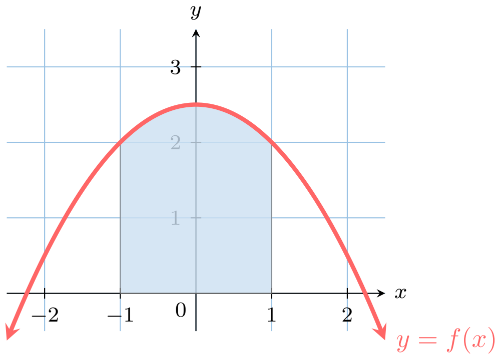
INTEGRALS

A THE DEFINITE INTEGRAL AS AN AREA

A.1 DEFINITION OF THE DEFINITE INTEGRAL

A.1.1 IDENTIFYING THE DEFINITE INTEGRAL FOR A GIVEN AREA

MCQ 1:



The shaded area is represented by which definite integral?

☐ $\int_0^2 f(x)\,\mathrm{d}x$

☐ $\int_{-1}^2 f(x)\,\mathrm{d}x$

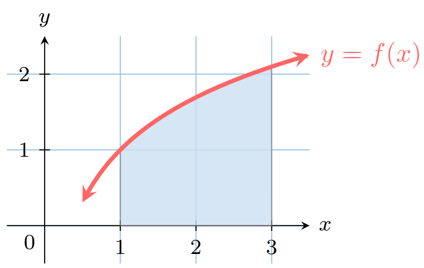
☒ $\int_{-1}^1 f(x)\,\mathrm{d}x$

Answer: The definite integral $\int_a^b f(x)\,\mathrm{d}x$ represents the area bounded by the curve of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

- From the graph, we can see that the shaded region starts at the vertical line $x = -1$ and ends at the vertical line $x = 1$.
- Therefore, the lower limit of integration is $a = -1$ and the upper limit of integration is $b = 1$.

The correct definite integral is $\int_{-1}^1 f(x)\,\mathrm{d}x$.

MCQ 2:



The shaded area is represented by which definite integral?

☒ $\int_1^3 f(x)\,\mathrm{d}x$

☐ $\int_0^3 f(x)\,\mathrm{d}x$

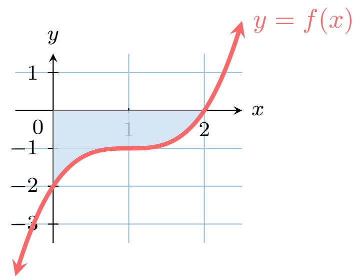
☐ $\int_1^2 f(x)\,\mathrm{d}x$

Answer: The definite integral $\int_a^b f(x)\,\mathrm{d}x$ represents the area bounded by the curve of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

- From the graph, the shaded region starts at the vertical line $x = 1$ and ends at the vertical line $x = 3$.
- Therefore, the lower limit of integration is $a = 1$ and the upper limit of integration is $b = 3$.

The correct definite integral is $\int_1^3 f(x)\,\mathrm{d}x$.

MCQ 3:



The shaded area is represented by which definite integral?

☐ $\int_0^1 f(x)\,\mathrm{d}x$

☒ $\int_0^2 f(x)\,\mathrm{d}x$

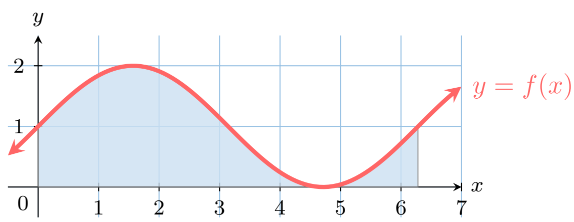
☐ $\int_1^2 f(x)\,\mathrm{d}x$

Answer: The definite integral $\int_a^b f(x)\,\mathrm{d}x$ represents the area bounded by the curve of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

- From the graph, the shaded region starts at the vertical line $x = 0$ and ends at the vertical line $x = 2$.
- Therefore, the lower limit of integration is $a = 0$ and the upper limit of integration is $b = 2$.

The correct definite integral is $\int_0^2 f(x)\,\mathrm{d}x$.

MCQ 4:



The shaded area is represented by which definite integral?

☐ $\int_0^{\pi} f(x)\,\mathrm{d}x$

☒ $\int_0^{2\pi} f(x)\,\mathrm{d}x$

☐ $\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$

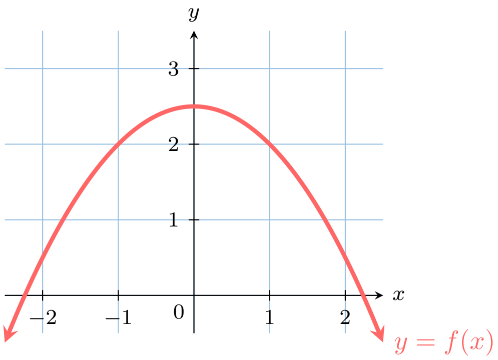
Answer: The definite integral $\int_a^b f(x)\,\mathrm{d}x$ represents the area bounded by the curve of the function $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$.

- From the graph, we can see that the shaded region starts at the vertical line $x = 0$ and ends at the vertical line $x = 2\pi \approx 6.28$.
- Therefore, the lower limit of integration is $a = 0$ and the upper limit of integration is $b = 2\pi$.

The correct definite integral is $\int_0^{2\pi} f(x)\,\mathrm{d}x$.

A.1.2 INTERPRETING THE SIGN OF A DEFINITE INTEGRAL

MCQ 5:

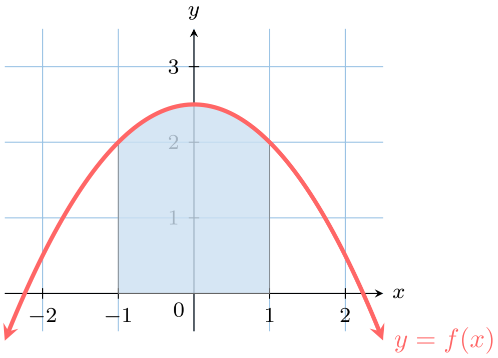


Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_{-1}^{1} f(x)\,\mathrm{d}x$.

☒ Positive

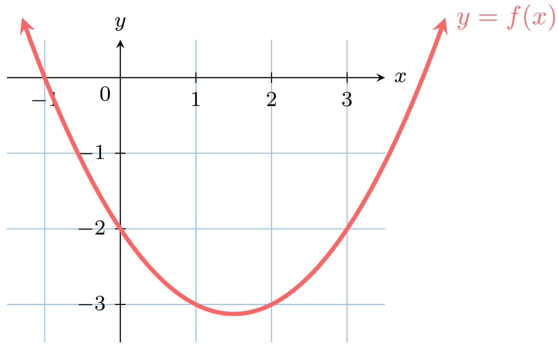
☐ Negative

Answer: The definite integral $\int_{-1}^{1} f(x)\,\mathrm{d}x$ represents the signed area between the curve $y = f(x)$ and the x-axis over the interval $[-1, 1]$.



From the graph, we can see that for all x in the interval $[-1, 1]$, the function $f(x)$ is above the x-axis, meaning $f(x) \geq 0$.
Since the function is non-negative over the entire interval of integration, the area it bounds with the x-axis is counted positively.
Therefore, the value of the definite integral is **positive**.

MCQ 6:

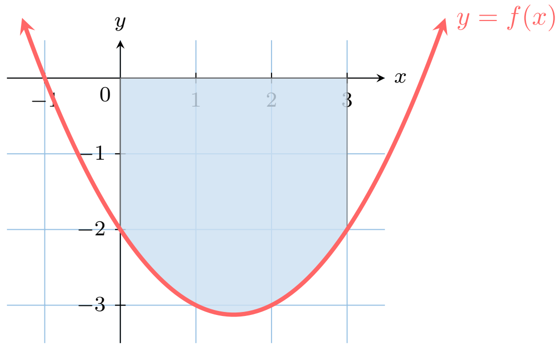


Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_0^3 f(x)\,\mathrm{d}x$.

☐ Positive

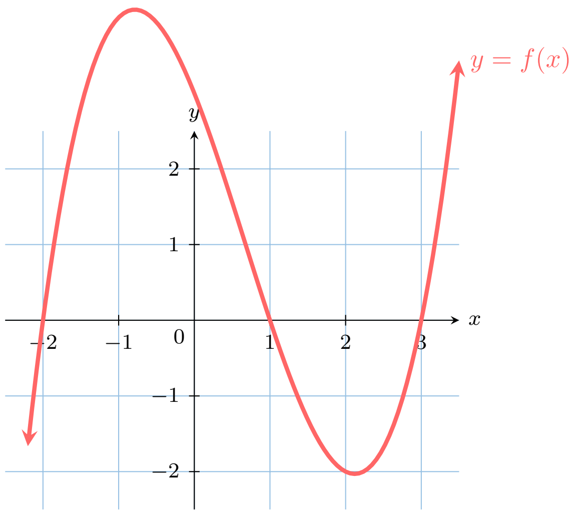
☒ Negative

Answer: The definite integral $\int_0^3 f(x)\,\mathrm{d}x$ represents the signed area between the curve $y = f(x)$ and the x-axis over the interval $[0, 3]$.



From the graph, we can see that for all x in the interval $[0, 3]$, the function $f(x)$ is below the x-axis, meaning $f(x) \leq 0$.
Since the function is non-positive over the entire interval of integration, the area it bounds with the x-axis is counted negatively.
Therefore, the value of the definite integral is **negative**.

MCQ 7:

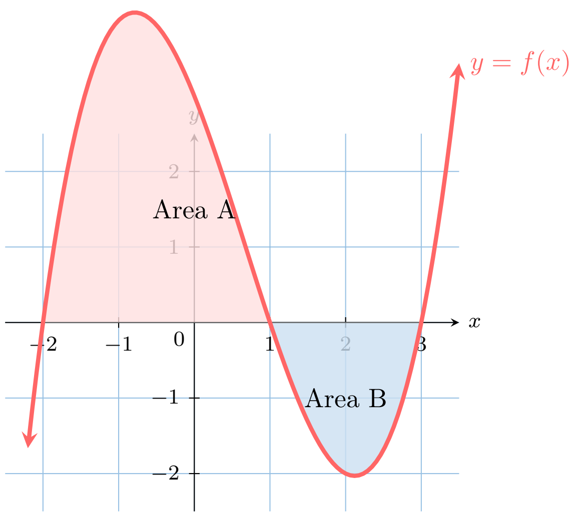


Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_{-2}^{3} f(x)\, \mathrm{d}x$.

