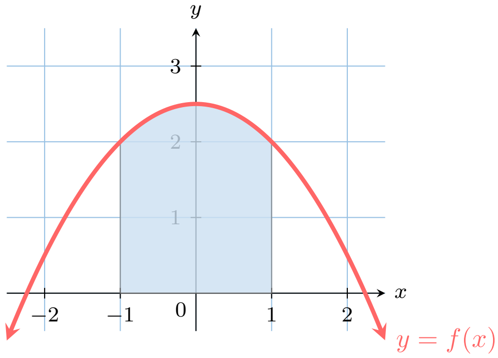
INTEGRALS

A THE DEFINITE INTEGRAL AS AN AREA

A.1 DEFINITION OF THE DEFINITE INTEGRAL

A.1.1 IDENTIFYING THE DEFINITE INTEGRAL FOR A GIVEN AREA

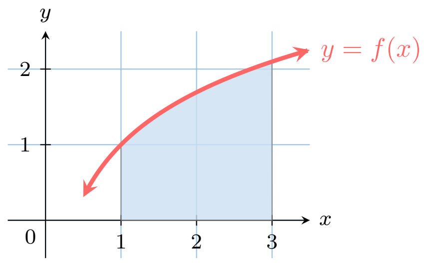
MCQ 1:



The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_0^2 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_{-1}^2 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_{-1}^1 f(x)\,\mathrm{d}x$

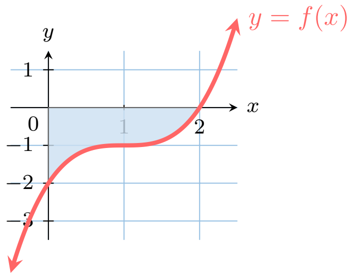
MCQ 2:



The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_1^3 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_0^3 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_1^2 f(x)\,\mathrm{d}x$

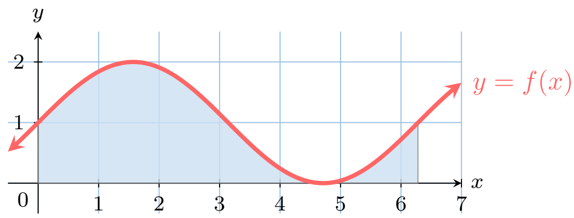
MCQ 3:



The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_0^2 f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_1^2 f(x)\,\mathrm{d}x$

MCQ 4:

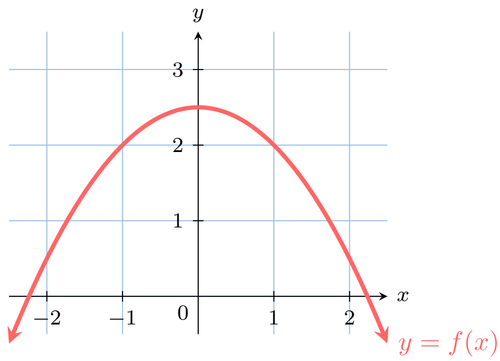


The shaded area is represented by which definite integral?

- ☐ $\displaystyle\int_0^{\pi} f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_0^{2\pi} f(x)\,\mathrm{d}x$
- ☐ $\displaystyle\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$

A.1.2 INTERPRETING THE SIGN OF A DEFINITE INTEGRAL

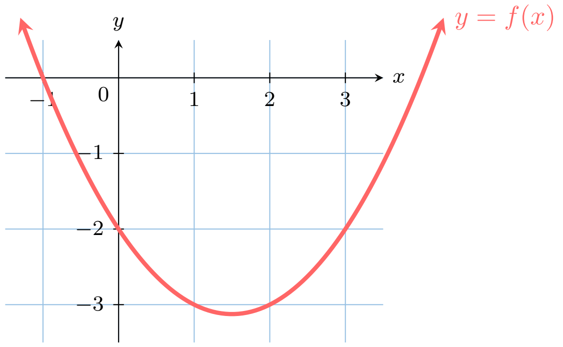
MCQ 5:



Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\displaystyle\int_{-1}^1 f(x)\,\mathrm{d}x$.

