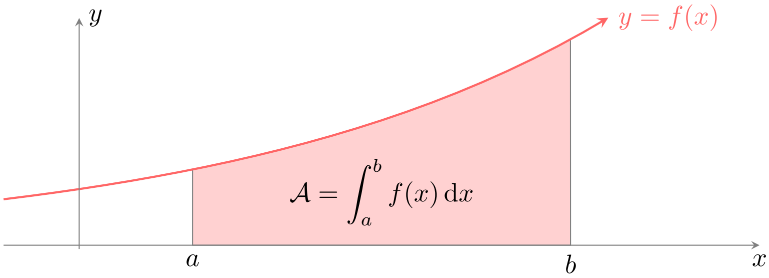
INTEGRALS

The measurement of area has been fundamental to science and society since antiquity. In ancient Egypt, surveyors used knotted ropes to construct right angles, allowing them to measure and restore the boundaries of rectangular fields washed away by the annual floods of the Nile. While finding the area of shapes with straight sides is straightforward, calculus provides a revolutionary tool for finding the area of regions bounded by curves.

In this chapter, we will develop a method to find the exact area, $\mathcal{A}$, of the region bounded by the graph of a function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$. This area is denoted by the **definite integral**:



A THE DEFINITE INTEGRAL AS AN AREA

A.1 DEFINITION OF THE DEFINITE INTEGRAL

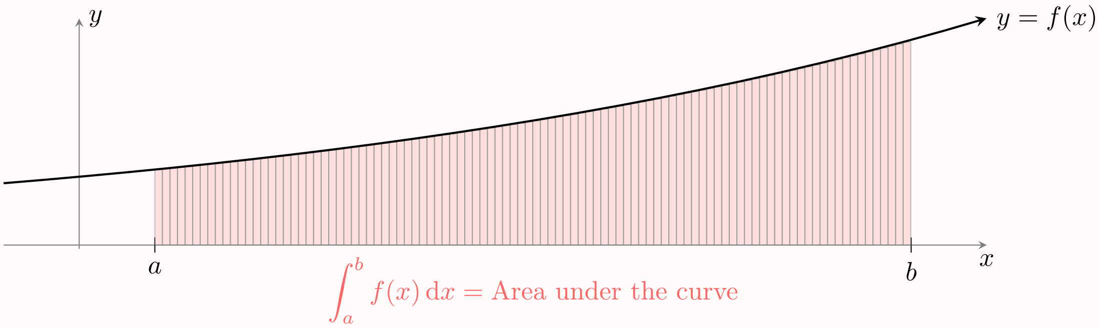
Definition Definite Integral

The **definite integral** of a continuous function f from a to b is the limit of the Riemann sum as the number of subintervals approaches infinity. It is denoted by:

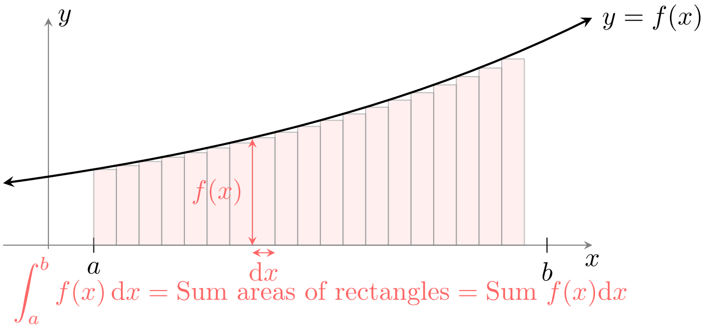
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

where the interval $[a, b]$ is divided into n subintervals of equal width $\Delta x = \dfrac{b-a}{n}$ and x_i is a sample point in the ith subinterval.

- a and b are the **limits of integration**.
- $f(x)$ is the **integrand**.



This notation, introduced by Leibniz, captures the idea of summing ($\int$ is an elongated 'S' for *summa*) the areas of infinitely many rectangles of height $f(x)$ and infinitesimal width dx.

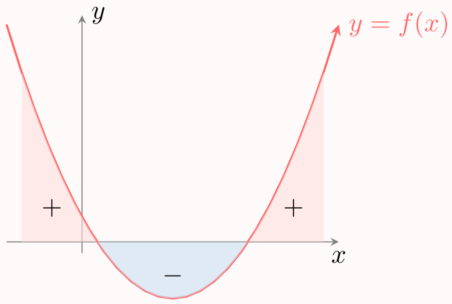


Definition Signed Area

The definite integral calculates the **signed area**.