☐ Positive

☒ Negative

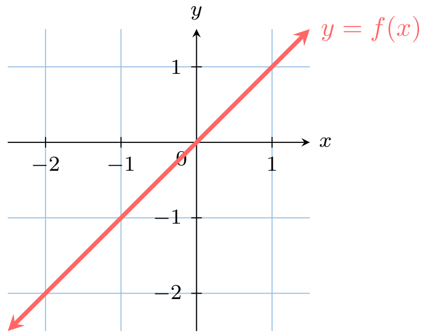
Answer: The definite integral represents the sum of the signed areas.



- Over the interval $[-2, 1]$, the function is above the x-axis. The area, A, is counted positively.
- Over the interval $[1, 3]$, the function is below the x-axis. The area, B, is counted negatively.

The total integral is $\int_{-2}^{3} f(x)\, \mathrm{d}x = (\text{Area A}) - (\text{Area B})$.By visual inspection of the graph, the negative area (Area B) is larger than the positive area (Area A). Therefore, the sum of the signed areas will be negative.The value of the definite integral is **negative**.

MCQ 8:

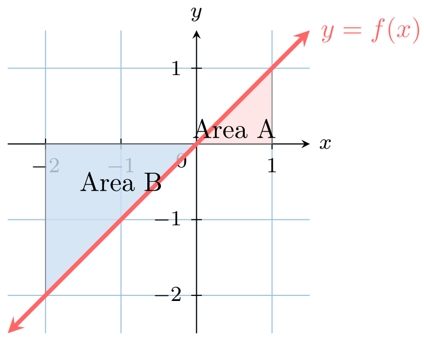


Considering the graph of the function $f(x) = x$ above, determine the sign of the definite integral $\int_{-2}^{1} f(x)\, \mathrm{d}x$.

☐ Positive

☒ Negative

Answer: The definite integral represents the sum of the signed areas between the line and the x-axis.



- Over the interval $[-2, 0]$, the function is below the x-axis. This negative area (Area B) is a triangle with base 2 and height 2, so its area is $\frac{1}{2}(2)(2) = 2$. The integral over this part is -2.
- Over the interval $[0, 1]$, the function is above the x-axis. This positive area (Area A) is a triangle with base 1 and height 1, so its area is $\frac{1}{2}(1)(1) = 0.5$. The integral over this part is $+0.5$.

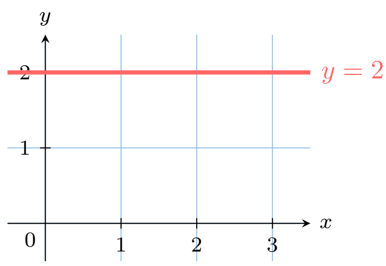
The total integral is the sum of these signed areas:

$$\int_{-2}^{1} f(x)\, \mathrm{d}x = (+0.5) + (-2) = -1.5$$

Since the negative area is larger than the positive area, the value of the definite integral is **negative**.

A.1.3 EVALUATING INTEGRALS USING GEOMETRIC FORMULAS

Ex 9:

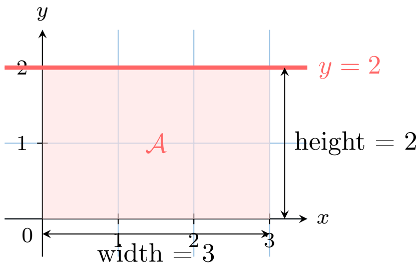


Using the geometric interpretation of the integral as an area, find:

$$\int_0^3 2\,\mathrm{d}x = \boxed{6}$$

Answer: The definite integral represents the area under the constant function $f(x) = 2$ from $x = 0$ to $x = 3$.
This area forms a rectangle.

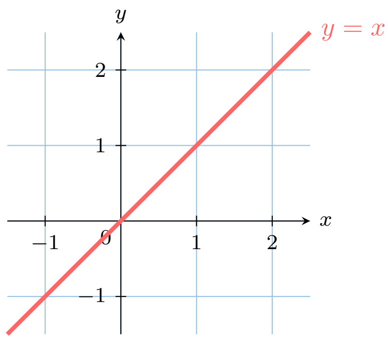
- The width of the rectangle is the length of the interval: $3 - 0 = 3$.
- The height of the rectangle is the constant value of the function: 2.



The area of a rectangle is given by the formula $\mathcal{A} = \text{width} \times \text{height}$.

$$\begin{aligned}\int_0^3 2\,\mathrm{d}x &= \text{Area(Rectangle)}\\ &= 3 \times 2\\ &= 6.\end{aligned}$$

Ex 10:

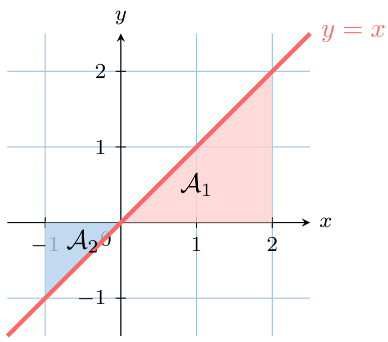


Using the geometric interpretation of the integral as a signed area, find:

$$\int_{-1}^2 x\,\mathrm{d}x = \boxed{\frac{3}{2}}$$

Answer: The definite integral represents the signed area between the curve $y = x$ and the x-axis from $x = -1$ to $x = 2$.

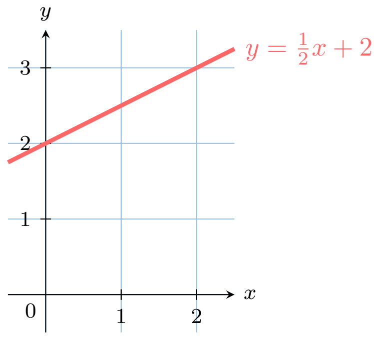
- The area above the x-axis (from $x = 0$ to $x = 2$) is a triangle with base 2 and height 2. This area is counted positively.
- The area below the x-axis (from $x = -1$ to $x = 0$) is a triangle with base 1 and height 1. This area is counted negatively.



The integral is the sum of these signed areas:

$$\begin{aligned}\int_{-1}^2 x\,\mathrm{d}x &= \overbrace{\text{Area}(\mathcal{A}_1)}^{\text{Positive Area}} - \overbrace{\text{Area}(\mathcal{A}_2)}^{\text{Negative Area}}\\ &= \frac{2 \times 2}{2} - \frac{1 \times 1}{2}\\ &= 2 - \frac{1}{2}\\ &= \frac{3}{2}\end{aligned}$$

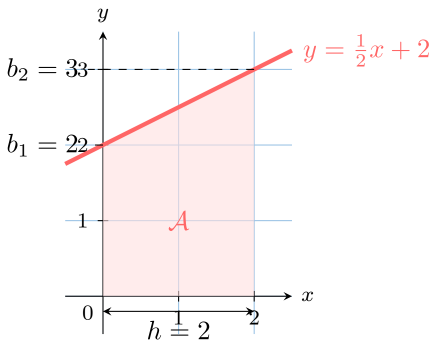
Ex 11:



Using the geometric interpretation of the integral as an area, find:

$$\int_0^2 \left(\frac{1}{2}x + 2\right)\,\mathrm{d}x = \boxed{5}$$

Answer: The definite integral represents the area under the line $y = \frac{1}{2}x + 2$ from $x = 0$ to $x = 2$. This area forms a trapezoid.
The vertices of the trapezoid are at $(0, 0)$, $(2, 0)$, $(2, 3)$, and $(0, 2)$.

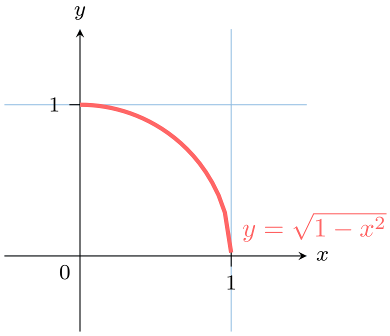


The area of a trapezoid is given by the formula $\mathcal{A} = \frac{1}{2}(b_1 + b_2)h$, where b_1 and b_2 are the parallel bases and h is the height.

- The height of the trapezoid is the width of the interval: $h = 2 - 0 = 2$.
- The first parallel base is the value of the function at $x = 0$: $b_1 = f(0) = \frac{1}{2}(0) + 2 = 2$.
- The second parallel base is the value of the function at $x = 2$: $b_2 = f(2) = \frac{1}{2}(2) + 2 = 3$.

$$\begin{aligned}\int_0^2 \left(\frac{1}{2}x + 2\right)\,\mathrm{d}x &= \text{Area(Trapezoid)}\\ &= \frac{1}{2}(b_1 + b_2)h\\ &= \frac{1}{2}(2 + 3)(2)\\ &= 5.\end{aligned}$$

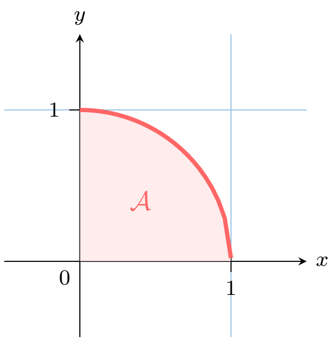
Ex 12:



Using the geometric interpretation of the integral as an area, find:

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \boxed{\frac{\pi}{4}}$$

Answer: The function is $f(x) = \sqrt{1-x^2}$. If we let $y = \sqrt{1-x^2}$, then squaring both sides gives $y^2 = 1 - x^2$, which can be rearranged to $x^2+y^2 = 1$. This is the equation of a circle centered at the origin with a radius of $r = 1$. Since $y = \sqrt{1-x^2}$ must be non-negative, the function describes the upper semi-circle.
The definite integral represents the area under this curve from $x = 0$ to $x = 1$. This is the area of the quarter-circle in the first quadrant.



