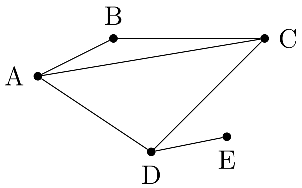
GRAPH THEORY

A DEFINITIONS

A.1 DETERMINING THE DEGREE OF VERTICES

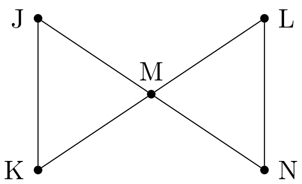
Ex 1:



Count the degree of the vertices:

- $\deg(A) = \square$
- $\deg(B) = \square$
- $\deg(C) = \square$
- $\deg(D) = \square$
- $\deg(E) = \square$

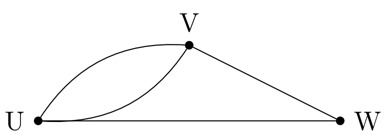
Ex 2:



Count the degree of the vertices:

- $\deg(J) = \square$
- $\deg(K) = \square$
- $\deg(L) = \square$
- $\deg(M) = \square$
- $\deg(N) = \square$

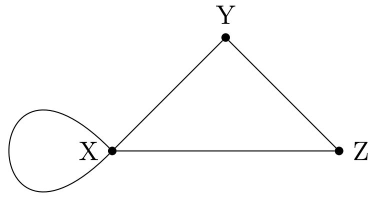
Ex 3:



Count the degree of the vertices:

- $\deg(U) = \square$
- $\deg(V) = \square$
- $\deg(W) = \square$

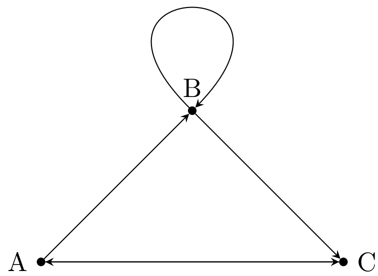
Ex 4:



Count the degree of the vertices (remember that a loop counts for 2):

- $\deg(X) = \square$
- $\deg(Y) = \square$
- $\deg(Z) = \square$

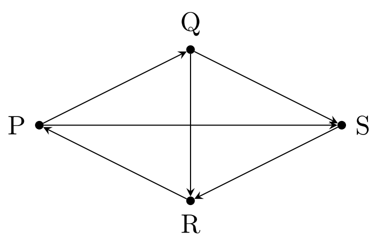
Ex 5:



Determine the in-degree ($\deg_{in}$) and out-degree ($\deg_{out}$) of the vertices:

- Vertex A: $\deg_{in}(A) = \square$, $\deg_{out}(A) = \square$
- Vertex B: $\deg_{in}(B) = \square$, $\deg_{out}(B) = \square$
- Vertex C: $\deg_{in}(C) = \square$, $\deg_{out}(C) = \square$

Ex 6:

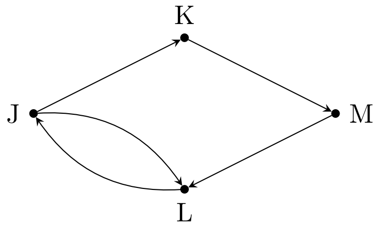


Count the in-degree and out-degree:

- $\deg_{in}(P) = \square$, $\deg_{out}(P) = \square$
- $\deg_{in}(Q) = \square$, $\deg_{out}(Q) = \square$
- $\deg_{in}(R) = \square$, $\deg_{out}(R) = \square$
- $\deg_{in}(S) = \square$, $\deg_{out}(S) = \square$

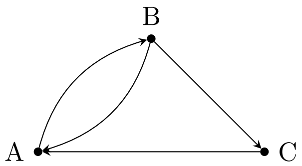
A.2 IDENTIFYING PATHS AND CIRCUITS

Ex 7: Consider the following directed graph (digraph):



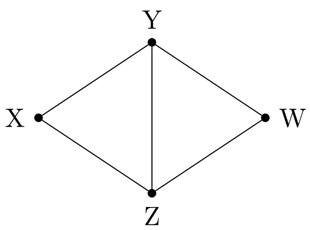
1. Is it possible to go directly from K to J?
2. Find a path from K to J.

