown on the graph below.



Find the total distance travelled by the car in the first 6 seconds.

$$\text{Total distance} = \boxed{60}\ \text{m}$$

Answer: The total distance is the sum of the absolute areas between the velocity graph and the time axis. We split the graph into two parts: the area above the axis (A, from $t = 0$ to $t = 4$) and the area below the axis (B, from $t = 4$ to $t = 6$).



- **Area A (Triangle above axis)**: $0 \leq t \leq 4$ The car moves in the positive direction.

$$A_A = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 4 \times 20 = 40 \text{ m}$$

- **Area B (Triangle below axis)**: $4 \leq t \leq 6$ The car moves in the negative direction. For total distance, we take the absolute value of this area.

$$A_B = \left|\frac{1}{2} \times \text{base} \times \text{height}\right| = \left|\frac{1}{2} \times (6-4) \times (-20)\right| = 20 \text{ m}$$

Total Distance:The total distance is the sum of the absolute areas.

$$\text{Distance} = A_A + A_B = 40 + 20 = 60 \text{ m}$$

C.2 FINDING TOTAL DISTANCE WITH DIRECTION CHANGE

Ex 17: A particle moves in a straight line with velocity function $v(t) = 4 - 2t$ m/s for $0 \leq t \leq 3$.

1. Find the time when the particle **reverses direction**.

$$t = \boxed{2} \text{ s}$$

2. Hence, find the **total distance** travelled from 0 s to 3 s.

$$\text{distance} = \boxed{5} \text{ m}$$

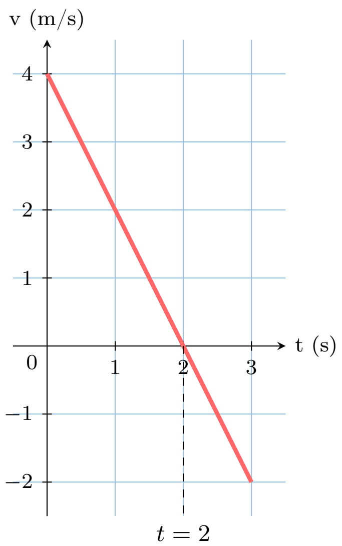
Answer:

1. The particle reverses direction when $v(t) = 0$:

$$\begin{aligned} 4 - 2t &= 0 \\ 2t &= 4 \\ t &= 2 \text{ s.} \end{aligned}$$

2. A sign diagram for $v(t) = 4 - 2t$ shows:
   - $v(t) \geq 0$ on $[0, 2]$
   - $v(t) \leq 0$ on $[2, 3]$

$$\begin{aligned} \text{Distance} &= \int_0^3 |v(t)|\, dt \\ &= \int_0^2 (4 - 2t)\, dt + \int_2^3 -(4 - 2t)\, dt \\ &= \left[4t - t^2\right]_0^2 + \left[-4t + t^2\right]_2^3 \\ &= (8 - 4) - (0) + \big((-12 + 9) - (-8 + 4)\big) \\ &= 4 + (-3 - (-4)) \\ &= 4 + 1 \\ &= 5 \text{ m.} \end{aligned}$$



Ex 18: A particle moves in a straight line with velocity function

$$v(t) = -t^2 + 4t - 3 \quad \text{m/s}, \quad 0 \leq t \leq 3.$$

1. Find the first time when the particle **reverses direction**.

$$t = \boxed{1} \text{ s}$$

2. Hence, find the **total distance** travelled from 0 s to 3 s.

$$\text{distance} = \boxed{\frac{8}{3}} \text{ m}$$

Answer:

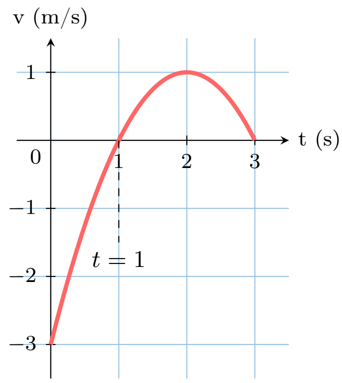
1. The particle reverses direction when $v(t) = 0$:

$$-t^2 + 4t - 3 = 0 \;\Rightarrow\; (t-1)(t-3) = 0 \;\Rightarrow\; t = 1,\ 3.$$

   On the interval $[0, 3]$, the relevant reversal is at $t = 1$ s.