The area of a full circle is πr^2. For a unit circle ($r = 1$), the area is π. The area of the quarter-circle is therefore $\frac{1}{4}$ of the total area.

$$\begin{aligned}\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x &= \text{Area(Quarter Unit Circle)}\\ &= \frac{1}{4} \times \pi r^2\\ &= \frac{1}{4} \times \pi (1)^2\\ &= \frac{\pi}{4}.\end{aligned}$$

A.2 PROPERTIES OF THE DEFINITE INTEGRAL

A.2.1 APPLYING THE PROPERTIES OF DEFINITE INTEGRALS

Ex 13: For a function f, $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 2$ and $\displaystyle\int_1^2 f(x)\,\mathrm{d}x = 1$, find:

$$\int_0^2 f(x)\,\mathrm{d}x = \boxed{3}$$
$$\int_0^0 f(x)\,\mathrm{d}x = \boxed{0}$$
$$\int_0^2 4f(x)\,\mathrm{d}x = \boxed{12}$$

Answer:

- To find $\displaystyle\int_0^2 f(x)\,\mathrm{d}x$, we use the additivity property of integrals (Chasles's Relation):

$$\begin{aligned}\int_0^2 f(x)\,\mathrm{d}x &= \int_0^1 f(x)\,\mathrm{d}x + \int_1^2 f(x)\,\mathrm{d}x\\ &= 2+1\\ &= 3\end{aligned}$$

- To find $\displaystyle\int_0^0 f(x)\,\mathrm{d}x$, we use the property that the integral over a zero-width interval is zero:

$$\int_0^0 f(x)\,\mathrm{d}x = 0$$

- To find $\displaystyle\int_0^2 4f(x)\,\mathrm{d}x$, we use the linearity property of integrals (constant multiple rule):

$$\begin{aligned}\int_0^2 4f(x)\,\mathrm{d}x &= 4\int_0^2 f(x)\,\mathrm{d}x\\ &= 4 \times 3 \quad \text{(using the result from the first part)}\\ &= 12\end{aligned}$$

Ex 14: Given that $\displaystyle\int_1^3 f(x)\,\mathrm{d}x = 4$ and $\displaystyle\int_1^3 g(x)\,\mathrm{d}x = -2$, find:

$$\int_1^3 (f(x)+g(x))\,\mathrm{d}x = \boxed{2}$$
$$\int_1^3 (2f(x)-3g(x))\,\mathrm{d}x = \boxed{14}$$

Answer:

- To find $\displaystyle\int_1^3 (f(x)+g(x))\,\mathrm{d}x$, we use the linearity property (sum rule):

$$\begin{aligned}\int_1^3 (f(x)+g(x))\,\mathrm{d}x &= \int_1^3 f(x)\,\mathrm{d}x + \int_1^3 g(x)\,\mathrm{d}x\\ &= 4 + (-2)\\ &= 2\end{aligned}$$

- To find $\displaystyle\int_1^3 (2f(x)-3g(x))\,\mathrm{d}x$, we use the linearity property (sum and constant multiple rules):

$$\begin{aligned}\int_1^3 (2f(x)-3g(x))\,\mathrm{d}x &= \int_1^3 2f(x)\,\mathrm{d}x - \int_1^3 3g(x)\,\mathrm{d}x\\ &= 2\int_1^3 f(x)\,\mathrm{d}x - 3\int_1^3 g(x)\,\mathrm{d}x\\ &= 2(4) - 3(-2)\\ &= 8-(-6) = 14\end{aligned}$$

Ex 15: Given that $\displaystyle\int_0^3 f(x)\,\mathrm{d}x = -5$ and $\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 2$, find the value of $\displaystyle\int_1^3 f(x)\,\mathrm{d}x$.

$$\int_1^3 f(x)\,\mathrm{d}x = \boxed{-7}$$

Answer: We use the additivity property of integrals (Chasles's Relation), which states that for any c between a and b:

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

In our case, we have $a = 0, c = 1, b = 3$.

$$\int_0^3 f(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x + \int_1^3 f(x)\,\mathrm{d}x$$

We can now substitute the given values into this equation:

$$-5 = 2 + \int_1^3 f(x)\,\mathrm{d}x$$

To find the unknown integral, we rearrange the equation:

$$\begin{aligned}\int_1^3 f(x)\,\mathrm{d}x &= -5 - 2\\ &= -7\end{aligned}$$

Ex 16: Given that $\int_2^5 f(x)\,\mathrm{d}x = 10$ and $\int_2^5 g(x)\,\mathrm{d}x = 3$, find:

$$\int_2^5 (f(x) - g(x))\,\mathrm{d}x = \boxed{7}$$
$$\int_2^5 5g(x)\,\mathrm{d}x = \boxed{15}$$

Answer:

- To find $\int_2^5 (f(x) - g(x))\,\mathrm{d}x$, we use the linearity property (difference rule):

$$\begin{aligned}\int_2^5 (f(x) - g(x))\,\mathrm{d}x &= \int_2^5 f(x)\,\mathrm{d}x - \int_2^5 g(x)\,\mathrm{d}x\\ &= 10 - 3\\ &= 7\end{aligned}$$

- To find $\int_2^5 5g(x)\,\mathrm{d}x$, we use the linearity property (constant multiple rule):

$$\begin{aligned}\int_2^5 5g(x)\,\mathrm{d}x &= 5\int_2^5 g(x)\,\mathrm{d}x\\ &= 5 \times 3\\ &= 15\end{aligned}$$

Ex 17: Given that $\int_{-1}^4 h(x)\,\mathrm{d}x = 6$ and $\int_2^4 h(x)\,\mathrm{d}x = 5$, find the value of $\int_{-1}^2 h(x)\,\mathrm{d}x$.

$$\int_{-1}^2 h(x)\,\mathrm{d}x = \boxed{1}$$

Answer: We use the additivity property of integrals (Chasles's Relation):

$$\int_{-1}^4 h(x)\,\mathrm{d}x = \int_{-1}^2 h(x)\,\mathrm{d}x + \int_2^4 h(x)\,\mathrm{d}x$$

We can now substitute the given values into this equation:

$$6 = \int_{-1}^2 h(x)\,\mathrm{d}x + 5$$

To find the unknown integral, we rearrange the equation:

$$\begin{aligned}\int_{-1}^2 h(x)\,\mathrm{d}x &= 6 - 5\\ &= 1\end{aligned}$$

B THE FUNDAMENTAL THEOREM OF CALCULUS

B.1 ANTIDERIVATIVES

B.1.1 VERIFYING ANTIDERIVATIVES BY DIFFERENTIATION

MCQ 18: Is the function $F(x) = 2x$ an antiderivative of the function $f(x) = 2$?

☒ Yes

☐ No

Answer: To check if $F(x)$ is an antiderivative of $f(x)$, we must find the derivative of $F(x)$ and see if it equals $f(x)$.

$$\begin{aligned}F'(x) &= \frac{d}{dx}(2x)\\ &= 2\end{aligned}$$

Since $F'(x) = f(x)$, the statement is **true**. $F(x) = 2x$ is an antiderivative of $f(x) = 2$.

MCQ 19: Is the function $F(x) = \frac{1}{4}x^4$ an antiderivative of the function $f(x) = x^3$?

☒ Yes

☐ No

Answer: To check if $F(x)$ is an antiderivative of $f(x)$, we must find the derivative of $F(x)$ and see if it equals $f(x)$.

$$\begin{aligned}F'(x) &= \frac{d}{dx}\left(\frac{1}{4}x^4\right)\\ &= \frac{1}{4}\cdot(4x^3)\\ &= x^3\end{aligned}$$

Since $F'(x) = f(x)$, the statement is **true**.

MCQ 20: Is the function $F(x) = e^{3x}$ an antiderivative of the function $f(x) = e^{3x}$?

☐ Yes

☒ No

Answer: To check if $F(x)$ is an antiderivative of $f(x)$, we must find the derivative of $F(x)$.

$$\begin{aligned} F'(x) &= \frac{d}{dx}(e^{3x}) \\ &= e^{3x} \cdot 3 \\ &= 3e^{3x} \end{aligned}$$

Since $F'(x) \neq f(x)$, the statement is **false**. The correct antiderivative would be $\frac{1}{3}e^{3x}$.

MCQ 21: Is the function $F(x) = -\cos(x)$ an antiderivative of the function $f(x) = \sin(x)$?

☒ Yes

☐ No

Answer: To check if $F(x)$ is an antiderivative of $f(x)$, we must find the derivative of $F(x)$.

$$\begin{aligned} F'(x) &= \frac{d}{dx}(-\cos(x)) \\ &= -(-\sin(x)) \\ &= \sin(x) \end{aligned}$$

Since $F'(x) = f(x)$, the statement is **true**.

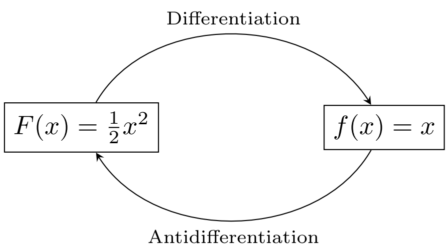
B.1.2 FINDING ANTIDERIVATIVES BY INSPECTION

Ex 22: Find an antiderivative of $f(x) = x$.

$$F(x) = \boxed{\frac{1}{2}x^2}$$

Answer: We are looking for a function $F(x)$ such that $F'(x) = f(x) = x$. We can use the method of "guess and check".

- **Guess:** We know that differentiation reduces the power of x by one. To get a result of x^1, we should start with a function involving x^2. Let's guess $F_{guess}(x) = x^2$.
- **Differentiate (Check):** Let's find the derivative of our guess:
  $$\frac{d}{dx}(x^2) = 2x$$
  This result, $2x$, is not equal to our target function, $f(x) = x$. However, it is off by a constant factor of 2.
- **Adjust:** Since our derivative is twice as large as the target function, we need to divide our original guess by 2.
  Let's try the adjusted function, $F(x) = \frac{1}{2}x^2$. Checking the derivative:
  $$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{2}x^2\right) \\ &= \frac{1}{2}(2x) \\ &= x \end{aligned}$$
  This matches our target function $f(x)$. Therefore, an antiderivative of x is $\frac{1}{2}x^2$.