Ex 8: Consider the following directed graph (digraph):



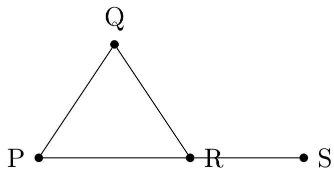
1. Is it possible to go directly from A to C?
2. Find a circuit starting and ending at A.

Ex 9: Consider the following undirected graph:



1. Is it possible to go directly from X to W?
2. Find a cycle starting and ending at Y that passes through Z.

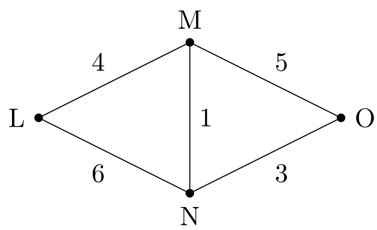
Ex 10: Consider the following undirected graph:



1. Is it possible to go from P to S without passing through R?
2. Find a path from Q to S.

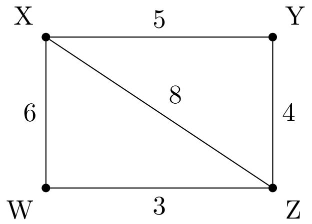
A.3 ANALYZING PATHS IN WEIGHTED GRAPHS

Ex 11: Consider the following weighted graph representing travel costs between cities:



1. What is the weight of edge MN?
2. Calculate the total weight of the path $L \to M \to O \to N \to L$.
3. Find the lowest total weight of the path from L to O.

Ex 12: Consider the following weighted graph representing driving times (in hours) between towns:

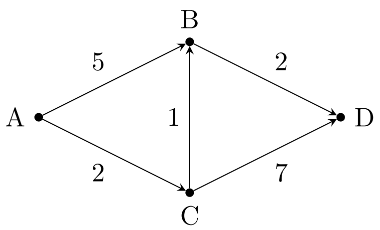


1. What is the weight of edge XZ?

2. Calculate the total weight of the path $W \to X \to Y \to Z$.

3. Find the lowest total weight of the path from W to Y.

Ex 13: Consider the following weighted directed graph representing flight costs between airports. Note that the connections are one-way (indicated by arrows).

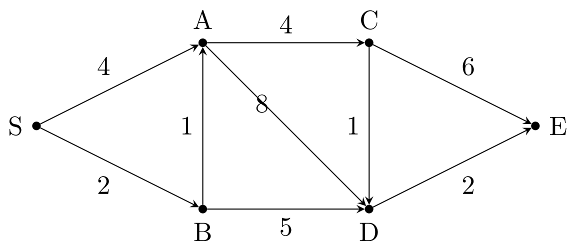


1. What is the weight of the directed edge from C to B?

2. Calculate the total weight of the path $A \to B \to D$.

3. Find the lowest total weight of a path from A to D.

Ex 14: Consider the following weighted directed graph representing latency (in ms) in a complex computer network from a source server S to a destination server E.



1. What is the weight of the directed edge from B to A?

2. Calculate the total weight of the path $S \to B \to A \to C \to D \to E$.

3. Find the lowest total weight of a path from S to E.

A.4 MODELLING SITUATIONS WITH GRAPHS

Ex 15: A computer network consists of 4 computers (labeled A, B, C, and D) connected to a central server (labeled S). Every computer is connected directly to the server, but not to each other.
Draw a graph to illustrate this network.

Ex 16: Four friends (Alice, Bob, Charlie, and David) meet for dinner. Everyone shakes hands with everyone else exactly once. Draw a graph to represent these handshakes, where vertices represent the people and edges represent the handshakes.

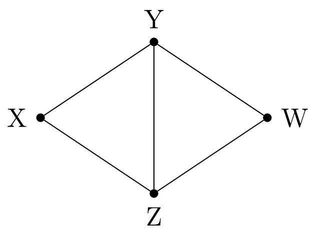
Ex 17: A train line serves four stations: North Station, Central Station, South Station, and Airport. The track goes from North to Central, then from Central to South, and finally from South to Airport. The train travels the same path in both directions. Draw a graph to illustrate the connections between these stations.