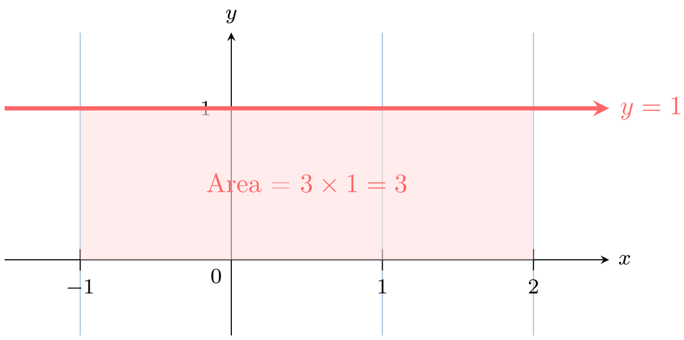
- Area above the x-axis is counted as **positive**.
- Area below the x-axis is counted as **negative**.

The integral is the sum of these signed areas.



Ex: Determine the integral $\displaystyle\int_{-1}^{2} 1\,\mathrm{d}x$ by interpreting it as an area.

Answer: The integral represents the area under the constant function $f(x) = 1$ from $x = -1$ to $x = 2$. This forms a rectangle with width $2 - (-1) = 3$ and height 1.



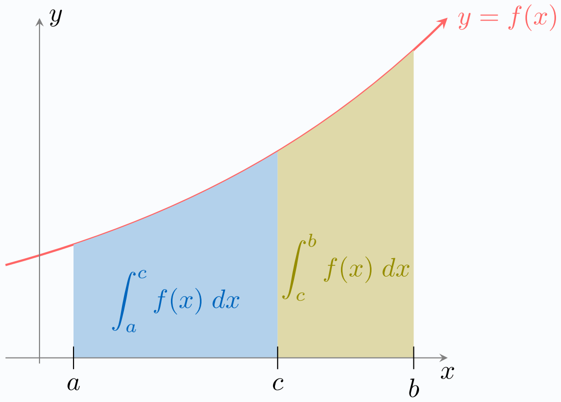
Therefore, $\displaystyle\int_{-1}^{2} 1\,\mathrm{d}x = 3$.

A.2 PROPERTIES OF THE DEFINITE INTEGRAL

Proposition Properties of Integration

Let f and g be continuous functions and k be a constant.

1. **Zero-Width Interval:**
$$\int_a^a f(x)\,dx = 0.$$
2. **Additivity of Intervals (Chasles's Relation):** For any c between a and b:
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



3. **Linearity:**

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

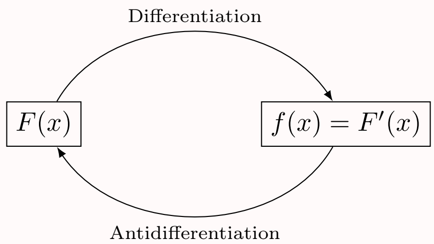
B THE FUNDAMENTAL THEOREM OF CALCULUS

So far, we have defined the definite integral as the limit of a sum—a geometric concept of area. Separately, differentiation is a process of finding rates of change. The **Fundamental Theorem of Calculus** provides a profound and powerful link between these two seemingly unrelated ideas: integration and differentiation are inverse processes of each other.

B.1 ANTIDERIVATIVES

Definition Antiderivative

A function F is an **antiderivative** of a function f if $F'(x) = f(x)$. The process of finding an antiderivative is called **antidifferentiation** or **indefinite integration**.



Since the derivative of a constant is zero, any function f does not have a unique antiderivative. For example, if $F(x) = x^2$ is an antiderivative of $f(x) = 2x$, then $G(x) = x^2 + 5$ is also an antiderivative, since $G'(x) = 2x + 0 = 2x$. All antiderivatives of a function differ only by a constant.

Definition Indefinite Integral

The family of all antiderivatives of a function f is called the **indefinite integral** of f. It is denoted by:

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

where F is any particular antiderivative of f and C is an arbitrary constant called the **constant of integration**.

B.2 FINDING ANTIDERIVATIVES

The process of finding antiderivatives relies on reversing the rules of differentiation. Just as we have a table of derivatives for common functions, we can create a corresponding table of antiderivatives.

Proposition **Antiderivatives of Common Functions**

Function $f(x)$	**An Antiderivative** $F(x)$
k (a constant)	kx
$x^n, n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
e^x	e^x
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\cos(x)$	$\sin(x)$
$\sin(x)$	$-\cos(x)$

Proof

This table is obtained by reading the table of standard derivatives in reverse.

Ex: Find an antiderivative of $f(x) = \frac{1}{x^2}$.

Answer: First, we rewrite the function using a negative exponent: $f(x) = x^{-2}$.