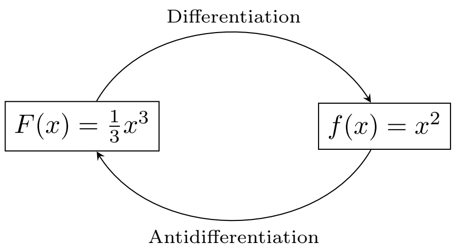


Ex 23: Find an antiderivative of $f(x) = x^2$.

$$F(x) = \boxed{\frac{1}{3}x^3}$$

Answer: We are looking for a function $F(x)$ such that $F'(x) = f(x) = x^2$. We can use the method of "guess and check".

- **Guess:** We know that differentiation reduces the power of x by one. To get a result of x^2, we should start with a function involving x^3. Let's guess $F_{guess}(x) = x^3$.
- **Differentiate (Check):** Let's find the derivative of our guess:
  $$\frac{d}{dx}(x^3) = 3x^2$$
  This result, $3x^2$, is not equal to our target function, $f(x) = x^2$. However, it is off by a constant factor of 3.
- **Adjust:** Since our derivative is three times as large as the target function, we need to divide our original guess by 3.
  Let's try the adjusted function, $F(x) = \frac{1}{3}x^3$. Checking the derivative:
  $$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{3}x^3\right) \\ &= \frac{1}{3}(3x^2) \\ &= x^2 \end{aligned}$$
  This matches our target function $f(x)$. Therefore, an antiderivative of x^2 is $\frac{1}{3}x^3$.



Ex 24: Find an antiderivative of $f(x) = x^{-2}$.

$$F(x) = \boxed{-x^{-1}}$$

Answer: We are looking for a function $F(x)$ such that $F'(x) = f(x) = x^{-2}$. We use the "guess and check" method.

- **Guess:** To get a result of x^{-2} after differentiation, we should start with a function involving x^{-1} (since differentiation reduces the power by one). Let's guess $F_{guess}(x) = x^{-1}$.
- **Differentiate (Check):** Let's find the derivative of our guess:
  $$\frac{d}{dx}(x^{-1}) = -1 \cdot x^{-1-1} = -x^{-2}$$
  This result, $-x^{-2}$, is not equal to our target function, $f(x) = x^{-2}$. However, it is off by a constant factor of -1.
- **Adjust:** Since our derivative is the negative of the target function, we need to multiply our original guess by -1.
  Let's try the adjusted function, $F(x) = -x^{-1}$. Checking the derivative:
  $$\begin{aligned} F'(x) &= \frac{d}{dx}\left(-x^{-1}\right) \\ &= -(-1 \cdot x^{-2}) \\ &= x^{-2} \end{aligned}$$

This matches our target function $f(x)$. Therefore, an antiderivative of x^{-2} is $-x^{-1}$ (or $-\frac{1}{x}$).

$$\begin{tikzcd} \boxed{F(x) = -x^{-1}} \arrow[rr, bend left, "\text{Differentiation}"] & & \boxed{f(x) = x^{-2}} \arrow[ll, bend left, "\text{Antidifferentiation}"] \end{tikzcd}$$

Ex 25: Find an antiderivative of $f(x) = e^{2x}$.

$$F(x) = \boxed{\frac{1}{2}e^{2x}}$$

Answer: We are looking for a function $F(x)$ such that $F'(x) = f(x) = e^{2x}$. We use the "guess and check" method.

- **Guess:** The derivative of an exponential function is another exponential function. A natural first guess is $F_{guess}(x) = e^{2x}$.
- **Differentiate (Check):** Let's find the derivative of our guess using the chain rule:

$$\frac{d}{dx}(e^{2x}) = e^{2x} \cdot 2 = 2e^{2x}$$

  This result, $2e^{2x}$, is not equal to our target function, $f(x) = e^{2x}$. However, it is off by a constant factor of 2.
- **Adjust:** Since our derivative is twice as large as the target function, we need to divide our original guess by 2.
  Let's try the adjusted function, $F(x) = \frac{1}{2}e^{2x}$. Checking the derivative:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) \\ &= \frac{1}{2}(e^{2x} \cdot 2) \\ &= e^{2x} \end{aligned}$$

  This matches our target function $f(x)$. Therefore, an antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

$$\begin{tikzcd} \boxed{F(x) = \frac{1}{2}e^{2x}} \arrow[rr, bend left, "\text{Differentiation}"] & & \boxed{f(x) = e^{2x}} \arrow[ll, bend left, "\text{Antidifferentiation}"] \end{tikzcd}$$

B.2 FINDING ANTIDERIVATIVES

B.2.1 FINDING ANTIDERIVATIVES OF BASIC FUNCTIONS

Ex 26: Find the indefinite integral of $f(x) = x^4$.

$$\int x^4\,dx = \boxed{\frac{1}{5}x^5 + C}$$

Answer: We use the power rule for integration, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, with $n = 4$.

$$\begin{aligned} \int x^4\,dx &= \frac{x^{4+1}}{4+1} + C \\ &= \frac{x^5}{5} + C \end{aligned}$$

Ex 27: Find the indefinite integral of $f(x) = \cos(x)$.

$$\int \cos(x)\,dx = \boxed{\sin(x) + C}$$

Answer: From the table of common antiderivatives, we know that the antiderivative of $\cos(x)$ is $\sin(x)$.Therefore, the indefinite integral is:

$$\int \cos(x)\,dx = \sin(x) + C$$

Ex 28: Find the indefinite integral of $f(x) = x^{-3}$.

$$\int x^{-3}\,dx = \boxed{-\frac{1}{2x^2} + C}$$

Answer: We use the power rule for integration, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, with $n = -3$.

$$\begin{aligned} \int x^{-3}\,dx &= \frac{x^{-3+1}}{-3+1} + C \\ &= \frac{x^{-2}}{-2} + C \\ &= -\frac{1}{2}x^{-2} + C \\ &= -\frac{1}{2x^2} + C \end{aligned}$$

Ex 29: Find the indefinite integral of $f(x) = \frac{1}{x^2}$.

$$\int \frac{1}{x^2}\,dx = \boxed{-\frac{1}{x} + C}$$

Answer: First, we rewrite the function in power form: $f(x) = x^{-2}$.We then use the power rule for integration, $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, with $n = -2$.

$$\begin{aligned} \int x^{-2}\,dx &= \frac{x^{-2+1}}{-2+1} + C \\ &= \frac{x^{-1}}{-1} + C \\ &= -x^{-1} + C \\ &= -\frac{1}{x} + C \end{aligned}$$

Ex 30: Find the indefinite integral of $f(x) = \frac{1}{\sqrt{x}}$.

$$\int \frac{1}{\sqrt{x}}\,dx = \boxed{2\sqrt{x} + C}$$

Answer: First, we rewrite the function in power form: $f(x) = \frac{1}{x^{\frac{1}{2}}} = x^{-\frac{1}{2}}$.We then use the power rule for integration, $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, with $n = -\frac{1}{2}$.

$$\begin{aligned}\int x^{-\frac{1}{2}}\,dx &= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= 2x^{\frac{1}{2}} + C \\ &= 2\sqrt{x} + C\end{aligned}$$

Ex 31: Find the indefinite integral of $f(x) = e^x$.

$$\int e^x\,dx = \boxed{e^x + C}$$

Answer: From the table of common antiderivatives, we know that the function e^x is its own antiderivative.Therefore, the indefinite integral is:

$$\int e^x\,dx = e^x + C$$

B.2.2 APPLYING THE LINEARITY OF INTEGRATION

Ex 32: Find the indefinite integral of $f(x) = 3x^2 - 4x + 5$.

$$\int (3x^2 - 4x + 5)\,dx = \boxed{x^3 - 2x^2 + 5x + C}$$

Answer: We use the linearity property to integrate term by term.

$$\begin{aligned}\int (3x^2 - 4x + 5)\,dx &= 3\int x^2\,dx - 4\int x\,dx + \int 5\,dx \\ &= 3\left(\frac{x^3}{3}\right) - 4\left(\frac{x^2}{2}\right) + 5x + C \\ &= x^3 - 2x^2 + 5x + C\end{aligned}$$

Ex 33: Find the indefinite integral of $f(x) = 2e^x + x^3$.

$$\int (2e^x + x^3)\,dx = \boxed{2e^x + \frac{1}{4}x^4 + C}$$

Answer: We use the linearity property to integrate term by term.

$$\begin{aligned}\int (2e^x + x^3)\,dx &= 2\int e^x\,dx + \int x^3\,dx \\ &= 2e^x + \frac{x^4}{4} + C\end{aligned}$$

Ex 34: Find the indefinite integral of $f(x) = 4\sin(x) - 7$.

$$\int (4\sin(x) - 7)\,dx = \boxed{-4\cos(x) - 7x + C}$$

Answer: We use the linearity property to integrate term by term.

$$\begin{aligned}\int (4\sin(x) - 7)\,dx &= 4\int \sin(x)\,dx - \int 7\,dx \\ &= 4(-\cos(x)) - 7x + C \\ &= -4\cos(x) - 7x + C\end{aligned}$$

Ex 35: Find the indefinite integral of $f(x) = 4\sqrt{x} + \frac{6}{x^3}$.

$$\int \left(4\sqrt{x} + \frac{6}{x^3}\right)\,dx = \boxed{\frac{8}{3}x\sqrt{x} - \frac{3}{x^2} + C}$$

Answer: First, we rewrite the function in power form: $f(x) = 4x^{1/2} + 6x^{-3}$. Then we integrate term by term.

$$\begin{aligned}\int (4x^{1/2} + 6x^{-3})\,dx &= 4\int x^{1/2}\,dx + 6\int x^{-3}\,dx \\ &= 4\left(\frac{x^{3/2}}{3/2}\right) + 6\left(\frac{x^{-2}}{-2}\right) + C \\ &= 4\left(\frac{2}{3}x^{3/2}\right) - 3x^{-2} + C \\ &= \frac{8}{3}x\sqrt{x} - \frac{3}{x^2} + C\end{aligned}$$

Ex 36: Find the indefinite integral of $f(x) = \frac{5}{x} - 2\cos(x)$.

$$\int \left(\frac{5}{x} - 2\cos(x)\right)\,dx = \boxed{5\ln|x| - 2\sin(x) + C}$$

Answer: We use the linearity property to integrate term by term.

$$\begin{aligned}\int \left(\frac{5}{x} - 2\cos(x)\right)\,dx &= 5\int \frac{1}{x}\,dx - 2\int \cos(x)\,dx \\ &= 5\ln|x| - 2\sin(x) + C\end{aligned}$$

B.2.3 APPLYING THE REVERSE CHAIN RULE

Ex 37: Find the indefinite integral of $f(x) = 2x(x^2+2)^3$.

$$\int 2x(x^2+2)^3\,dx = \boxed{\frac{1}{4}(x^2+2)^4 + C}$$

Answer: We recognize that the integrand is of the form $u'(x)[u(x)]^n$.