Ex 18: In a small business, a CEO manages two team leaders (L_1 and L_2). Team leader L_1 supervises two employees (E_1 and E_2), while team leader L_2 supervises one employee (E_3).
Draw a graph representing this hierarchy.

Ex 19: A food delivery company has a distribution center (D) and three delivery zones (Z_1, Z_2, Z_3). The delivery trucks leave the distribution center to go to Z_1 or Z_2. From Z_1, they can go to Z_2 or Z_3. From Z_2, they can only go to Z_3. Once in Z_3, they return directly to the distribution center D.
Draw a directed graph to illustrate these delivery routes.

A.5 CLASSIFYING SEQUENCES OF VERTICES

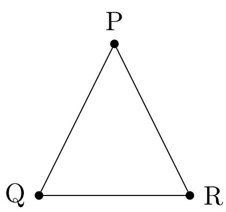
MCQ 20: Consider the following graph:



How is the sequence $X \to Y \to Z \to W$ classified?

- ☐ A path
- ☐ A trail
- ☐ A cycle
- ☐ A circuit

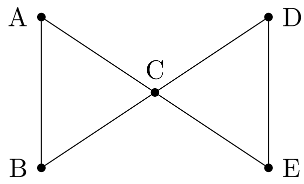
MCQ 21: Consider the following graph:



How is the sequence $P \to Q \to R \to P$ classified?

- ☐ A path
- ☐ A trail
- ☐ A cycle
- ☐ A circuit

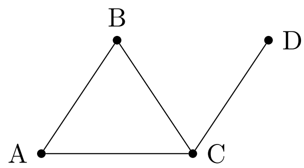
MCQ 22: Consider the following graph ("Bowtie"):



How is the sequence $A \to B \to C \to D \to E \to C \to A$ classified?

- ☐ A path
- ☐ A trail
- ☐ A cycle
- ☐ A circuit

MCQ 23: Consider the following graph:



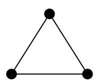
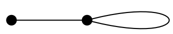
How is the sequence $D \to C \to B \to A \to C$ classified?

- ☐ A path
- ☐ A trail
- ☐ A cycle
- ☐ A circuit

B PROPERTIES OF GRAPHS

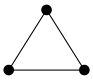
B.1 IDENTIFYING GRAPH PROPERTIES

MCQ 24: Which of the following graphs is a **simple graph**?

- ☐ 
- ☐ 
- ☐ 
- ☐ 

MCQ 25: Which of the following graphs is **connected**?

☐

☐ 

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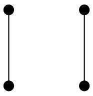
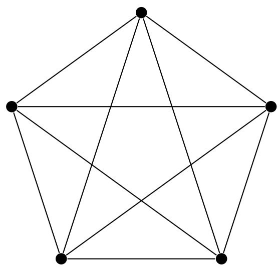
☐ 

Ex 26: Consider the complete graph K_5 shown below.



1. How many vertices does it have?

☐

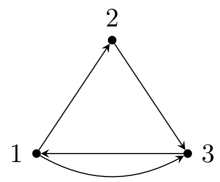
2. How many edges does it have?

☐

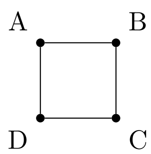
C ADJACENCY MATRICES

C.1 WRITING ADJACENCY MATRICES

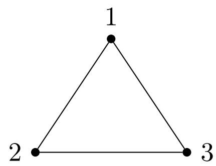
Ex 27: Write the adjacency matrix **M** for the following directed graph:



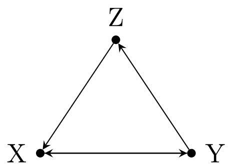
Ex 28: Write the adjacency matrix **M** for the following undirected graph:



Ex 29: Write the adjacency matrix **M** for the following undirected graph:



Ex 30: Write the adjacency matrix **M** for the following directed graph:



C.2 DETERMINING THE NUMBER OF WALKS

Ex 31: Consider the directed graph H given by the following adjacency matrix (vertices are ordered alphabetically: P, Q, R):

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Draw the directed graph H.