An antiderivative of the power function $f(x) = x^n$ (for $n \neq -1$) is $F(x) = \dfrac{x^{n+1}}{n+1}$. With $n = -2$, we have

$$\begin{aligned} F(x) &= \frac{x^{-2+1}}{-2+1} \\ &= \frac{x^{-1}}{-1} \\ &= -\frac{1}{x} \end{aligned}$$

Proposition **Linearity of Integration**

Let u and v be two functions with antiderivatives U and V, and let k be a constant.

- An antiderivative of $u + v$ is $U + V$.
- An antiderivative of ku is kU.

This allows us to find the antiderivative of a sum of functions term by term.

Ex: Find an antiderivative of $f(x) = 2x + 3$.

Answer: We use the linearity property to find the antiderivative of each term separately.

- An antiderivative of $2x$ is $2 \cdot \dfrac{x^2}{2} = x^2$.
- An antiderivative of 3 is $3x$.

Combining these, an antiderivative of $f(x) = 2x + 3$ is $F(x) = x^2 + 3x$.

Proposition **Antiderivatives of Composite Functions (Reverse Chain Rule)**

Let u be a differentiable function of x.

Function $f(x)$	**An Antiderivative** $F(x)$
$u'(x)[u(x)]^n, \quad n \neq -1$	$\dfrac{[u(x)]^{n+1}}{n+1}$
$\dfrac{u'(x)}{u(x)}$	$\ln\lvert u(x)\rvert$
$u'(x)e^{u(x)}$	$e^{u(x)}$

Ex: Find an antiderivative of $f(x) = \frac{2x}{x^2+1}$.

Answer: The function $f(x)$ is of the form $\dfrac{u'(x)}{u(x)}$.

Let $u(x) = x^2 + 1$. Then its derivative is $u'(x) = 2x$.

So, we can write $f(x) = \dfrac{u'(x)}{u(x)}$.

From the table, an antiderivative is $F(x) = \ln\lvert u(x)\rvert$.

Substituting back, we get $F(x) = \ln\lvert x^2 + 1\rvert$. Since $x^2 + 1$ is always positive, we can remove the absolute value bars:

$$F(x) = \ln(x^2 + 1)$$

Theorem Fundamental Theorem of Calculus

If f is a continuous function on the interval $[a, b]$ and F is any antiderivative of f, then:

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

This result is often written using the notation $[F(x)]_a^b = F(b) - F(a)$.

Proof

The theorem states that integration and differentiation are inverse processes. We can understand this by examining the "area function".

1. **Defining the Area Function:** Let's define a function, $A(x)$, as the area under the curve $y = f(t)$ from a fixed starting point a to a variable endpoint x:

$$A(x) = \int_a^x f(t)\,dt.$$

   Our goal is to show that the derivative of this area function, $A'(x)$, is simply the original function $f(x)$.

2. **The Area of a Thin Strip:** By its definition, the area of a thin vertical strip between x and $x + h$ is the difference between the total area up to $x + h$ and the total area up to x:

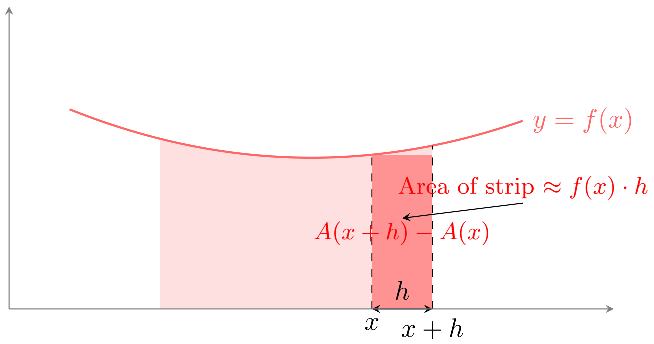
$$\text{Area of strip} = A(x + h) - A(x).$$

3. **Approximating the Area of the Strip:** As shown in the diagram, if the width h is very small, this thin strip is almost a perfect rectangle.

   - The **width** of the rectangle is h.
   - The **height** of the rectangle is approximately $f(x)$.

   So, we can approximate the strip's area:

$$\text{Area of strip} \approx f(x) \cdot h.$$



4. **Connecting to the Derivative:** We now have two expressions for the area of the strip:

$$A(x + h) - A(x) \approx f(x) \cdot h.$$

   Dividing both sides by h, we get the difference quotient for the function $A(x)$:

$$\frac{A(x + h) - A(x)}{h} \approx f(x).$$

5. **Taking the Limit:** This approximation becomes a perfect equality as the width of the strip, h, approaches zero. Taking the limit of both sides:

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x).$$

   The expression on the left is, by definition, the derivative of the area function, $A'(x)$:

$$A'(x) = f(x).$$

6. **Finding the Formula:** Let $F(x)$ be any antiderivative of $f(x)$. Since all antiderivatives of a function differ only by a constant, we know that:

$$A(x) = F(x) + C$$

for some constant C. To find C, we can evaluate this at $x = a$:

$$A(a) = \int_a^a f(t)\, dt = 0$$

so

$$F(a) + C = 0 \implies C = -F(a).$$

Thus the area function is $A(x) = F(x) - F(a)$.
To find the total area up to $x = b$, we simply evaluate $A(b)$:

$$\int_a^b f(x)\, dx = A(b) = F(b) - F(a).$$

Remark This theorem is fundamental because it connects the geometric concept of area (the definite integral) with the algebraic process of antidifferentiation. It gives us a powerful method to calculate exact areas without using the limit of a Riemann sum.