- Let the "inner function" be $u(x) = x^2 + 2$.
- Then its derivative is $u'(x) = 2x$.

The function $f(x) = 2x(x^2+2)^3$ perfectly matches the form $u'(x)[u(x)]^3$.We use the formula for the antiderivative of $u'u^n$, which is $\frac{[u(x)]^{n+1}}{n+1}$, with $n = 3$:

$$\begin{aligned}\int 2x(x^2+2)^3\,dx &= \frac{(x^2+2)^{3+1}}{3+1} + C \\ &= \frac{(x^2+2)^4}{4} + C\end{aligned}$$

Ex 38: Find the indefinite integral of $f(x) = 2xe^{x^2}$.

$$\int 2xe^{x^2}\,dx = \boxed{e^{x^2} + C}$$

Answer: We recognize that the integrand is of the form $u'(x)e^{u(x)}$.

- Let the "inner function" in the exponent be $u(x) = x^2$.
- Then its derivative is $u'(x) = 2x$.

The function $f(x) = 2xe^{x^2}$ perfectly matches the form $u'(x)e^{u(x)}$.We use the formula for the antiderivative of $u'e^u$, which is $e^{u(x)}$:

$$\int 2xe^{x^2}\,dx = e^{x^2} + C$$

Ex 39: Find the indefinite integral of $f(x) = x^2(x^3+1)^4$.

$\int x^2(x^3+1)^4\,dx = \boxed{\frac{1}{15}(x^3+1)^5+C}$

Answer: We look for a pattern of the form $k \cdot u'(x)[u(x)]^n$.

- Let the "inner function" be $u(x) = x^3+1$.
- Then its derivative is $u'(x) = 3x^2$.

Our function $f(x) = x^2(x^3+1)^4$ is very close to the $u'u^n$ form, but it's missing the factor of 3. We can introduce this factor by multiplying and dividing by 3:

$$\begin{aligned} f(x) &= \frac{1}{3}\cdot 3\cdot x^2(x^3+1)^4 \\ &= \frac{1}{3}\cdot \underbrace{(3x^2)}_{u'(x)}\underbrace{(x^3+1)^4}_{[u(x)]^4} \end{aligned}$$

Now we can integrate using the formula $\int k\cdot u'u^n\,dx = k\frac{u^{n+1}}{n+1}+C$, with $k=\frac{1}{3}$ and $n=4$:

$$\begin{aligned} \int x^2(x^3+1)^4\,dx &= \frac{1}{3}\int 3x^2(x^3+1)^4\,dx \\ &= \frac{1}{3}\left(\frac{(x^3+1)^{4+1}}{4+1}\right)+C \\ &= \frac{1}{3}\frac{(x^3+1)^5}{5}+C \\ &= \frac{1}{15}(x^3+1)^5+C \end{aligned}$$

Ex 40: Find the indefinite integral of $f(x) = \frac{x}{x^2+1}$.

$\int \frac{x}{x^2+1}\,dx = \boxed{\frac{1}{2}\ln(x^2+1)+C}$

Answer: We look for a pattern of the form $k\cdot\frac{u'(x)}{u(x)}$.

- Let the "inner function" in the denominator be $u(x) = x^2+1$.
- Then its derivative is $u'(x) = 2x$.

Our function $f(x) = \frac{x}{x^2+1}$ is very close to the $\frac{u'}{u}$ form, but the numerator is missing a factor of 2. We can introduce this factor by multiplying the expression by $\frac{2}{2}$:

$$f(x) = \frac{1}{2}\cdot\frac{2x}{x^2+1} = \frac{1}{2}\cdot\frac{u'(x)}{u(x)}$$

Now we can integrate using the formula $\int k\cdot\frac{u'}{u}\,dx = k\ln|u|+C$, with $k=\frac{1}{2}$.

$$\begin{aligned} \int \frac{x}{x^2+1}\,dx &= \frac{1}{2}\int\frac{2x}{x^2+1}\,dx \\ &= \frac{1}{2}\ln|x^2+1|+C \end{aligned}$$

Since x^2+1 is always positive, we can write the final answer without the absolute value bars:

$$\frac{1}{2}\ln(x^2+1)+C$$

B.2.4 FINDING A SPECIFIC ANTIDERIVATIVE USING AN INITIAL CONDITION

Ex 41: Find the function $f(x)$ given that $f'(x) = x+1$ and $f(0) = 1$.

$f(x) = \boxed{\frac{1}{2}x^2+x+1}$

Answer: The problem requires us to find a specific antiderivative of $f'(x)$ that satisfies the initial condition $f(0) = 1$.

1. **Find the general antiderivative.** We first find the indefinite integral of $f'(x)$:
$$\begin{aligned} f(x) &= \int (x+1)\,dx \\ &= \frac{x^2}{2}+x+C \end{aligned}$$
This is the set of all possible functions whose derivative is $x+1$.
2. **Use the initial condition to find C.** We are given that $f(0) = 1$. We substitute $x = 0$ into our general antiderivative and set it equal to 1:
$$\begin{aligned} f(0) &= \frac{(0)^2}{2}+(0)+C \\ 1 &= 0+0+C \\ C &= 1 \end{aligned}$$
3. **Write the specific function.** Now that we have found the constant of integration, we can write the unique function that satisfies both conditions:
$$f(x) = \frac{1}{2}x^2+x+1$$

Ex 42: Find the function $f(x)$ given that $f'(x) = e^x$ and $f(0) = 3$.

$f(x) = \boxed{e^x+2}$

Answer: The problem requires us to find a specific antiderivative of $f'(x)$ that satisfies the initial condition $f(0) = 3$.

1. **Find the general antiderivative.** We first find the indefinite integral of $f'(x)$:
$$\begin{aligned} f(x) &= \int e^x\,dx \\ &= e^x+C \end{aligned}$$
This is the family of all possible functions whose derivative is e^x.
2. **Use the initial condition to find C.** We are given that $f(0) = 3$. We substitute $x = 0$ into our general antiderivative and set it equal to 3:
$$\begin{aligned} f(0) &= e^0+C \\ 3 &= 1+C \\ C &= 2 \end{aligned}$$

3. **Write the specific function.** Now that we have found the constant of integration, we can write the unique function that satisfies both conditions:

$$f(x) = e^x + 2$$

Ex 43: Find the function $f(x)$ given that $f'(x) = \cos(x)$ and $f(\pi) = 1$.

$$f(x) = \boxed{\sin(x) + 1}$$

Answer: The problem requires us to find a specific antiderivative of $f'(x)$ that satisfies the initial condition $f(\pi) = 1$.

1. **Find the general antiderivative.** We first find the indefinite integral of $f'(x)$:

$$\begin{aligned} f(x) &= \int \cos(x)\,dx \\ &= \sin(x) + C \end{aligned}$$

This is the family of all possible functions whose derivative is $\cos(x)$.

2. **Use the initial condition to find C.** We are given that $f(\pi) = 1$. We substitute $x = \pi$ into our general antiderivative and set it equal to 1:

$$\begin{aligned} f(\pi) &= \sin(\pi) + C \\ 1 &= 0 + C \\ C &= 1 \end{aligned}$$

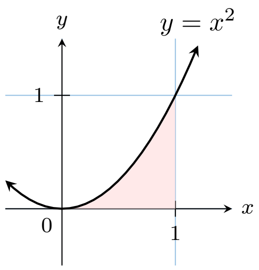
3. **Write the specific function.** Now that we have found the constant of integration, we can write the unique function that satisfies both conditions:

$$f(x) = \sin(x) + 1$$

B.3 FUNDAMENTAL THEOREM OF CALCULUS

B.3.1 CALCULATING AREA USING THE FUNDAMENTAL THEOREM

Ex 44:



Find the area of the region enclosed by the x-axis, the curve $y = x^2$, and the lines $x = 0$ and $x = 1$.

$$\text{Area} = \boxed{\frac{1}{3}}\ \text{units}^2$$

Answer: The area, $\mathcal{A}$, is given by the definite integral of the function $f(x) = x^2$ from $a = 0$ to $b = 1$.

$$\mathcal{A} = \int_0^1 x^2\,dx$$

We use the Fundamental Theorem of Calculus to evaluate the integral.

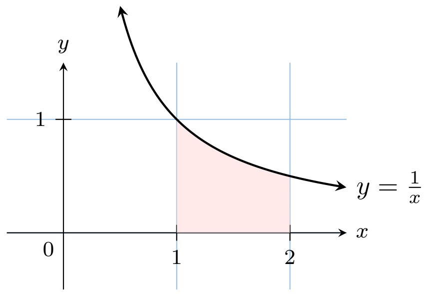
1. **Find an antiderivative:** An antiderivative of $f(x) = x^2$ is $F(x) = \frac{x^3}{3}$.

2. **Evaluate at the limits:**

$$\begin{aligned} \mathcal{A} &= \int_0^1 x^2\,dx \\ &= \left[\frac{x^3}{3}\right]_0^1 \\ &= \frac{1^3}{3} - \frac{0^3}{3} \\ &= \frac{1}{3} - 0 \\ &= \frac{1}{3} \end{aligned}$$

The area of the shaded region is $\frac{1}{3}$ square units.