2. Sign analysis: $v(t) < 0$ on $(0, 1)$ and $v(t) > 0$ on $(1, 3)$.

$$\begin{aligned} \text{Distance} &= \int_0^1 |v(t)|\, dt + \int_1^3 |v(t)|\, dt \\ &= \int_0^1 -(v(t))\, dt + \int_1^3 v(t)\, dt \\ &= \int_0^1 (t^2 - 4t + 3)\, dt + \int_1^3 (-t^2 + 4t - 3)\, dt \\ &= \left[\tfrac{1}{3}t^3 - 2t^2 + 3t\right]_0^1 + \left[-\tfrac{1}{3}t^3 + 2t^2 - 3t\right]_1^3 \\ &= \left(\tfrac{1}{3} - 2 + 3\right) + \left((0) - \left(-\tfrac{1}{3} + 2 - 3\right)\right) \\ &= \tfrac{4}{3} + \tfrac{4}{3} = \tfrac{8}{3} \text{ m.} \end{aligned}$$



C.3 FINDING VELOCITY AND TOTAL DISTANCE FROM DISPLACEMENT

Ex 19: A particle P moves along a straight line. Its displacement, s metres, from a fixed origin O at time t seconds is given by the function:

$$s(t) = t^3 - 6t^2 + 9t, \quad \text{for } t \geq 0$$

1. Find an expression for the velocity, $v(t)$, of the particle.

$$v(t) = \boxed{3t^2 - 12t + 9}$$

2. Find the times at which the particle is instantaneously at rest.

$$t = \boxed{1} \text{ s and } t = \boxed{3} \text{ s}$$

3. Find the total distance travelled by the particle in the first 4 seconds of its motion.

$$\text{Total distance} = \boxed{12} \text{ m}$$

Answer:

1. **Velocity Function**
   Differentiate the displacement function $s(t)$ with respect to t:
   $$v(t) = s'(t) = 3t^2 - 12t + 9$$
2. **Times when at Rest**
   The particle is at rest when $v(t) = 0$.
   $$3t^2 - 12t + 9 = 0$$
   $$3(t^2 - 4t + 3) = 0$$
   $$3(t-1)(t-3) = 0$$
   The particle is at rest at $t = 1$ s and $t = 3$ s.
3. **Total Distance in first 4 seconds**
   The particle changes direction at $t = 1$ and $t = 3$. We must calculate the distance travelled in the intervals $[0, 1]$, $[1, 3]$, and $[3, 4]$.
   - Displacement at key times:
     - $s(0) = 0$
     - $s(1) = 1^3 - 6(1)^2 + 9(1) = 4$ m
     - $s(3) = 3^3 - 6(3)^2 + 9(3) = 27 - 54 + 27 = 0$ m
     - $s(4) = 4^3 - 6(4)^2 + 9(4) = 64 - 96 + 36 = 4$ m
   - Distance travelled in each interval:
     - From $t = 0$ to $t = 1$: $|s(1) - s(0)| = |4 - 0| = 4$ m.
     - From $t = 1$ to $t = 3$: $|s(3) - s(1)| = |0 - 4| = 4$ m.
     - From $t = 3$ to $t = 4$: $|s(4) - s(3)| = |4 - 0| = 4$ m.

   Total distance $= 4 + 4 + 4 = 12$ m.