Ex: Find $\int_0^2 x^2 \mathrm{d}x$.

Answer:

1. **Find an antiderivative:** An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$.
2. **Apply the Fundamental Theorem:**

$$\begin{aligned}\int_0^2 x^2 \mathrm{d}x &= \left[\frac{x^3}{3}\right]_0^2 \\ &= F(2) - F(0) \\ &= \frac{2^3}{3} - \frac{0^3}{3} \\ &= \frac{8}{3} - 0 = \frac{8}{3}\end{aligned}$$

This means the exact area under the curve $y = x^2$ over $[0, 2]$ is $\frac{8}{3}$.

C TECHNIQUES FOR INTEGRATION

While we can integrate basic functions by reversing the rules of differentiation, many functions require more advanced techniques. This section covers two key methods that are the reverse of the chain rule and the product rule: **integration by substitution** and **integration by parts**.

C.1 INTEGRATION BY REVERSE CHAIN RULE

Method Integration by Inspection

A common strategy is to make an educated guess for the antiderivative, differentiate it, and adjust any constant factors. This is often called "integration by inspection" or "guess and check".

1. **Guess:** Look at the function and guess a possible antiderivative.
2. **Differentiate:** Differentiate your guess.
3. **Adjust:** Compare your result with the original integrand and multiply by a constant factor to correct any discrepancies.

Ex: Find the integral of $\sin(3x)$.

Answer:

1. **Guess:** The integral of sin is $-\cos$. A good guess for the antiderivative is $F(x) = -\cos(3x)$.
2. **Differentiate:** Using the chain rule, the derivative of our guess is $F'(x) = -(-\sin(3x)) \cdot 3 = 3\sin(3x)$.
3. **Adjust:** Our result, $3\sin(3x)$, is 3 times larger than the original integrand, $\sin(3x)$. We must therefore divide our initial guess by 3.

The correct antiderivative is $-\frac{1}{3}\cos(3x)$. So,

$$\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + C$$

C.2 INTEGRATION BY SUBSTITUTION

Integration by substitution is a powerful technique that reverses the chain rule for differentiation. It is used when an integrand contains both a function and its derivative.

Consider the integral $\int (2x^3+1)^7(6x^2)\,dx$. We can see that the expression contains an "inner function", $u = 2x^3+1$, and its derivative, $u'(x) = 6x^2$. This structure is a clear indicator for substitution.

Let's define a new variable, $u = 2x^3+1$. Differentiating this with respect to x gives $u'(x) = 6x^2$. In differential form, we can write $du = 6x^2\,dx$. Now we can substitute both u and du into the original integral:

$$\begin{aligned}\int \underbrace{(2x^3+1)^7}_{u^7}\underbrace{(6x^2)\,dx}_{du} &= \int u^7\,du \\ &= \frac{1}{8}u^8 + C\end{aligned}$$

Finally, we substitute back to express the result in terms of x:

$$\int (2x^3+1)^7(6x^2)\,dx = \frac{1}{8}(2x^3+1)^8 + C$$

Proposition Integration by Substitution

- **Indefinite Integral:**

$$\int f(u(x))u'(x)\,dx = \int f(u)\,du$$

- **Definite Integral:**

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du$$

Ex: Find $\int_0^1 2x(x^2+1)^3\,dx$.

Answer:

- **Substitution:** Let $u = x^2+1$. Then $du = 2x\,dx$.
- **Change limits:**
  - When $x = 0$, $u = 0^2+1 = 1$.
  - When $x = 1$, $u = 1^2+1 = 2$.
- **Integrate:** We substitute to get a simpler integral in terms of u:

$$\begin{aligned}\int_0^1 \underbrace{(x^2+1)^3}_{u^3}\underbrace{2x\,dx}_{du} &= \int_1^2 u^3\,du \\ &= \left[\frac{u^4}{4}\right]_1^2 \\ &= \frac{2^4}{4} - \frac{1^4}{4} \\ &= 4 - \frac{1}{4} \\ &= \frac{15}{4}\end{aligned}$$