Ex 45:



Find the area of the region enclosed by the x-axis, the curve $y = \frac{1}{x}$, and the lines $x = 1$ and $x = 2$.

$$\text{Area} = \boxed{\ln(2)}\ \text{units}^2$$

Answer: The area, $\mathcal{A}$, is given by the definite integral of the function $f(x) = \frac{1}{x}$ from $a = 1$ to $b = 2$.

$$\mathcal{A} = \int_1^2 \frac{1}{x}\,dx$$

We use the Fundamental Theorem of Calculus to evaluate the integral.

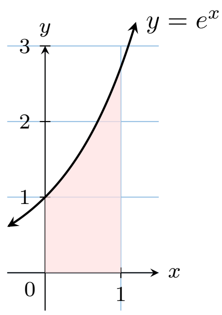
1. **Find an antiderivative:** An antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln|x|$.

2. **Evaluate at the limits:**

$$\begin{aligned} \mathcal{A} &= \int_1^2 \frac{1}{x}\,dx \\ &= [\ln|x|]_1^2 \\ &= \ln|2| - \ln|1| \\ &= \ln(2) - 0 \\ &= \ln(2) \end{aligned}$$

The area of the shaded region is $\ln(2)$ square units.

Ex 46:



Find the area of the region enclosed by the x-axis, the curve $y = e^x$, and the lines $x = 0$ and $x = 1$.

$$\text{Area} = \boxed{e-1} \text{ units}^2$$

Answer: The area, $\mathcal{A}$, is given by the definite integral of the function $f(x) = e^x$ from $a = 0$ to $b = 1$.

$$\mathcal{A} = \int_0^1 e^x \, dx$$

We use the Fundamental Theorem of Calculus to evaluate the integral.

1. **Find an antiderivative:** An antiderivative of $f(x) = e^x$ is $F(x) = e^x$.

2. **Evaluate at the limits:**

$$\begin{aligned}\mathcal{A} &= \int_0^1 e^x \, dx \\ &= [e^x]_0^1 \\ &= F(1) - F(0) \\ &= e^1 - e^0 \\ &= e - 1\end{aligned}$$

The area of the shaded region is $e - 1$ square units.

B.3.2 EVALUATING DEFINITE INTEGRALS: LEVEL 1

Ex 47: Find the value of the definite integral:

$$\int_0^3 x \, dx = \boxed{\frac{9}{2}}$$

Answer: We use the Fundamental Theorem of Calculus to evaluate the integral.

1. **Find an antiderivative:** An antiderivative of $f(x) = x$ is $F(x) = \frac{x^2}{2}$.

2. **Evaluate at the limits of integration:** We calculate $F(3) - F(0)$.

$$\begin{aligned}\int_0^3 x \, dx &= \left[\frac{x^2}{2}\right]_0^3 \\ &= \frac{3^2}{2} - \frac{0^2}{2} \\ &= \frac{9}{2} - 0 \\ &= \frac{9}{2}\end{aligned}$$

The value of the definite integral is $\frac{9}{2}$.

Ex 48: Find the value of the definite integral:

$$\int_0^\pi \sin(x) \, dx = \boxed{2}$$

Answer: We use the Fundamental Theorem of Calculus to evaluate the integral.

1. **Find an antiderivative:** An antiderivative of $f(x) = \sin(x)$ is $F(x) = -\cos(x)$.

2. **Evaluate at the limits of integration:** We calculate $F(\pi) - F(0)$.

$$\begin{aligned}\int_0^\pi \sin(x) \, dx &= [-\cos(x)]_0^\pi \\ &= (-\cos(\pi)) - (-\cos(0)) \\ &= (-(-1)) - (-1) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

The value of the definite integral is 2.

Ex 49: Find the value of the definite integral:

$$\int_0^2 e^x \, dx = \boxed{e^2 - 1}$$

Answer: We use the Fundamental Theorem of Calculus to evaluate the integral.

1. **Find an antiderivative:** An antiderivative of $f(x) = e^x$ is $F(x) = e^x$.

2. **Evaluate at the limits of integration:** We calculate $F(2) - F(0)$.

$$\begin{aligned}\int_0^2 e^x \, dx &= [e^x]_0^2 \\ &= e^2 - e^0 \\ &= e^2 - 1\end{aligned}$$

The value of the definite integral is $e^2 - 1$.

Ex 50: Find the value of the definite integral:

$$\int_1^e \frac{1}{x} \, dx = \boxed{1}$$

Answer: We use the Fundamental Theorem of Calculus to evaluate the integral.

1. **Find an antiderivative:** An antiderivative of $f(x) = \frac{1}{x}$ is $F(x) = \ln|x|$.

2. **Evaluate at the limits of integration:** We calculate $F(e) - F(1)$.

$$\begin{aligned}\int_1^e \frac{1}{x} \, dx &= [\ln|x|]_1^e \\ &= \ln|e| - \ln|1| \\ &= \ln(e) - \ln(1) \\ &= 1 - 0 \\ &= 1\end{aligned}$$

The value of the definite integral is 1.

B.3.3 EVALUATING DEFINITE INTEGRALS: LEVEL 2

Ex 51: Find the value of the definite integral:

$$\int_1^2 (3x^2 + 2x - 1) \, dx = \boxed{9}$$

Answer: We use the linearity of integration and the Fundamental Theorem.

1. **Find an antiderivative:** An antiderivative of $f(x) = 3x^2 + 2x - 1$ is $F(x) = x^3 + x^2 - x$.

2. **Evaluate at the limits of integration:** We calculate $F(2) - F(1)$.

$$\begin{aligned}\int_1^2 (3x^2 + 2x - 1)\,dx &= \left[x^3 + x^2 - x\right]_1^2 \\ &= (2^3 + 2^2 - 2) - (1^3 + 1^2 - 1) \\ &= (8 + 4 - 2) - (1 + 1 - 1) \\ &= 10 - 1 \\ &= 9\end{aligned}$$

Ex 52: Find the value of the definite integral:

$$\int_{\pi/2}^{\pi} (2\sin(x) + \cos(x))\,dx = \boxed{1}$$

Answer: We use the linearity of integration and the Fundamental Theorem.

1. **Find an antiderivative:** An antiderivative of $f(x) = 2\sin(x) + \cos(x)$ is $F(x) = -2\cos(x) + \sin(x)$.

2. **Evaluate at the limits of integration:** We calculate $F(\pi) - F(\pi/2)$.

$$\begin{aligned}\int_{\pi/2}^{\pi} (2\sin(x) + \cos(x))\,dx &= [-2\cos(x) + \sin(x)]_{\pi/2}^{\pi} \\ &= (-2\cos(\pi) + \sin(\pi)) - (-2\cos(\pi/ \\ &= (-2(-1) + 0) - (-2(0) + 1) \\ &= 2 - 1 \\ &= 1\end{aligned}$$

Ex 53: Find the value of the definite integral:

$$\int_1^3 \frac{6}{x^3}\,dx = \boxed{\frac{8}{3}}$$

Answer: We use the Fundamental Theorem of Calculus. First, we rewrite the integrand in power form.

$$\int_1^3 6x^{-3}\,dx$$

1. **Find an antiderivative:** We use the power rule for integration.

$$F(x) = 6 \cdot \frac{x^{-2}}{-2} = -3x^{-2} = -\frac{3}{x^2}$$

2. **Evaluate at the limits of integration:** We calculate $F(3) - F(1)$.

$$\begin{aligned}\int_1^3 6x^{-3}\,dx &= \left[-\frac{3}{x^2}\right]_1^3 \\ &= \left(-\frac{3}{3^2}\right) - \left(-\frac{3}{1^2}\right) \\ &= -\frac{3}{9} - (-3) \\ &= -\frac{1}{3} + 3 \\ &= \frac{8}{3}\end{aligned}$$

The value of the definite integral is $\frac{8}{3}$.

Ex 54: Find the value of the definite integral:

$$\int_0^1 2xe^{x^2}\,dx = \boxed{e - 1}$$

Answer: We use the Fundamental Theorem of Calculus to evaluate the integral.

1. **Find an antiderivative:** The integrand $f(x) = 2xe^{x^2}$ is of the form $u'(x)e^{u(x)}$, which is the derivative of $e^{u(x)}$.
   - Let $u(x) = x^2$.
   - Then $u'(x) = 2x$.

   So, an antiderivative of $f(x) = 2xe^{x^2}$ is $F(x) = e^{x^2}$.

2. **Evaluate at the limits of integration:** We calculate $F(1) - F(0)$.

$$\begin{aligned}\int_0^1 2xe^{x^2}\,dx &= \left[e^{x^2}\right]_0^1 \\ &= e^{1^2} - e^{0^2} \\ &= e^1 - 1 \\ &= e - 1\end{aligned}$$

The value of the definite integral is $e - 1$.

B.3.4 DEFINING FUNCTIONS USING DEFINITE INTEGRALS

Ex 55: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_{\pi/2}^{x} \cos(t)\,dt$$

$$F(x) = \boxed{\sin(x) - 1}$$

Answer: We evaluate the definite integral where the upper limit is a variable, x. We use the Fundamental Theorem of Calculus.

1. **Find an antiderivative of the integrand.** The integrand is $f(t) = \cos(t)$. An antiderivative with respect to t is:

$$G(t) = \sin(t)$$

2. **Evaluate at the limits of integration.** The limits are $\pi/2$ and x.

$$\begin{aligned}F(x) &= \int_{\pi/2}^{x} \cos(t)\,dt \\ &= [\sin(t)]_{\pi/2}^{x} \\ &= G(x) - G(\pi/2) \\ &= \sin(x) - \sin(\pi/2) \\ &= \sin(x) - 1\end{aligned}$$

The function is $F(x) = \sin(x) - 1$.

Ex 56: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0$$

$$F(x) = \boxed{\ln(x)}$$

Answer: We evaluate the definite integral where the upper limit is a variable, x. We use the Fundamental Theorem of Calculus.