Ex 20: A particle P moves along a straight line. Its displacement, s metres, from a fixed origin O at time t seconds is given by the function:

$$s(t) = 2t^3 - 15t^2 + 24t + 10, \quad \text{for } t \geq 0$$

1. Find an expression for the velocity, $v(t)$, of the particle.
   $$v(t) = \boxed{6t^2 - 30t + 24}$$
2. Find the times at which the particle is instantaneously at rest.
   $$t = \boxed{1} \text{ s and } t = \boxed{4} \text{ s}$$
3. Find the total distance travelled by the particle in the first 5 seconds of its motion.
   $$\text{Total distance} = \boxed{49} \text{ m}$$

Answer:

1. **Velocity Function**
   Differentiate the displacement function $s(t)$ with respect to t:
   $$v(t) = s'(t) = 6t^2 - 30t + 24$$
2. **Times when at Rest**
   The particle is at rest when $v(t) = 0$.
   $$6t^2 - 30t + 24 = 0$$
   $$6(t^2 - 5t + 4) = 0$$
   $$6(t-1)(t-4) = 0$$
   The particle is at rest at $t = 1$ s and $t = 4$ s.
3. **Total Distance in first 5 seconds**
   The particle changes direction at $t = 1$ and $t = 4$. We must calculate the distance travelled in the intervals $[0, 1]$, $[1, 4]$, and $[4, 5]$.
   - Displacement at key times:
     - $s(0) = 10$ m
     - $s(1) = 2(1)^3 - 15(1)^2 + 24(1) + 10 = 2 - 15 + 24 + 10 = 21$ m
     - $s(4) = 2(4)^3 - 15(4)^2 + 24(4) + 10 = 128 - 240 + 96 + 10 = -6$ m
     - $s(5) = 2(5)^3 - 15(5)^2 + 24(5) + 10 = 250 - 375 + 120 + 10 = 5$ m
   - Distance travelled in each interval:
     - From $t = 0$ to $t = 1$: $|s(1) - s(0)| = |21 - 10| = 11$ m.
     - From $t = 1$ to $t = 4$: $|s(4) - s(1)| = |-6 - 21| = |-27| = 27$ m.
     - From $t = 4$ to $t = 5$: $|s(5) - s(4)| = |5 - (-6)| = |11| = 11$ m.

   Total distance $= 11 + 27 + 11 = 49$ m.

D ACCELERATION

D.1 FINDING VELOCITY AND ACCELERATION

Ex 21: A particle moves with velocity function $v(t) = 10t - t^2$ cm/s, for $t \geq 0$. Find:

1. the velocity of the particle when $t = 2$ seconds.
   $$\boxed{16} \text{ cm/s}$$
2. the average acceleration of the particle from $t = 1$ to $t = 3$ seconds.
   $$\boxed{6} \text{ cm/s}^2$$
3. the acceleration function $a(t)$.
   $$a(t) = \boxed{10 - 2t}$$
4. the instantaneous acceleration of the particle when $t = 3$ seconds.
   $$\boxed{4} \text{ cm/s}^2$$

Answer:

1. **Velocity at $t = 2$ s:**
   $$v(2) = 10(2) - (2)^2 = 20 - 4 = 16 \text{ cm/s}$$

2. **Average acceleration from $t = 1$ to $t = 3$ s:**

$$\begin{aligned}\text{Avg. acceleration} &= \frac{v(3) - v(1)}{3 - 1} \\ &= \frac{(10(3) - 3^2) - (10(1) - 1^2)}{2} \\ &= \frac{(30 - 9) - (10 - 1)}{2} \\ &= \frac{21 - 9}{2} \\ &= \frac{12}{2} = 6 \text{ cm/s}^2\end{aligned}$$

3. **Acceleration function**:

$$a(t) = v'(t) = \frac{d}{dt}(10t - t^2) = 10 - 2t$$

4. **Instantaneous acceleration at $t = 3$ s:**

$$a(3) = 10 - 2(3) = 10 - 6 = 4 \text{ cm/s}^2$$

Ex 22: An object moves in a straight line with displacement function $s(t) = t^3 - t^2 - 5$ metres at time t seconds, for $t \geq 0$.