2. Find the number of walks of length 2 from vertex P to vertex R.

3. Find the number of walks of length 3 from vertex R to vertex Q.

Ex 32: Consider the undirected graph K given by the following adjacency matrix (vertices are ordered alphabetically: X, Y, Z):

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

1. Draw the undirected graph K.

2. Find the number of walks of length 2 from vertex X to vertex Y.

3. Find the number of walks of length 3 from vertex Y to vertex Z.

Ex 33: Consider the graph G given by the following adjacency matrix (vertices are ordered alphabetically: A, B, C, D):

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

1. Draw the non-directed graph G.

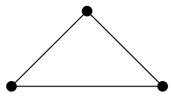
2. Find the number of walks of length 2 from vertex A to vertex D.

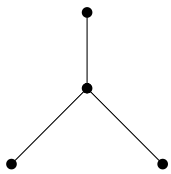
3. Find the number of walks of length 3 from vertex B to itself.

D TREES AND MINIMUM SPANNING TREES

D.1 IDENTIFYING TREES

MCQ 34: Which of the following graphs are **trees**?

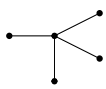
☐ 

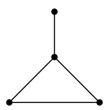
☐ 

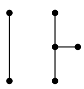
☐

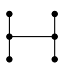
☐

MCQ 35: Which of the following graphs are **trees**?

☐ 

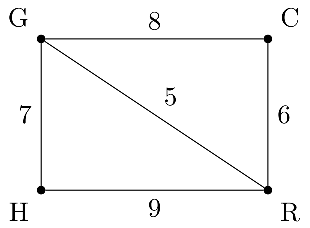
☐ 

☐ 

☐ 

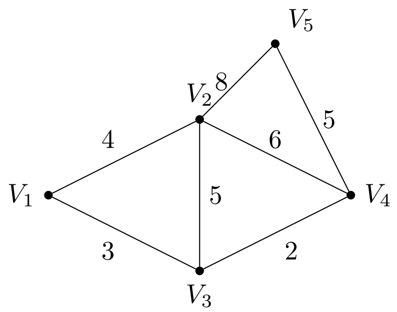
D.2 OPTIMIZING NETWORKS

Ex 36: A city council plans to build bike lanes connecting four parks: Green Park (G), Central Park (C), Riverside Park (R), and Hilltop Park (H). The costs (in thousands of euros) to build lanes between these parks are given in the weighted graph below. The council wants to connect all parks with the minimum total cost.



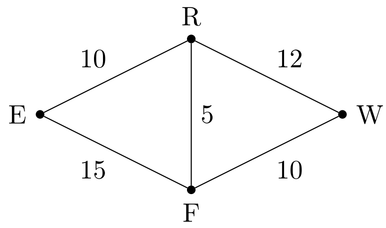
1. List the edges in ascending order of weight.
2. Determine the edges that form the Minimum Spanning Tree (MST).
3. Calculate the minimum cost to connect all the parks.

Ex 37: A utility company needs to connect five rural villages (V_1, V_2, V_3, V_4, V_5) to the power grid. The distances (in km) between the villages are given in the weighted graph below.
The company wants to minimize the total length of power lines installed.



1. List the edges in ascending order of weight.
2. Determine the edges that form the Minimum Spanning Tree (MST).
3. Calculate the minimum total length of the power lines.

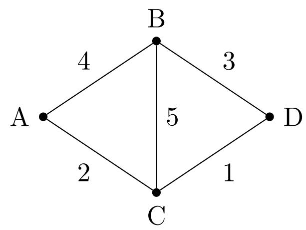
Ex 38: An amusement park wants to install a fiber optic network to connect four main zones: the Entrance (E), the Roller Coasters (R), the Food Court (F), and the Water Park (W). The costs of laying the cables between these zones (in thousands of dollars) are shown in the weighted graph below.
The park wants to connect all zones with the minimum possible cost.



1. List the edges in ascending order of weight.
2. Determine the edges that form the Minimum Spanning Tree (MST).
3. Calculate the minimum cost to connect all the zones.