1. **Find an antiderivative of the integrand.** The integrand is $f(t) = \frac{1}{t}$. An antiderivative is:

$$G(t) = \ln|t|$$

2. **Evaluate at the limits of integration.** The limits are 1 and x.

$$\begin{aligned} F(x) &= \int_1^x \frac{1}{t}\,dt \\ &= [\ln|t|]_1^x \\ &= G(x) - G(1) \\ &= \ln|x| - \ln|1| \end{aligned}$$

Since the problem states $x > 0$, we can remove the absolute value bars. Also, $\ln(1) = 0$.

$$F(x) = \ln(x) - 0 = \ln(x)$$

The function is $F(x) = \ln(x)$.

Ex 57: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_0^x (u^2+1)\,du$$

$$F(x) = \boxed{\frac{1}{3}x^3 + x}$$

Answer: We are asked to evaluate a definite integral where the upper limit is a variable, x. The result will be a function of x. We use the Fundamental Theorem of Calculus.

1. **Find an antiderivative of the integrand.** The integrand is the function $f(u) = u^2+1$. An antiderivative with respect to the variable u is:

$$G(u) = \frac{u^3}{3} + u$$

2. **Evaluate at the limits of integration.** The limits are 0 and x.

$$\begin{aligned} F(x) &= \int_0^x (u^2+1)\,du \\ &= \left[\frac{u^3}{3} + u\right]_0^x \\ &= G(x) - G(0) \\ &= \left(\frac{x^3}{3} + x\right) - \left(\frac{0^3}{3} + 0\right) \\ &= \frac{x^3}{3} + x \end{aligned}$$

The function is $F(x) = \frac{x^3}{3} + x$.

B.3.5 STUDYING SEQUENCES DEFINED BY INTEGRALS

Ex 58: A sequence (u_n) is defined for $n \geq 0$ by the integral:

$$u_n = \int_0^1 x^n\,dx$$

1. Calculate the first three terms of the sequence: u_0, u_1, and u_2.
2. Find a general formula for u_n.

- $u_0 = \boxed{1}$
- $u_1 = \boxed{\frac{1}{2}}$
- $u_2 = \boxed{\frac{1}{3}}$
- $u_n = \boxed{\frac{1}{n+1}}$

Answer: We evaluate the definite integral for each case using the Fundamental Theorem of Calculus.

1. **Calculate the first three terms:**
   - For $n = 0$:
   $$u_0 = \int_0^1 x^0\,dx = \int_0^1 1\,dx = [x]_0^1 = 1 - 0 = 1$$
   - For $n = 1$:
   $$u_1 = \int_0^1 x^1\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$$
   - For $n = 2$:
   $$u_2 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$
2. **Find the general formula for u_n:** We calculate the integral for a general integer $n \geq 0$. An antiderivative of x^n is $\frac{x^{n+1}}{n+1}$.

$$\begin{aligned} u_n &= \int_0^1 x^n\,dx \\ &= \left[\frac{x^{n+1}}{n+1}\right]_0^1 \\ &= \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} \\ &= \frac{1}{n+1} - 0 \\ &= \frac{1}{n+1} \end{aligned}$$

Ex 59: A sequence (u_n) is defined for $n \geq 0$ by the integral:

$$u_n = \int_0^1 \frac{x^n}{1+x}\,dx$$

1. Calculate u_0.
2. Prove that for any integer $n \geq 0$, the recurrence relation $u_{n+1} + u_n = \frac{1}{n+1}$ holds.
3. Hence, deduce the value of u_1.

Answer:

1. **Calculation of u_0:**

$$\begin{aligned} u_0 &= \int_0^1 \frac{x^0}{1+x}\,dx = \int_0^1 \frac{1}{1+x}\,dx \\ &= \big[\ln|1+x|\big]_0^1 \\ &= \ln(2) - \ln(1) \\ &= \ln(2) \end{aligned}$$

2. **Proof of the recurrence relation:** We start with the left-hand side of the relation:

$$\begin{aligned}
u_{n+1} + u_n &= \int_0^1 \frac{x^{n+1}}{1+x}\,dx + \int_0^1 \frac{x^n}{1+x}\,dx \\
&= \int_0^1 \left(\frac{x^{n+1}+x^n}{1+x}\right) dx \quad \text{(by linearity)} \\
&= \int_0^1 \frac{x^n(x+1)}{1+x}\,dx \quad \text{(factoring the numerator)} \\
&= \int_0^1 x^n\,dx \quad \text{(simplifying the fraction)} \\
&= \left[\frac{x^{n+1}}{n+1}\right]_0^1 \\
&= \frac{1^{n+1}}{n+1} - \frac{0^{n+1}}{n+1} \\
&= \frac{1}{n+1}
\end{aligned}$$

This proves that $u_{n+1} + u_n = \frac{1}{n+1}$.

3. **Deduction of u_1:** We use the recurrence relation proven in the previous step with $n = 0$:

$$u_{0+1} + u_0 = \frac{1}{0+1} \implies u_1 + u_0 = 1$$

We can now solve for u_1 and substitute the value of u_0 we found in the first part:

$$\begin{aligned}
u_1 &= 1 - u_0 \\
&= 1 - \ln(2)
\end{aligned}$$

Ex 60: A sequence (u_n) is defined for any integer $n > 0$ by the integral:

$$u_n = \int_0^1 \frac{e^{nx}}{1+e^x}\,dx$$

1. Calculate u_1.
2. Prove that for any integer $n > 0$, the following recurrence relation holds:
$$u_{n+1} + u_n = \frac{e^n - 1}{n}$$
3. Hence, deduce the value of u_2.

Answer:

1. **Calculation of u_1:**

$$u_1 = \int_0^1 \frac{e^{1\cdot x}}{1+e^x}\,dx = \int_0^1 \frac{e^x}{1+e^x}\,dx$$

This integrand is of the form $\frac{u'(x)}{u(x)}$ where $u(x) = 1 + e^x$ and $u'(x) = e^x$.

$$\begin{aligned}
u_1 &= \left[\ln|1+e^x|\right]_0^1 \\
&= \ln(1+e^1) - \ln(1+e^0) \quad (\text{since } 1+e^x > 0) \\
&= \ln(1+e) - \ln(2)
\end{aligned}$$

2. **Proof of the recurrence relation (for $n > 0$):** We start with the left-hand side of the relation:

$$\begin{aligned}
u_{n+1} + u_n &= \int_0^1 \frac{e^{(n+1)x}}{1+e^x}\,dx + \int_0^1 \frac{e^{nx}}{1+e^x}\,dx \\
&= \int_0^1 \frac{e^{nx}e^x + e^{nx}}{1+e^x}\,dx \quad \text{(by linearity)} \\
&= \int_0^1 \frac{e^{nx}(e^x+1)}{1+e^x}\,dx \quad \text{(factoring the numerator)} \\
&= \int_0^1 e^{nx}\,dx \quad \text{(simplifying the fraction)} \\
&= \left[\frac{e^{nx}}{n}\right]_0^1 \\
&= \frac{e^{n(1)}}{n} - \frac{e^{n(0)}}{n} \\
&= \frac{e^n - 1}{n}
\end{aligned}$$

This proves that $u_{n+1} + u_n = \frac{e^n-1}{n}$.

3. **Deduction of u_2:** We use the recurrence relation with $n = 1$:

$$u_{1+1} + u_1 = \frac{e^1 - 1}{1} \implies u_2 + u_1 = e - 1$$

We solve for u_2 and substitute the value of u_1 found in the first part:

$$\begin{aligned}
u_2 &= e - 1 - u_1 \\
&= e - 1 - (\ln(1+e) - \ln(2)) \\
&= e - 1 - \ln\left(\frac{1+e}{2}\right)
\end{aligned}$$

C TECHNIQUES FOR INTEGRATION

C.1 INTEGRATION BY REVERSE CHAIN RULE

C.1.1 FINDING INTEGRALS FROM DERIVATIVES

Ex 61:

1. Find the derivative of $\arcsin(x)$.

$$\tfrac{d}{dx}(\arcsin(x)) = \boxed{\frac{1}{\sqrt{1-x^2}}}$$

2. Hence, find the indefinite integral $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{\arcsin(x) + C}$$

Answer:

1. **Finding the derivative of $\arcsin(x)$:**
Let $y = \arcsin(x)$. By definition, this means $\sin(y) = x$. We now differentiate this equation implicitly with respect to x:

$$\frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$

Using the chain rule on the left side, we get:

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

We need to express $\cos(y)$ in terms of x. Using the Pythagorean identity, $\cos^2(y) + \sin^2(y) = 1$, we have $\cos^2(y) = 1 - \sin^2(y)$.
Since $\sin(y) = x$, this becomes $\cos^2(y) = 1 - x^2$. The range of the arcsin function is $[-\pi/2, \pi/2]$, an interval where $\cos(y)$ is non-negative. Thus, we take the positive square root: $\cos(y) = \sqrt{1-x^2}$.
Substituting this back gives the derivative:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$

2. **Finding the integral:**
The instruction "Hence" means we must use the result from part 1. We are asked to find:

$$\int \frac{1}{\sqrt{1-x^2}}\, dx$$

From part 1, we just proved that the derivative of $\arcsin(x)$ is precisely this integrand. Since integration is the process of finding the antiderivative (the reverse of differentiation), we can conclude directly:

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x) + C$$

Ex 62:

1. Find the derivative of $f(x) = \arctan(x)$.

$$\tfrac{d}{dx}(\arctan(x)) = \boxed{\frac{1}{1+x^2}}$$

2. Hence, find the indefinite integral $\displaystyle\int \frac{1}{1+x^2}\, dx$.

$$\int \frac{1}{1+x^2}\, dx = \boxed{\arctan(x) + C}$$

Answer:

1. **Finding the derivative of** $\arctan(x)$**:**
Let $y = \arctan(x)$, which means $\tan(y) = x$. Differentiating implicitly with respect to x:

$$\frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x) \implies \sec^2(y) \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

Using the identity $\sec^2(y) = 1 + \tan^2(y)$, and since $\tan(y) = x$, we get $\sec^2(y) = 1 + x^2$.