1. Find the object's displacement, velocity, and acceleration when $t = 2$ seconds.

$$s(2) = \boxed{-1} \text{ m}$$
$$v(2) = \boxed{8} \text{ m/s}$$
$$a(2) = \boxed{10} \text{ m/s}^2$$

2. Find the time at which the object has zero acceleration.

$$t = \boxed{1/3} \text{ s}$$

Answer: First, we find the velocity and acceleration functions by differentiating $s(t)$.

$$v(t) = s'(t) = 3t^2 - 2t$$

$$a(t) = v'(t) = 6t - 2$$

a) **Values at $t = 2$ s:**
We substitute $t = 2$ into each function.

- **Displacement**: $s(2) = (2)^3 - (2)^2 - 5 = 8 - 4 - 5 = -1$ m.
- **Velocity**: $v(2) = 3(2)^2 - 2(2) = 12 - 4 = 8$ m/s.
- **Acceleration**: $a(2) = 6(2) - 2 = 12 - 2 = 10$ m/s^2.

b) **Time of zero acceleration**:
Set the acceleration function $a(t)$ equal to zero and solve for t.

$$\begin{aligned}6t - 2 &= 0 \\ 6t &= 2 \\ t &= \frac{2}{6} = \frac{1}{3} \text{ s}\end{aligned}$$

D.2 FINDING VELOCITY AND DISTANCE FROM ACCELERATION

Ex 23: A rocket is launched from rest. It accelerates vertically according to the function $a(t) = 2e^{-t/50}$ m/s^2, for $t \geq 0$.

1. Find the velocity function $v(t)$ of the rocket.
2. How long will it take for the rocket to reach a speed of 80 m/s?
3. How far will the rocket have travelled in this time?

Answer:

1. **Velocity Function**:
We find $v(t)$ by integrating $a(t)$.

$$v(t) = \int 2e^{-t/50}\, dt = 2 \cdot \frac{e^{-t/50}}{-1/50} + C = -100e^{-t/50} + C$$

The rocket is initially at rest, so $v(0) = 0$.

$$v(0) = -100e^0 + C = 0 \implies -100 + C = 0 \implies C = 100$$

Thus, $v(t) = 100 - 100e^{-t/50} = 100(1 - e^{-t/50})$ m/s.

2. **Time to reach 80 m/s**:
Set $v(t) = 80$ and solve for t.

$$\begin{aligned}100(1 - e^{-t/50}) &= 80 \\ 1 - e^{-t/50} &= 0.8 \\ e^{-t/50} &= 0.2 \\ -t/50 &= \ln(0.2) = \ln(1/5) = -\ln(5) \\ t &= 50\ln(5) \text{ s} \quad (\approx 80.5 \text{ s})\end{aligned}$$

3. **Distance Travelled**:
Since the rocket is always accelerating from rest, its velocity is always positive, so distance is the integral of $v(t)$. Let $T = 50\ln(5)$.

$$\begin{aligned}s(T) &= \int_0^T (100 - 100e^{-t/50})\, dt \\ &= \left[100t - 100\frac{e^{-t/50}}{-1/50}\right]_0^T \\ &= \left[100t + 5000e^{-t/50}\right]_0^T \\ &= (100T + 5000e^{-T/50}) - (0 + 5000e^0) \\ &= 100(50\ln 5) + 5000e^{-\ln 5} - 5000 \\ &= 5000\ln 5 + 5000(1/5) - 5000 \\ &= 5000\ln 5 + 1000 - 5000 \\ &= 5000\ln 5 - 4000 \text{ m} \quad (\approx 4047 \text{ m})\end{aligned}$$

Ex 24: A particle is initially at the origin and at rest. It starts to move along a straight line with acceleration $a(t) = \frac{6}{(t+1)^2}$ m/s^2, for $t \geq 0$.