- ☐ Positive
- ☐ Negative

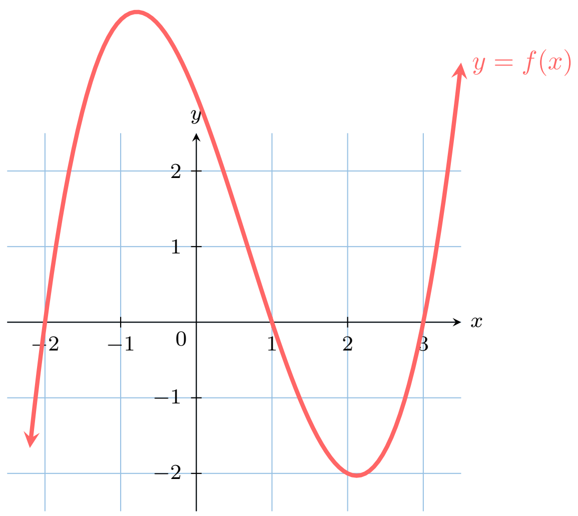
MCQ 6:



Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_0^3 f(x)\, dx$.

- ☐ Positive
- ☐ Negative

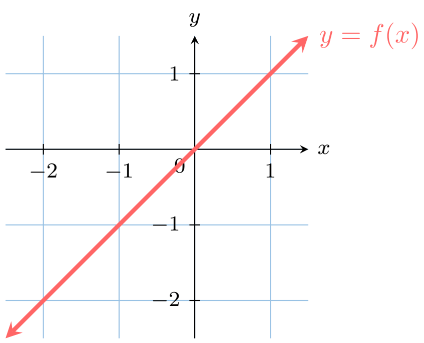
MCQ 7:



Considering the graph of the function $f(x)$ above, determine the sign of the definite integral $\int_{-2}^3 f(x)\, dx$.

- ☐ Positive
- ☐ Negative

MCQ 8:

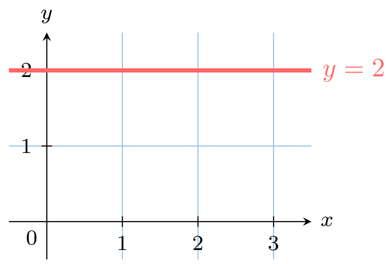


Considering the graph of the function $f(x) = x$ above, determine the sign of the definite integral $\int_{-2}^1 f(x)\, dx$.

- ☐ Positive
- ☐ Negative

A.1.3 EVALUATING INTEGRALS USING GEOMETRIC FORMULAS

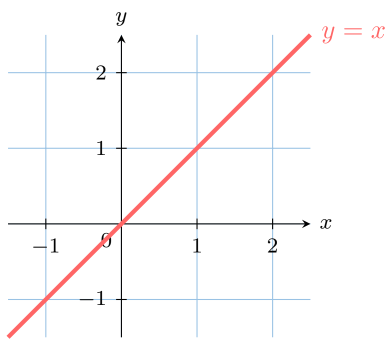
Ex 9:



Using the geometric interpretation of the integral as an area, find:

$$\int_0^3 2\, dx = \square$$

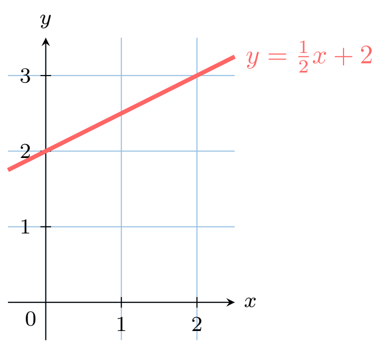
Ex 10:



Using the geometric interpretation of the integral as a signed area, find:

$$\int_{-1}^2 x\, dx = \square$$

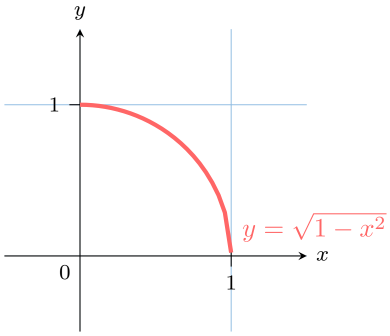
Ex 11:



Using the geometric interpretation of the integral as an area, find:

$$\int_0^2 \left(\frac{1}{2}x + 2\right)\, dx = \square$$

Ex 12:



Using the geometric interpretation of the integral as an area, find:

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x = \square$$

A.2 PROPERTIES OF THE DEFINITE INTEGRAL

A.2.1 APPLYING THE PROPERTIES OF DEFINITE INTEGRALS

Ex 13: For a function f, $\int_0^1 f(x)\,\mathrm{d}x = 2$ and $\int_1^2 f(x)\,\mathrm{d}x = 1$, find:

$$\int_0^2 f(x)\,\mathrm{d}x = \square$$
$$\int_0^0 f(x)\,\mathrm{d}x = \square$$
$$\int_0^2 4f(x)\,\mathrm{d}x = \square$$