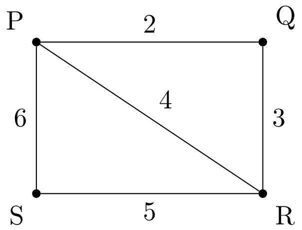
D.3 DETERMINING THE MINIMUM SPANNING TREE

Ex 39: Consider the weighted graph below:



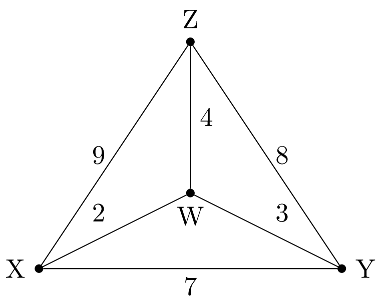
1. List the edges in ascending order of weight.
2. Using Kruskal's algorithm or Prim's algorithm, find the total weight of the Minimum Spanning Tree (MST).

Ex 40: Consider the weighted graph below:



1. List the edges in ascending order of weight.
2. Using Kruskal's algorithm or Prim's algorithm, find the total weight of the Minimum Spanning Tree (MST).

Ex 41: Consider the weighted graph below with a central node:

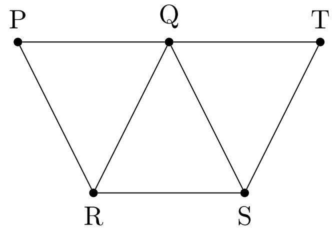


1. List the edges in ascending order of weight.
2. Using Kruskal's algorithm or Prim's algorithm, find the total weight of the Minimum Spanning Tree (MST).

E EULERIAN GRAPHS

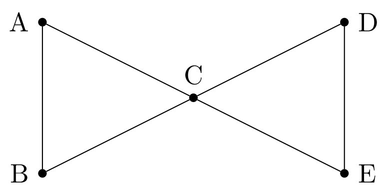
E.1 IDENTIFYING EULERIAN CIRCUITS AND TRAILS

Ex 42: Consider the following graph:



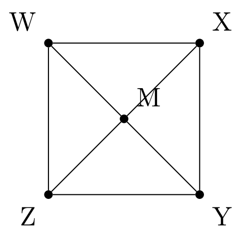
Does this graph contain an Eulerian circuit? If not, does it contain an Eulerian trail? Justify your answer.

Ex 43: Consider the following graph:



Does this graph contain an Eulerian circuit? If not, does it contain an Eulerian trail? Justify your answer.

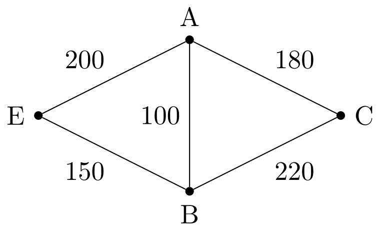
Ex 44: Consider the following graph:



Does this graph contain an Eulerian circuit? If not, does it contain an Eulerian trail? Justify your answer.

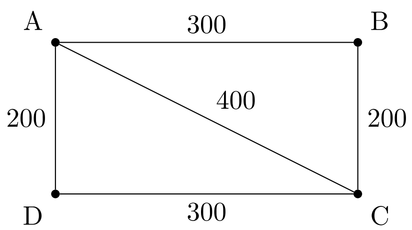
E.2 SOLVING THE CHINESE POSTMAN PROBLEM

Ex 45: A park ranger needs to inspect every path in a park shown by the graph below. The lengths of the paths are given in meters. The ranger starts at the entrance E and must return to E.



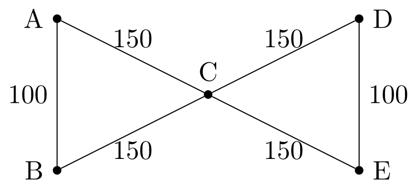
1. Explain why it is impossible to traverse every path exactly once and return to E.
2. Solve the Chinese Postman Problem to find the minimum distance the ranger must travel.

Ex 46: A snowplow must clear snow from all the roads connecting four intersections A, B, C, and D. The distances in meters are shown in the graph below. The snowplow starts at the garage located at intersection A and must return to A at the end of the shift.



1. Explain why an Eulerian circuit does not exist in this graph.
2. Solve the Chinese Postman Problem to find the minimum distance the snowplow must travel to clear every road and return to A.

Ex 47: A maintenance worker needs to paint the lines on the paths of a small park, represented by the graph below. The lengths are in meters. The worker starts at the central intersection C and must return to C after painting every path exactly once.