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

2. **Finding the integral:**
We are asked to find $\int \frac{1}{1+x^2}\, dx$. From part 1, we know that the derivative of $\arctan(x)$ is this exact function. Therefore, by reversing the process:

$$\int \frac{1}{1+x^2}\, dx = \arctan(x) + C$$

Ex 63:

1. Find the derivative of $f(x) = \ln(\cos(x))$.

$$\tfrac{d}{dx}(\ln(\cos(x))) = \boxed{-\tan(x)}$$

2. Hence, find the indefinite integral $\displaystyle\int \tan(x)\, dx$.

$$\int \tan(x)\, dx = \boxed{-\ln|\cos(x)| + C}$$

Answer:

1. **Finding the derivative of** $\ln(\cos(x))$**:**
We use the chain rule. Let $u = \cos(x)$, then $y = \ln(u)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot (-\sin(x)) = \frac{1}{\cos(x)} \cdot (-\sin(x)) = -\tan(x)$$

2. **Finding the integral:**
From part 1, we know that $\frac{d}{dx}(\ln(\cos(x))) = -\tan(x)$. This means $\int -\tan(x)\, dx = \ln(\cos(x)) + C$. To find $\int \tan(x)\, dx$, we can multiply by -1:

$$\int \tan(x)\, dx = -\int -\tan(x)\, dx = -\ln(\cos(x)) + C$$

Since the argument of a logarithm must be positive, we use the absolute value for the general case: $-\ln|\cos(x)| + C$.

Ex 64:

1. Find the derivative of $f(x) = x\ln(x) - x$.

$$\tfrac{d}{dx}(x\ln(x) - x) = \boxed{\ln(x)}$$

2. Hence, find the indefinite integral $\displaystyle\int \ln(x)\, dx$.

$$\int \ln(x)\, dx = \boxed{x\ln(x) - x + C}$$

Answer:

1. **Finding the derivative of** $x\ln(x) - x$**:**
We use the product rule for the term $x\ln(x)$.

$$\frac{d}{dx}(x\ln(x) - x) = \left[(1) \cdot \ln(x) + x \cdot \left(\frac{1}{x}\right)\right] - 1$$

$$= (\ln(x) + 1) - 1 = \ln(x)$$

2. **Finding the integral:**
We are asked to find $\int \ln(x)\, dx$. From part 1, we know that the derivative of $x\ln(x) - x$ is $\ln(x)$. Therefore, by reversing the process:

$$\int \ln(x)\, dx = x\ln(x) - x + C$$

C.2 INTEGRATION BY SUBSTITUTION

C.2.1 INTEGRATING BY SUBSTITUTION FOR INDEFINITE INTEGRALS

Ex 65: Find the indefinite integral of $f(x) = 2x\cos(x^2)$.

$$\int 2x\cos(x^2)\,dx = \boxed{\sin(x^2) + C}$$

Answer: We use the method of integration by substitution.

- **Substitution:** We identify an "inner function" and its derivative.
  - Let $u = x^2$.
  - Then, differentiating with respect to x gives $\frac{du}{dx} = 2x$.
  - Rearranging for dx, we get $du = 2x\,dx$.
- **Integrate:** We substitute to get a simpler integral in terms of u. The integral becomes:

$$\begin{aligned}\int \underbrace{\cos(x^2)}_{\cos(u)}\underbrace{2x\,dx}_{du} &= \int \cos(u)\,du \\ &= \sin(u) + C\end{aligned}$$

- **Substitute back:** Finally, we replace u with x^2 to express the result in terms of the original variable x.

$$\sin(x^2) + C$$

Ex 66: Find the indefinite integral of $f(x) = 3x^2(x^3+5)^4$.

$$\int 3x^2(x^3+5)^4\,dx = \boxed{\frac{1}{5}(x^3+5)^5 + C}$$

Answer: We use the method of integration by substitution.

- **Substitution:** We identify an "inner function" and its derivative.
  - Let $u = x^3 + 5$.
  - Then, differentiating with respect to x gives $\frac{du}{dx} = 3x^2$.
  - Rearranging for dx, we get $du = 3x^2\,dx$.
- **Integrate:** We substitute to get a simpler integral in terms of u. The integral becomes:

$$\begin{aligned}\int \underbrace{(x^3+5)^4}_{u^4}\underbrace{3x^2\,dx}_{du} &= \int u^4\,du \\ &= \frac{u^5}{5} + C\end{aligned}$$

- **Substitute back:** Finally, we replace u with $x^3 + 5$.

$$\frac{1}{5}(x^3+5)^5 + C$$

Ex 67: Find the indefinite integral of $f(x) = \frac{4x^3}{x^4+1}$.

$$\int \frac{4x^3}{x^4+1}\,dx = \boxed{\ln(x^4+1) + C}$$

Answer: We use the method of integration by substitution.

- **Substitution:** We identify an "inner function" (the denominator) and its derivative.
  - Let $u = x^4 + 1$.
  - Then, differentiating gives $\frac{du}{dx} = 4x^3$.
  - Rearranging, we get $du = 4x^3\,dx$.
- **Integrate:** We substitute to get a simpler integral of the form $\int \frac{1}{u}du$.

$$\begin{aligned}\int \frac{1}{\underbrace{x^4+1}_{u}}\underbrace{4x^3\,dx}_{du} &= \int \frac{1}{u}\,du \\ &= \ln|u| + C\end{aligned}$$

- **Substitute back:** We replace u with x^4+1. Since x^4+1 is always positive, the absolute value bars are not necessary.

$$\ln(x^4+1) + C$$

Ex 68: Find the indefinite integral of $f(x) = \cos^3(x)\sin(x)$.

$$\int \cos^3(x)\sin(x)\,dx = \boxed{-\frac{1}{4}\cos^4(x) + C}$$

Answer: We use the method of integration by substitution.

- **Substitution:** We identify an "inner function" and its derivative.
  - Let $u = \cos(x)$.
  - Then, differentiating with respect to x gives $\frac{du}{dx} = -\sin(x)$.
  - Rearranging for the differential, we get $-du = \sin(x)\,dx$.
- **Integrate:** We substitute to get a simpler integral in terms of u. The integral becomes:

$$\begin{aligned}\int \underbrace{\cos^3(x)}_{u^3}\underbrace{\sin(x)\,dx}_{-du} &= \int u^3\,(-du) \\ &= -\int u^3\,du \\ &= -\frac{u^4}{4} + C\end{aligned}$$

- **Substitute back:** Finally, we replace u with $\cos(x)$ to express the result in terms of the original variable x.

$$-\frac{1}{4}\cos^4(x) + C$$

C.2.2 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

Ex 69: Find the value of the definite integral $\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx$.

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx = \boxed{0}$$

Answer: We use integration by substitution for a definite integral.

- **Substitution:** Let $u = x^2$. Then $du = 2x\,dx$.
- **Change limits:**
  - When $x = 0$, $u = 0^2 = 0$.
  - When $x = \sqrt{\pi}$, $u = (\sqrt{\pi})^2 = \pi$.
- **Integrate:** We substitute to get a simpler integral in terms of u with the new limits:

$$\begin{aligned}\int_0^{\sqrt{\pi}} \cos(x^2)(2x\;dx) &= \int_0^{\pi} \cos(u)\,du \\ &= [\sin(u)]_0^{\pi} \\ &= \sin(\pi) - \sin(0) \\ &= 0 - 0 = 0\end{aligned}$$

Ex 70: Find the value of the definite integral $\displaystyle\int_0^1 \frac{x}{x^2+1}\,dx$.

$$\int_0^1 \frac{x}{x^2+1}\;dx = \boxed{\frac{1}{2}\ln(2)}$$

Answer: We use integration by substitution. Let $u = x^2 + 1$, so $du = 2x\,dx$. This means $x\,dx = \frac{1}{2}du$.

- **Substitution and changing limits:**
  - When $x = 0$, $u = 0^2 + 1 = 1$.
  - When $x = 1$, $u = 1^2 + 1 = 2$.
- **Integrate:**

$$\begin{aligned}\int_0^1 \frac{x}{x^2+1}\;dx &= \int_1^2 \frac{1}{u}\left(\frac{1}{2}du\right) \\ &= \frac{1}{2}\int_1^2 \frac{1}{u}\,du \\ &= \frac{1}{2}\left[\ln|u|\right]_1^2 \\ &= \frac{1}{2}(\ln(2) - \ln(1)) \\ &= \frac{1}{2}\ln(2)\end{aligned}$$

Ex 71: Find the value of the definite integral $\displaystyle\int_0^{\pi/2} \cos^3(x)\sin(x)\,dx$.

$$\int_0^{\pi/2} \cos^3(x)\sin(x)\;dx = \boxed{\frac{1}{4}}$$

Answer: We use integration by substitution. Let $u = \cos(x)$, so $du = -\sin(x)\,dx$, which means $\sin(x)\,dx = -du$.

- **Substitution and changing limits:**
  - When $x = 0$, $u = \cos(0) = 1$.
  - When $x = \pi/2$, $u = \cos(\pi/2) = 0$.
- **Integrate:** Notice the change in the order of the limits.

$$\begin{aligned}\int_0^{\pi/2} \cos^3(x)\sin(x)\;dx &= \int_1^0 u^3(-du) \\ &= -\int_1^0 u^3\,du \\ &= \int_0^1 u^3\,du \quad \text{(reversing limits changes t} \\ &= \left[\frac{u^4}{4}\right]_0^1 \\ &= \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4}\end{aligned}$$

Ex 72: Find the value of the definite integral $\displaystyle\int_0^1 6xe^{x^2}\,dx$.

$$\int_0^1 6xe^{x^2}\;dx = \boxed{3(e-1)}$$

Answer: We use integration by substitution. Let $u = x^2$, so $du = 2x\,dx$.The integrand is $6xe^{x^2} = 3\cdot(2x)e^{x^2}$, so we can write $6x\,dx = 3\,du$.

- **Substitution and changing limits:**
  - When $x = 0$, $u = 0^2 = 0$.
  - When $x = 1$, $u = 1^2 = 1$.
- **Integrate:** We substitute to get a simpler integral in terms of u with the new limits:

$$\begin{aligned}\int_0^1 6xe^{x^2}\;dx &= \int_0^1 e^{x^2}(6x\;dx) \\ &= \int_0^1 e^u(3\,du) \\ &= 3\int_0^1 e^u\,du \\ &= 3\left[e^u\right]_0^1 \\ &= 3(e^1 - e^0) \\ &= 3(e-1)\end{aligned}$$