Ex 14: Given that $\int_1^3 f(x)\,\mathrm{d}x = 4$ and $\int_1^3 g(x)\,\mathrm{d}x = -2$, find:

$$\int_1^3 (f(x)+g(x))\,\mathrm{d}x = \square$$
$$\int_1^3 (2f(x)-3g(x))\,\mathrm{d}x = \square$$

Ex 15: Given that $\int_0^3 f(x)\,\mathrm{d}x = -5$ and $\int_0^1 f(x)\,\mathrm{d}x = 2$, find the value of $\int_1^3 f(x)\,\mathrm{d}x$.

$$\int_1^3 f(x)\,\mathrm{d}x = \square$$

Ex 16: Given that $\int_2^5 f(x)\,\mathrm{d}x = 10$ and $\int_2^5 g(x)\,\mathrm{d}x = 3$, find:

$$\int_2^5 (f(x)-g(x))\,\mathrm{d}x = \square$$
$$\int_2^5 5g(x)\,\mathrm{d}x = \square$$

Ex 17: Given that $\int_{-1}^4 h(x)\,\mathrm{d}x = 6$ and $\int_2^4 h(x)\,\mathrm{d}x = 5$, find the value of $\int_{-1}^2 h(x)\,\mathrm{d}x$.

$$\int_{-1}^2 h(x)\,\mathrm{d}x = \square$$

B THE FUNDAMENTAL THEOREM OF CALCULUS

B.1 ANTIDERIVATIVES

B.1.1 VERIFYING ANTIDERIVATIVES BY DIFFERENTIATION

MCQ 18: Is the function $F(x) = 2x$ an antiderivative of the function $f(x) = 2$?

☐ Yes

☐ No

MCQ 19: Is the function $F(x) = \frac{1}{4}x^4$ an antiderivative of the function $f(x) = x^3$?

☐ Yes

☐ No

MCQ 20: Is the function $F(x) = e^{3x}$ an antiderivative of the function $f(x) = e^{3x}$?

☐ Yes

☐ No

MCQ 21: Is the function $F(x) = -\cos(x)$ an antiderivative of the function $f(x) = \sin(x)$?

☐ Yes

☐ No

B.1.2 FINDING ANTIDERIVATIVES BY INSPECTION

Ex 22: Find an antiderivative of $f(x) = x$.

$$F(x) = \square$$

Ex 23: Find an antiderivative of $f(x) = x^2$.

$$F(x) = \square$$

Ex 24: Find an antiderivative of $f(x) = x^{-2}$.

$$F(x) = \square$$

Ex 25: Find an antiderivative of $f(x) = e^{2x}$.

$$F(x) = \square$$

B.2 FINDING ANTIDERIVATIVES

B.2.1 FINDING ANTIDERIVATIVES OF BASIC FUNCTIONS

Ex 26: Find the indefinite integral of $f(x) = x^4$.

$$\int x^4\,dx = \square$$

Ex 27: Find the indefinite integral of $f(x) = \cos(x)$.

$$\int \cos(x)\,dx = \square$$

Ex 28: Find the indefinite integral of $f(x) = x^{-3}$.

$$\int x^{-3}\,dx = \square$$

Ex 29: Find the indefinite integral of $f(x) = \frac{1}{x^2}$.

$$\int \frac{1}{x^2}\,dx = \square$$

Ex 30: Find the indefinite integral of $f(x) = \frac{1}{\sqrt{x}}$.

$$\int \frac{1}{\sqrt{x}}\,dx = \square$$

Ex 31: Find the indefinite integral of $f(x) = e^x$.

$$\int e^x\,dx = \square$$

B.2.2 APPLYING THE LINEARITY OF INTEGRATION

Ex 32: Find the indefinite integral of $f(x) = 3x^2 - 4x + 5$.

$$\int (3x^2 - 4x + 5)\,dx = \square$$

Ex 33: Find the indefinite integral of $f(x) = 2e^x + x^3$.

$$\int (2e^x + x^3)\,dx = \square$$

Ex 34: Find the indefinite integral of $f(x) = 4\sin(x) - 7$.

$$\int (4\sin(x) - 7)\,dx = \square$$

Ex 35: Find the indefinite integral of $f(x) = 4\sqrt{x} + \frac{6}{x^3}$.

$$\int \left(4\sqrt{x} + \frac{6}{x^3}\right)\,dx = \square$$

Ex 36: Find the indefinite integral of $f(x) = \frac{5}{x} - 2\cos(x)$.