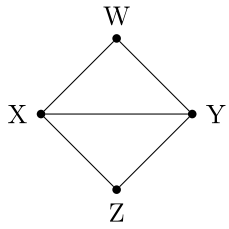


1. Determine the degree of each vertex.
2. Does this graph have an Eulerian circuit? If so, calculate the total distance the worker will travel.

F HAMILTONIAN GRAPHS

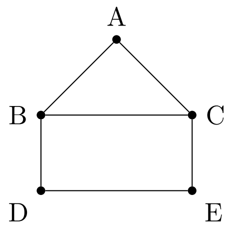
F.1 IDENTIFYING HAMILTONIAN PATHS AND CYCLES

Ex 48: Consider the following graph:



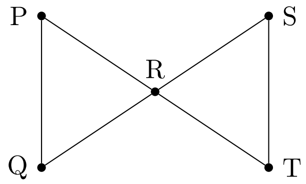
1. Find a Hamiltonian path from W to Z.
2. Does this graph contain a Hamiltonian cycle? If yes, list the vertices in order.

Ex 49: Consider the following graph:



1. Find a Hamiltonian path from D to E.
2. Does this graph contain a Hamiltonian cycle? If yes, list the vertices in order.

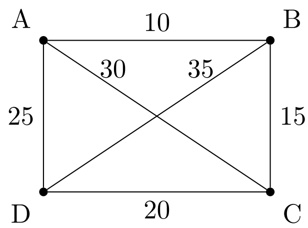
Ex 50: Consider the "Bowtie" graph below:



1. Find a Hamiltonian path from P to T.
2. Does this graph contain a Hamiltonian cycle? Explain why or why not.

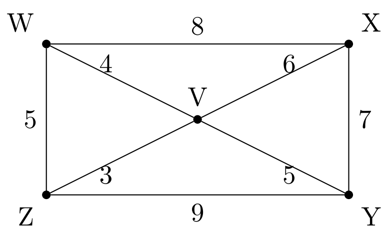
F.2 FINDING OPTIMAL TOUR BOUNDS

Ex 51: A delivery driver needs to visit 4 locations (A, B, C, D) starting and ending at the depot A. The distances (in km) are given by the weighted complete graph below.



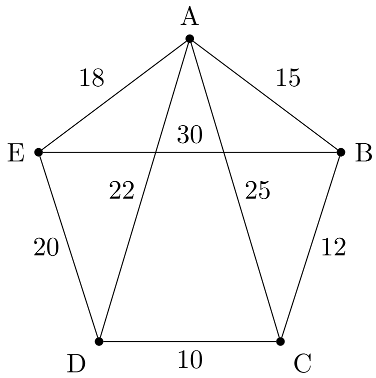
Use the **Nearest Neighbour Algorithm** starting at vertex A to find an upper bound for the optimal tour.

Ex 52: A network administrator wants to find the shortest loop to check 5 servers arranged in a star topology with a ring connection. The graph below shows the latency (ms). The central server is V.



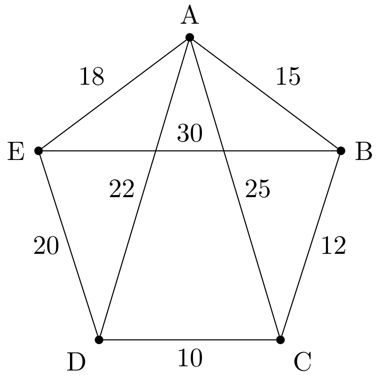
Use the **Deleted Vertex Algorithm** (deleting the central vertex V) to find a lower bound for the optimal tour.

Ex 53: A politician is campaigning and needs to visit 5 towns: A, B, C, D, E. The travel times (in minutes) are given in the weighted graph below. The politician starts at town A and must visit every town exactly once before returning to A.



Use the **Nearest Neighbour Algorithm** starting at vertex A to find an upper bound for the travel time.

Ex 54: A politician is campaigning and needs to visit 5 towns: A, B, C, D, E. The travel times (in minutes) are given in the weighted graph below. The politician wants to estimate the minimum time for a tour.



Use the **Deleted Vertex Algorithm** (deleting vertex A) to find a lower bound for the optimal tour.