1. Find the velocity function $v(t)$ of the particle.

2. How long will it take for the particle to reach a speed of 4 m/s?

3. How far will the particle have travelled in this time?

Answer:

1. **Velocity Function**:
We find $v(t)$ by integrating $a(t) = 6(t+1)^{-2}$.

$$v(t) = \int 6(t+1)^{-2}\,dt = \frac{6(t+1)^{-1}}{-1} + C = -\frac{6}{t+1} + C$$

The particle is initially at rest, so $v(0) = 0$.

$$v(0) = -\frac{6}{0+1} + C = 0 \implies -6 + C = 0 \implies C = 6$$

Thus, $v(t) = 6 - \frac{6}{t+1}$ m/s.

2. **Time to reach 4 m/s**:
Set $v(t) = 4$ and solve for t.

$$\begin{aligned} 6 - \frac{6}{t+1} &= 4 \\ 2 &= \frac{6}{t+1} \\ 2(t+1) &= 6 \\ t+1 &= 3 \\ t &= 2 \text{ s} \end{aligned}$$

3. **Distance Travelled**:
Since the particle starts from rest and its acceleration is always positive, its velocity is always positive for $t > 0$. Therefore, the distance travelled is the integral of $v(t)$.

$$\begin{aligned} s(2) &= \int_0^2 \left(6 - \frac{6}{t+1}\right) dt \\ &= [6t - 6\ln|t+1|]_0^2 \\ &= (6(2) - 6\ln(2+1)) - (6(0) - 6\ln(0+1)) \\ &= (12 - 6\ln 3) - (0 - 6\ln 1) \\ &= 12 - 6\ln 3 \text{ m} \quad (\approx 5.41 \text{ m}) \end{aligned}$$

Ex 25: A particle starts from rest at the origin and moves along a straight line. Its acceleration is given by $a(t) = \cos(\frac{\pi}{4}t)$ m/s^2, for $t \geq 0$.

1. Find the velocity function $v(t)$ of the particle.

2. What is the maximum speed of the particle?

3. Find the total distance travelled by the particle in the first 4 seconds.

Answer:

1. **Velocity Function**:
We find $v(t)$ by integrating $a(t)$.

$$v(t) = \int \cos\left(\frac{\pi}{4}t\right) dt = \frac{\sin(\frac{\pi}{4}t)}{\pi/4} + C = \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) + C$$

The particle is initially at rest, so $v(0) = 0$.

$$v(0) = \frac{4}{\pi}\sin(0) + C = 0 \implies C = 0$$

Thus, $v(t) = \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right)$ m/s.

2. **Maximum Speed**:
The speed is $S(t) = |v(t)| = \left|\frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right)\right|$. The maximum value of the sine function is 1.
Therefore, the maximum speed is:

$$S_{max} = \frac{4}{\pi} \times 1 = \frac{4}{\pi} \text{ m/s} \quad (\approx 1.27 \text{ m/s})$$

3. **Distance Travelled in first 4 seconds**:
For the interval $0 \leq t \leq 4$, the argument of the sine function $\frac{\pi}{4}t$ goes from 0 to π. In this range, $\sin(\frac{\pi}{4}t)$ is always non-negative, so $v(t) \geq 0$.
Thus, the total distance is the integral of $v(t)$.

$$\begin{aligned} \text{Distance} &= \int_0^4 \frac{4}{\pi}\sin\left(\frac{\pi}{4}t\right) dt \\ &= \frac{4}{\pi}\left[\frac{-\cos(\frac{\pi}{4}t)}{\pi/4}\right]_0^4 \\ &= \frac{4}{\pi}\left[-\frac{4}{\pi}\cos\left(\frac{\pi}{4}t\right)\right]_0^4 \\ &= -\frac{16}{\pi^2}\left[\cos\left(\frac{\pi}{4}t\right)\right]_0^4 \\ &= -\frac{16}{\pi^2}(\cos(\pi) - \cos(0)) \\ &= -\frac{16}{\pi^2}(-1-1) \\ &= -\frac{16}{\pi^2}(-2) \\ &= \frac{32}{\pi^2} \text{ m} \quad (\approx 3.24 \text{ m}) \end{aligned}$$