$$\int \left(\frac{5}{x} - 2\cos(x)\right)\,dx = \square$$

B.2.3 APPLYING THE REVERSE CHAIN RULE

Ex 37: Find the indefinite integral of $f(x) = 2x(x^2+2)^3$.

$$\int 2x(x^2+2)^3\,dx = \square$$

Ex 38: Find the indefinite integral of $f(x) = 2xe^{x^2}$.

$$\int 2xe^{x^2}\,dx = \square$$

Ex 39: Find the indefinite integral of $f(x) = x^2(x^3+1)^4$.

$$\int x^2(x^3+1)^4\,dx = \square$$

Ex 40: Find the indefinite integral of $f(x) = \frac{x}{x^2+1}$.

$$\int \frac{x}{x^2+1}\,dx = \square$$

B.2.4 FINDING A SPECIFIC ANTIDERIVATIVE USING AN INITIAL CONDITION

Ex 41: Find the function $f(x)$ given that $f'(x) = x + 1$ and $f(0) = 1$.

$$f(x) = \square$$

Ex 42: Find the function $f(x)$ given that $f'(x) = e^x$ and $f(0) = 3$.

$$f(x) = \square$$

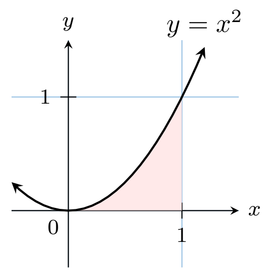
Ex 43: Find the function $f(x)$ given that $f'(x) = \cos(x)$ and $f(\pi) = 1$.

$$f(x) = \square$$

B.3 FUNDAMENTAL THEOREM OF CALCULUS

B.3.1 CALCULATING AREA USING THE FUNDAMENTAL THEOREM

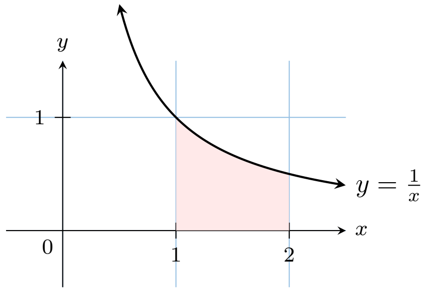
Ex 44:



Find the area of the region enclosed by the x-axis, the curve $y = x^2$, and the lines $x = 0$ and $x = 1$.

Area = $\square$ units^2

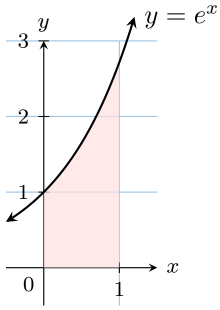
Ex 45:



Find the area of the region enclosed by the x-axis, the curve $y = \frac{1}{x}$, and the lines $x = 1$ and $x = 2$.

$$\text{Area} = \boxed{}\ \text{units}^2$$

Ex 46:



Find the area of the region enclosed by the x-axis, the curve $y = e^x$, and the lines $x = 0$ and $x = 1$.

$$\text{Area} = \boxed{}\ \text{units}^2$$

B.3.2 EVALUATING DEFINITE INTEGRALS: LEVEL 1

Ex 47: Find the value of the definite integral:

$$\int_0^3 x\,dx = \boxed{}$$

Ex 48: Find the value of the definite integral:

$$\int_0^\pi \sin(x)\,dx = \boxed{}$$

Ex 49: Find the value of the definite integral:

$$\int_0^2 e^x\,dx = \boxed{}$$

Ex 50: Find the value of the definite integral:

$$\int_1^e \frac{1}{x}\,dx = \boxed{}$$

B.3.3 EVALUATING DEFINITE INTEGRALS: LEVEL 2

Ex 51: Find the value of the definite integral:

$$\int_1^2 (3x^2 + 2x - 1)\,dx = \boxed{}$$

Ex 52: Find the value of the definite integral:

$$\int_{\pi/2}^\pi (2\sin(x) + \cos(x))\,dx = \boxed{}$$

Ex 53: Find the value of the definite integral:

$$\int_1^3 \frac{6}{x^3}\,dx = \boxed{}$$

Ex 54: Find the value of the definite integral:

$$\int_0^1 2xe^{x^2}\,dx = \boxed{}$$

B.3.4 DEFINING FUNCTIONS USING DEFINITE INTEGRALS

Ex 55: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_{\pi/2}^x \cos(t)\,dt$$

$$F(x) = \boxed{}$$

Ex 56: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_1^x \frac{1}{t}\,dt \quad \text{for } x > 0$$

$$F(x) = \boxed{}$$

Ex 57: Find the function $F(x)$ defined by the definite integral:

$$F(x) = \int_0^x (u^2 + 1)\,du$$

$$F(x) = \boxed{}$$

B.3.5 STUDYING SEQUENCES DEFINED BY INTEGRALS

Ex 58: A sequence (u_n) is defined for $n \geq 0$ by the integral:

$$u_n = \int_0^1 x^n\,dx$$

1. Calculate the first three terms of the sequence: u_0, u_1, and u_2.
2. Find a general formula for u_n.

- $u_0 = \boxed{}$
- $u_1 = \boxed{}$
- $u_2 = \boxed{}$
- $u_n = \boxed{}$

Ex 59: A sequence (u_n) is defined for $n \geq 0$ by the integral:

$$u_n = \int_0^1 \frac{x^n}{1+x}\,dx$$

1. Calculate u_0.

2. Prove that for any integer $n \geq 0$, the recurrence relation $u_{n+1} + u_n = \frac{1}{n+1}$ holds.
3. Hence, deduce the value of u_1.

Ex 60: A sequence (u_n) is defined for any integer $n > 0$ by the integral:

$$u_n = \int_0^1 \frac{e^{nx}}{1+e^x}\,dx$$

1. Calculate u_1.
2. Prove that for any integer $n > 0$, the following recurrence relation holds:
$$u_{n+1} + u_n = \frac{e^n - 1}{n}$$
3. Hence, deduce the value of u_2.

C TECHNIQUES FOR INTEGRATION

C.1 INTEGRATION BY REVERSE CHAIN RULE

C.1.1 FINDING INTEGRALS FROM DERIVATIVES

Ex 61:

1. Find the derivative of $\arcsin(x)$.
$$\frac{d}{dx}(\arcsin(x)) = \boxed{}$$
2. Hence, find the indefinite integral $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx$.
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \boxed{}$$

Ex 62:

1. Find the derivative of $f(x) = \arctan(x)$.
$$\frac{d}{dx}(\arctan(x)) = \boxed{}$$
2. Hence, find the indefinite integral $\displaystyle\int \frac{1}{1+x^2}\,dx$.
$$\int \frac{1}{1+x^2}\,dx = \boxed{}$$

Ex 63:

1. Find the derivative of $f(x) = \ln(\cos(x))$.
$$\frac{d}{dx}(\ln(\cos(x))) = \boxed{}$$
2. Hence, find the indefinite integral $\displaystyle\int \tan(x)\,dx$.
$$\int \tan(x)\,dx = \boxed{}$$

Ex 64:

1. Find the derivative of $f(x) = x\ln(x) - x$.
$$\frac{d}{dx}(x\ln(x) - x) = \boxed{}$$
2. Hence, find the indefinite integral $\displaystyle\int \ln(x)\,dx$.
$$\int \ln(x)\,dx = \boxed{}$$

C.2 INTEGRATION BY SUBSTITUTION

C.2.1 INTEGRATING BY SUBSTITUTION FOR INDEFINITE INTEGRALS

Ex 65: Find the indefinite integral of $f(x) = 2x\cos(x^2)$.

$$\int 2x\cos(x^2)\,dx = \boxed{}$$

Ex 66: Find the indefinite integral of $f(x) = 3x^2(x^3+5)^4$.

$$\int 3x^2(x^3+5)^4\,dx = \boxed{}$$

Ex 67: Find the indefinite integral of $f(x) = \frac{4x^3}{x^4+1}$.

$$\int \frac{4x^3}{x^4+1}\,dx = \boxed{}$$

Ex 68: Find the indefinite integral of $f(x) = \cos^3(x)\sin(x)$.

$$\int \cos^3(x)\sin(x)\,dx = \boxed{}$$

C.2.2 EVALUATING DEFINITE INTEGRALS BY SUBSTITUTION

Ex 69: Find the value of the definite integral $\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx$.

$$\int_0^{\sqrt{\pi}} 2x\cos(x^2)\,dx = \square$$

Ex 70: Find the value of the definite integral $\int_0^1 \frac{x}{x^2+1}\,dx$.

$$\int_0^1 \frac{x}{x^2+1}\,dx = \square$$

Ex 71: Find the value of the definite integral $\int_0^{\pi/2} \cos^3(x)\sin(x)\,dx$.

$$\int_0^{\pi/2} \cos^3(x)\sin(x)\,dx = \square$$

Ex 72: Find the value of the definite integral $\int_0^1 6xe^{x^2}\,dx$.

$$\int_0^1 6xe^{x^2}\,dx = \square$$